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Riordan matrices and higher-dimensional lattice walks

Asamoah Nkwanta

Department of Mathematics, Morgan State University, Baltimore, MD 21251, USA

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ABSTRACT

An algebraic combinatorial method is used to count higher-dimensional lattice walks in $\mathbb{Z}^m$ that are of length n ending at height k. As a consequence of using the method, Sands' two-dimensional lattice walk counting problem is generalized to higher dimensions. In addition to Sands' problem, another subclass of higher-dimensional lattice walks is also counted. Catalan type solutions are obtained and the first moments of the walks are computed. The first moments are then used to compute the average heights of the walks. Asymptotic estimates are also given.

1. Introduction

We use properties of the Riordan group to prove that certain matrix entries of an infinite two-dimensional array count higher-dimensional lattice walks of length n ending at height k. The infinite array is constructed using Riordan matrices. A Riordan matrix is a special infinite lower-triangular matrix where the columns of the matrix are coefficients of certain formal power series. Furthermore, the set of all Riordan matrices forms a group called the *Riordan group* (Shapiro et al., 1991). The walks are defined on the integral lattice $\mathbb{Z}^m$ under the conditions that each walk starts at the origin, moves unit steps according to specified restrictions, and never pass below certain hyperplanes. The length n of a walk is the number of unitary steps and the height k, which corresponds to the endpoint of the walk, will either touch (but not pass below) a particular hyperplane or be a certain distance above a particular hyperplane. The infinite two-dimensional array which is denoted by

$$\mathcal{L}_{i,j} = P^i C_0 \hat{E}^j$$

is a triple product of Riordan matrices P^i, C_0, and $\hat{E}^j$ where P^i is the i th power of the well-known Pascal (triangle) matrix P (see Example 2.1 and Eq. (4)), C_0 is the aerated Catalan matrix defined by Eq. (5), and $\hat{E}^j$ (defined by Eq. (6)) is the j th power of the infinite lower-triangular matrix $\hat{E}$ which has all 1's on and below the main diagonal and 0's everywhere else. See Fig. 1 for the first few entries of the array $\mathcal{L}_{i,j}$.

The infinite array $\mathcal{L}_{i,j}$ was initially constructed in Nkwanta (1997) where the first few entries are explicitly given (also see Nkwanta, 2003). Some matrices that appear in $\mathcal{L}_{i,j}$ have extensive combinatorial applications (e.g., the Pascal, Catalan, Motzkin, hexagonal (Hex), and directed animal arrays). Combinatorial objects counted by these matrix entries of $\mathcal{L}_{i,j}$ include ballot sequences (Shapiro, 1976), random walks (Donaghey and Shapiro, 1977), lattice paths with various restrictions (Guy, 2000; Guy et al., 1992), hexagonal graphs with certain restrictions (Harary and Read, 1972), directed animals (Gouyou-Beauchamps and Viennot, 1988), and interval graphs (Hanlon, 1982). Thus, $\mathcal{L}_{i,j}$ is of combinatorial

E-mail address: asamoah.nkwanta@morgan.edu

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$$L_{i,j} := \begin{array}{llll} C_0 & \to C_0\widehat{E} & \to C_0\widehat{E}^2 & \to \cdots \\ \downarrow & \downarrow & \downarrow & \\ PC_0 & \to PC_0\widehat{E} & \to PC_0\widehat{E}^2 & \to \cdots \\ \downarrow & \downarrow & \downarrow & \\ P^2C_0 & \to P^2C_0\widehat{E} & \to P^2C_0\widehat{E}^2 & \to \cdots \\ \downarrow & \downarrow & \downarrow & \\ P^3C_0 & \to P^3C_0\widehat{E} & \to P^3C_0\widehat{E}^2 & \to \cdots \\ \downarrow & \downarrow & \downarrow & \\ \vdots & \vdots & \vdots & \end{array}$$

Fig. 1. Infinite array of Riordan matrices.

interest. A remarkable characteristic of $\mathcal{L}_{i,j}$ is that it unifies many well-known combinatorial arrays, generating functions, and counting sequences (see Nkwanta, 2003). In addition, $\mathcal{L}_{i,j}$ is itself a Riordan matrix (see Theorem 2.3).

The problem of counting random walks on lattices with unitary steps north, south, east, and west was first posed by physicists as an approximate model for Brownian motion (Hughes, 1995). Sands (1990) asked for a simple counting argument for the number of different walks with n steps such that each step moves one unit either north, south, east or west, starting from the origin and remaining in the upper half-plane. Surprisingly, by moving down the leftmost column of $\mathcal{L}_{i,j}$ we find the solution of Sands' problem. In addition, we extend Sands' problem to higher dimensions where there are countably many step directions and the walks never pass below certain hyperplanes. Moreover, a matrix solution is found for this problem and the solution contains a simple choice argument. We also find an exact formula and asymptotic estimate for the walks. Another subclass of walks called *generalized* (*or partial-t*) *Motzkin walks* is given since they are also counted by the entries of the leftmost column. In addition, more results are also obtained by moving down the first column of $\mathcal{L}_{i,j}$.

By moving down the first column we obtain another surprising result, this time for a more restrictive subset of Sands type walks which we call *power walks*. These walks are also generalized to higher dimensions and a Motzkin analog is given. Thus an interesting implication, in reference to the Motzkin analogs, is that the first two columns of $\mathcal{L}_{i,j}$ model in an extremely natural way generalized Motzkin walks (see Section 3). Given the higher-dimensional walks for the columns, the weighted row sums of $\mathcal{L}_{i,j}$ are computed and the first moments are established. Then, using the first moments, we compute the average heights of the walks and obtain solutions involving the Catalan numbers c_n given below by Eq. (1).

Algebraic combinatorics involves the use of techniques from algebra, topology, and geometry in the solution of combinatorial problems, or the use of combinatorial methods to attack problems in these areas (Billera et al., 1999). Because of this interplay with many fields of mathematics, algebraic combinatorics is an area in which a wide variety of ideas and methods come together (Billera et al., 1999). The *Riordan group method*, which is both combinatorial and algebraic, is the main method used in this paper. The algebraic concept involves the Riordan group and the combinatorial concepts involve recurrences, generating functions, and bijections.

This paper is organized as follows. In Section 2, we make the paper self-contained by briefly outlining the Riordan group and defining multiplication of Riordan matrices, defining the Riordan group method, and defining the first moments of a Riordan matrix. Readers familiar with the Riordan group may skip these minor (but important) details and go directly to Section 3. The results will begin here where we prove that the first two columns of $\mathcal{L}_{i,j}$ count certain subclasses of higher-dimensional lattice walks. Note that the higher-dimensional generalizations are the main results of this paper. In Section 4, the moment generating function of the sequence associated with the entries of the weighted row-sum vector associated with $\mathcal{L}_{i,j}$ is computed and the average heights of the higher-dimensional walks are derived. Some discussion of other approaches and open problems related to this work are mentioned in Section 5.

2. Riordan: group, matrices, and method

The Riordan group depends upon certain formal power series. Before the definition, let $\mathbb{N}$ denote the natural numbers and $\mathbb{C}$ the complex numbers.

Definition 2.1. An infinite matrix $L=(\ell_{n,k})_{n,k\in\mathbb{N}}$ with entries in $\mathbb{C}$ is called a *Riordan matrix* if the k th column satisfies

$$\sum_{n\geq 0}\ell_{n,k}z^n := g(z)(f(z))^k,$$

where $g(z)=1+g_1z+g_2z^2+\cdots$ and $f(z)=f_1z+f_2z^2+f_3z^3+\cdots$ belong to the ring of formal power series $\mathbb{C}[[z]]$ and $f_1\neq 0$.

The concept of representing columns of infinite matrices by coefficients of formal power series is not new and goes back to Schur's paper on Faber polynomials (Schur, 1945). A formal power series of the form

$$b(z) = b_0 + b_1 z + b_2 z^2 + \cdots = \sum_{n \geq 0} b_n z^n,$$

where z is an indeterminate is called the *ordinary generating function* of the sequence $\{b_n\}$. The *Catalan generating function is*

$$c(z) := \frac{1-\sqrt{1-4z}}{2z} = \sum_{n \geq 0} c_n z^n \quad \text{where } c_n = \frac{1}{n+1}\binom{2n}{n}. \tag{1}$$

Here c_n is the n th *Catalan number*. The numbers c_n occur in a wide variety of combinatorial problems and algebraic applications (see Stanley, 1999, pp. 219–229 and 231, respectively). A Riordan matrix can be defined by a pair of ordinary generating functions as $L=(g(z), f(z))$.

Example 2.1. Pascal's triangle (in lower triangular form) and Shapiro's Catalan array in Shapiro (1976) denoted, respectively, by

$$P = (1/(1-z), z/(1-z)) \quad \text{and} \quad P^2 C_0 = (c^2(z), zc^2(z))$$

are Riordan matrices.

Note that all the matrices presented in this paper are generated by using ordinary generating functions. However, Riordan matrices can also be generated by using exponential generating functions. See Berry (2007) and He et al. (2007) for new developments from this perspective.

We now give two important results for multiplying Riordan matrices. The first theorem is called the *Fundamental Theorem of the Riordan Group* (Nkwanta and Shapiro, 2005). The second theorem follows from the first by applying the fundamental theorem to an arbitrary Riordan matrix N, one column of N at a time.

Theorem 2.1 (*Shapiro et al., 1991; Nkwanta and Shapiro, 2005*). *If $L = (\ell_{n,k})_{n,k \in \mathbb{N}} = (g(z), f(z))$ is a Riordan matrix and $h(z)$ is the generating function of the sequence associated with the entries of the column vector $h = (h_k)_{k \in \mathbb{N}}$, then the product of L and $h(z)$, defined by $L \otimes h(z) = g(z)h(f(z))$, is the generating function of the sequence associated with the entries of the column vector $(\sum_{k=0}^{n} \ell_{n,k} h_k)_{n \in \mathbb{N}}$.*

Proof. See Shapiro et al. (1991, Equation 5) or Nkwanta and Shapiro (2005, Proof of Theorem 2.1). □

Let us denote by $L*N$, or by simple juxtaposition LN, the row-by-column product of two Riordan matrices.

Theorem 2.2 (*Shapiro et al., 1991; Sprugnoli, 1995*). *If*

$$L = (\ell_{n,k})_{n,k \in \mathbb{N}} = (g(z), f(z)) \quad \text{and} \quad N = (v_{n,k})_{n,k \in \mathbb{N}} = (h(z), l(z))$$

are Riordan matrices, then $L*N$ is

$$L * N = \left(\sum_{j=0}^{n} \ell_{n,j} v_{j,k} \right)_{n,k \in \mathbb{N}} = (g(z)h(f(z)), l(f(z))),$$

and the set $\mathbf{R}$ *of all Riordan matrices is a group under the operation of matrix multiplication.*

Proof. See Shapiro et al. (1991, Equation 6) and Sprugnoli (1995). □

Example 2.2. Consider the Pascal and Catalan matrices mentioned above. Then their product is the Hex matrix H given by

$$H = P^3 C_0 = \left(\frac{1-3z-\sqrt{1-6z+5z^2}}{2z^2}, \frac{1-3z-\sqrt{1-6z+5z^2}}{2z} \right).$$

See Figure 4 in Nkwanta (2003) for the first few entries.

The group, we denote by $(\mathbf{R},*)$, is called the Riordan group. This noncommutative group is a generalization of the renewal array theory that was introduced by Rogers (1978) to study generalizations of the ordinary Pascal, Catalan, and Motzkin arrays. The Riordan group is still developing (Cameron and Nkwanta, 2005; He et al., 2007; Shapiro, 2003; Wilson, 2005) and finding applications in areas of mathematics outside of combinatorics (Berry, 2007; Egorychev and Zima, 2005;

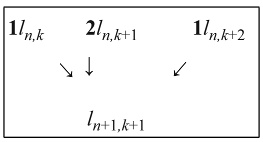

Fig. 2. A-sequence.

Huang, 2002; Nkwanta, 2008; Penson et al., 2004). For more details on the Riordan group see Shapiro et al. (1991) and Sprugnoli (1994, 1995).

2.1. Riordan group method

For a selected class of combinatorial objects, a counting approach to using the Riordan group is outlined by the following steps:

(1) Count a few cases and establish a sequence of numbers (integers).
(2) If possible set up the sequence as a Riordan matrix.
(3) Find and prove (combinatorially) the matrix formation rule.
(4) Use the formation rule to identify $g(z)$ and $f(z)$ for a given Riordan matrix $L=(g(z), f(z))$.
(5) Compute $L \otimes h(z)$.

One key step of the method is to find the matrix formation rule. A *formation rule*, which we denote by [**Z**;**A**], is a recurrence relation which defines the way entries of a Riordan matrix are computed. The notation [**Z**;**A**] means "**Z**" coincides with the formation of the zeroth column and "**A**" coincides with the formation of the other columns. In addition to this notation, formation rules are also denoted by dot diagrams (see Fig. 2). Also, see Merlini et al. (1997) for dot diagrams and related properties of Riordan matrices.

Following Rogers (1978) and Merlini et al. (1997), with minor adjustments, we give two useful characterizations of a given Riordan matrix. The formation rules which determine Riordan matrices are called **A**- and **Z**-sequences. The **Z**-sequence ($\mathbf{Z}=\{z_0,z_1,\ldots\}$) characterizes the zeroth column. This means every element $\ell_{n+1,0}$ can be expressed as a linear combination of all the elements in the preceding row, i.e.,

$$\ell_{n+1,0}=z_0\ell_{n,0}+z_1\ell_{n,1}+z_2\ell_{n,2}+\cdots.$$

The **A**-sequence ($\mathbf{A}=\{a_0,a_1,\ldots\}$, $a_0\neq 0$) characterizes the other columns. In this case every element $\ell_{n+1,k+1}$ can be expressed as a linear combination with coefficients in **A** of the elements in the preceding row, starting from the preceding column on, i.e.,

$$\ell_{n+1,k+1}=a_0\ell_{n,k}+a_1\ell_{n,k+1}+a_2\ell_{n,k+2}+\cdots.$$

Thus, if $A(y)$ and $Z(y)$ are the generating functions (in indeterminate y) of the **A**- and **Z**-sequences, respectively, then for a given Riordan matrix the generating functions $g(z)$ and $f(z)$ are the solutions of the functional equations

$$f(z)=zA(f(z)) \quad \text{and} \quad g(z)=\frac{g_0}{1-zZ(f(z))}. \tag{2}$$

Conversely, $A(y)$ and $Z(y)$ can be determined by letting $y=f(z)$ and eliminating z from

$$A(y)=y/z \quad \text{and} \quad Z(y)=\frac{g(z)-g_0}{zg(z)}. \tag{3}$$

Example 2.3. The formation rule of P^2C_0 is [**2**,**1**;**1**,**2**,**1**] where $Z(y)=2+y$ and $A(y)=1+2y+y^2$. In general, $\ell_{n+1,k+1}$ is computed as illustrated in Fig. 2.

Now consider the following Riordan matrices:

$$P^i=(1/(1-iz),z/(1-iz)), \tag{4}$$

$$C_0=\left(\frac{1-\sqrt{1-4z^2}}{2z^2},\frac{1-\sqrt{1-4z^2}}{2z}\right) \quad \text{and} \tag{5}$$

$$\hat{E}^j=(1/(1-z)^j,z). \tag{6}$$

Then, $\mathcal{L}_{i,j}$ is a Riordan matrix by the following theorem.

Theorem 2.3 (Nkwanta, 2003, 1997). *For* $i,j\geq 0$,

$$\mathcal{L}_{i,j}=P^iC_0\hat{E}^j=\left(\frac{k_i(z)}{(1-zk_i(z))^j},zk_i(z)\right) \quad \textit{where} \quad k_i(z)=\frac{1-iz-\sqrt{(1-iz)^2-4z^2}}{2z^2}. \tag{7}$$

Proof. It is sufficient to apply the product rule for Riordan matrices. See Nkwanta (2003, Theorem 3.8) or Nkwanta (1997). □

Note that the generating function given by Eq. (7) is a generalized Catalan generating function since

$$k_i(z) = \frac{1}{1-iz}c\left(\frac{z^2}{(1-iz)^2}\right),$$

where $c(z)$ is given by Eq. (1). Thus, we call $\mathcal{L}_{i,j}$ a generalized Catalan-type array. The **A**- and **Z**-sequences of $\mathcal{L}_{i,j}$ are now given.

Theorem 2.4 (Nkwanta, 2003). *The generating functions of the* **A**- *and* **Z**-*sequences of* $\mathcal{L}_{i,j}$ *are* $A_{i,j}(y) = 1+iy+y^2$ *and*

$$Z_{i,j}(y) = \frac{i+y+(1-(1-y)^j)}{y},$$

respectively.

Proof. See (Nkwanta, 2003, Theorem 3.10). □

2.2. First moments

The first moments are given by the weighted row sums of a Riordan matrix.

Definition 2.2. The *first moments* of a Riordan matrix L are denoted and computed by

$$L_{(mo)} = \begin{pmatrix} g & gf & gf^2 & \cdots & gf^k & \cdots \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix} = \sum_{k\geq 0} kgf^k,$$

where the column gf^k is the k th column vector associated with the coefficients of the k th generating function $g(z)f^k(z)$ and $v \leftrightarrow (0,1,\ldots)^T$ is the column vector associated with the coefficients of the generating function $v(z) = z/(1-z)^2$. Another way of characterizing Riordan matrices and moments is to consider the bivariate generating function $G(t,z) = g(z)/(1-tf(z))$. The first moments (averages) are then computed by

$$\left(\frac{\partial G}{\partial t}\right)_{t=1} = \frac{g(z)f(z)}{(1-f(z))^2}. \tag{8}$$

This is equivalent to L multiplied by $(0,1,\ldots)^T$ to obtain the moment vector $L_{(mo)}$. However, in this paper, we are interested in using the bivariate generating function $H(t,z) = tg(z)/(1-tf(z))$. As a result of this generating function the moment count starts at one. Thus, the first moments (averages) are computed by

$$\left(\frac{\partial H}{\partial t}\right)_{t=1} = \frac{g(z)}{(1-f(z))^2}. \tag{9}$$

This case, which we use to obtain the moment vector denoted $\overline{L}_{(mo)}$, is equivalent to L multiplied by the column vector $\overline{v} \leftrightarrow (1,2,\ldots)^T$. Note $\overline{v}$ denotes the column vector associated with the coefficients of the generating function $\overline{v}(z) = 1/(1-z)^2$. The first moments of L, in this case, are computed by

$$\overline{L}_{(mo)} = \begin{pmatrix} g & gf & gf^2 & \cdots & gf^k & \cdots \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ \vdots \end{pmatrix} = \sum_{k\geq 0} (k+1)gf^k.$$

Lemma 2.1. *Let* $L = (g(z), f(z))$ *be a Riordan matrix. Then, the first moment generating functions of the sequences associated with the entries of the moment vectors* $L_{(mo)}$ and $\overline{L}_{(mo)}$ *of* L *are, respectively, denoted by* $\ell(z)_{(mo)}$ *and* $\overline{\ell}(z)_{(mo)}$ *and given by*

$$\ell(z)_{(mo)} = L \otimes v(z) = \frac{g(z)f(z)}{(1-f(z))^2}$$

and

$$\overline{\ell}(z)_{(mo)} = L \otimes \overline{v}(z) = \frac{g(z)}{(1-f(z))^2}.$$

Proof. Use Theorem 2.1, Definition 2.2, and compare with Eqs. (8) and (9). Also see Sprugnoli (1994). □

Given the moment generating functions $\ell(z)_{(mo)}$ and $\overline{\ell}(z)_{(mo)}$, then

$$\mu(n) := \frac{[z^n](\ell(z)_{(mo)})}{[z^n]\left(\dfrac{g(z)f(z)}{(1-f(z))^2}\right)} \tag{10}$$

and

$$\overline{\mu}(n) := \frac{[z^n](\overline{\ell}(z)_{(mo)})}{[z^n]\left(\dfrac{g(z)}{(1-f(z))^2}\right)}. \tag{11}$$

These ratios are used to compute the expected value of the occurrence of a certain event of a certain combinatorial object with n properties counted by the entries of a given Riordan matrix. The symbol $[z^n]$ denotes the n th coefficient of a corresponding generating function. For the given cases, the generating functions of the sequences associated with the entries of the moment vectors of a given Riordan matrix $L=(g(z), f(z))$ are $g(z)\,f(z)\,/(1-f(z))^2$ and $g(z)/(1-f(z))^2$.

Example 2.4. Recall $P^2C_0=(c^2(z), zc^2(z))$. Then, the first moment generating function of the sequence associated with the entries of the moment vector $(P^2C_0)_{(mo)}$ of Riordan matrix P^2C_0 is given by the moment generating function $p^2c_0(z)_{(mo)}$ where

$$p^2c_0(z)_{(mo)} = \frac{c^2(z)}{(2-c(z))^2} = 1+4z+16z^2+64z^3+\cdots\ . \tag{12}$$

The ratio $\overline{\mu}(n)$ in closed form is

$$\overline{\mu}(n) = \frac{[z^n](p^2c_0(z)_{(mo)})}{[z^n](c(z)/\sqrt{1-4z})} = \frac{4^n}{\binom{2n+1}{n}} \tag{13}$$

Eq. (12) was obtained by Shapiro et al. (1983) and a lattice path interpretation of Eq. (12) is given by Callan (1999). Using Eq. (11), we see that Eq. (13) counts the average height above the line $y=-1$ of all walks counted by P^2C_0. Furthermore, Eq. (13) is generalized in Section 5 (see Proposition 4.3). For more details on moments and ordinary generating functions, see Sedgewick and Flajolet (1996, pp. 136–139) and Wilf (1990, pp. 108–109).

3. Lattice walks

The subject of counting walks on the lattice in Euclidean space is one of the most important areas of combinatorics (Gessel et al., 1998). We consider counting walks on the lattice $\mathbb{Z}^m$ for the first two columns of $\mathcal{L}_{i,j}$.

Definition 3.1. A *lattice* walk is a sequence of contiguous and reversible unit steps which traverses an m-dimensional integral lattice $\mathbb{Z}^m$.

For $m=2$ and 3, the step directions are given in Fig. 3.

All walks begin at the origin and move unit steps according to the following conditions. In one and two dimensions the walks are considered to be in the (x,y) plane and never pass below the x-axis. The length of each walk is the number of unitary steps, and the height corresponds to the y value of the endpoint (x,y) of the walk. In three dimensions the walks are considered to be in the three-dimensional Euclidean space and never pass below the (x,y) plane. The height of each walk corresponds to the z value of the endpoint (x,y,z) of the walk. In higher dimensions, $m>3$, the walks are considered to be in the m-dimensional Euclidean space and never pass below the $(m-1)$th hyperplane $x_1+\cdots+x_{m-1}=0$. The height of each walk corresponds to the value x_m of the endpoint $(x_1,\ldots,x_m)$ of the walk. The higher-dimensional step directions are defined in Fig. 4.

To avoid confusion with notation, for the remainder of the paper we use m when counting walks on $\mathbb{Z}^m$ and $i=d$ when counting walks given by the i th column of $\mathcal{L}_{i,j}$.

3.1. Lattice walks and the leftmost column

Lattice walk interpretations are given for the entries of $\mathcal{L}_{i,0}$ (recall Fig. 1). We start with C^0, PC_0, P^2C_0, P^3C_0, and P^4C_0 and count walks of a given length n and height k where entries of each matrix are indexed n by k $(0\le k\le n)$. In one dimension,

In $\mathbb{Z}^2$:	$(0, 1)$ = N (North)	$(0, -1)$ = S (South)
	$(1, 0)$ = E (East)	$(-1, 0)$ = W (West)
In $\mathbb{Z}^3$:	$(0, 0, 1)$ = N (North)	$(0, 0, -1)$ = S (South)
	$(0, 1, 0)$ = E (East)	$(0, -1, 0)$ = W (West)
	$(1, 0, 0)$ = F (Forward)	$(-1, 0, 0)$ = B (Backward)

Fig. 3. Unit steps.

In $\mathbb{Z}^m$:	$(0,0,1,0,\ldots,0)=\mathrm{N}$	$(0,0,-1,0,\ldots,0)=\mathrm{S}$
	$(0,1,0,\ldots,0,0)=\mathrm{E}$	$(0,-1,0,\ldots,0,0)=\mathrm{W}$
	$(1,0,0,\ldots,0,0)=\mathrm{F}$	$(-1,0,0,\ldots,0,0)=\mathrm{B}$
	$\vdots$	$\vdots$
	$(0,0,\ldots,0,0,1)=\mathrm{T}$	$(0,0,\ldots,0,0,-1)=\mathrm{U}$

Fig. 4. Higher-dimensional step directions.

the entries of C^0 count NS walks. In two dimensions, the entries of PC_0 and P^2C_0 count, respectively, NSE and NSEW walks. In three dimensions, the entries of P^3C_0 and P^4C_0 count, respectively, NSEWF and NSEWFB walks. As earlier mentioned, the solutions of these problems are known and can be found by using reflection and certain bijections that are constructed between planar (NSEW) and linear (NS) lattice walks (Breckenridge et al., 1991; Guy et al., 1992; Sands, 1991).

In this paper we provide an alternative method which gives the solution to Sands' problem and leads to higher-dimensional analogs of this problem. Moreover, we give a unified combinatorial interpretation of the leftmost column of $\mathcal{L}_{i,j}$. We start with Sands' problem and illustrate the Riordan group method in the proof of this problem. We then conclude this section with mentioning Motzkin walks since they are also counted by $\mathcal{L}_{i,0}$.

3.1.1. Sands' problem

Proposition 3.1. *The total number of NSEW walks of length n in* $\mathbb{Z}^2$, *beginning at the origin* (0,0) *and never passing below the x-axis, is* $\binom{2n+1}{n}$.

Proof (*Sketch*). The Riordan group method gives the following. *Step* (1): Let $\ell(n)$ denote the number of NSEW walks of length n. By hand, the first few values of $\ell(n)$ are $\ell(0)=1, \ell(1)=3, \ell(2)=10, \ell(3)=35$, and $\ell(4)=126$. Sloane (2001) and Sloane and Plouffe (1995) suggest $\{\ell(n)\}_{n\geq 0}=\{1,3,10,35,126,\ldots\}$, sequence # A001700. Hence, $\ell(n)$ is conjectured to equal the central binomial coefficient $\binom{2n+1}{n}$. *Step* (2): We form a Riordan matrix by letting $\ell(n,k)$ denote the number of NSEW walks of length n ending at height k. From this interpretation, we observe the first few entries of P^2C_0. *Step* (3): A recurrence is defined. The initial condition is $\ell(0,0)=1$. Then for $n,k\geq 1$, $\ell(n,k)$ satisfies

$$\ell(n,k)=\mathbf{1}\ell(n-1,k-1)+\mathbf{2}\ell(n-1,k)+\mathbf{1}\ell(n-1,k+1),$$
$$\ell(n,0)=\mathbf{2}\ell(n-1,0)+\mathbf{1}\ell(n-1,1) \quad \text{and}$$
$$\ell(n,k)=0 \quad \text{when } k\geq n+1.$$

From these recurrence relations we get the formation rule [**2**, **1**; **1**, **2**, **1**] which does indeed define P^2C_0. *Step* (4): By the formation rule, we obtain

$$P^2C_0=(c^2(z),zc^2(z)).$$

Step (5): Following Fundamental Theorem 2.1 and multiplying the pair by $1/(1-z)$, and using certain properties of $c(z)$ gives

$$\frac{c^2(z)}{1-zc^2(z)}=\frac{c(z)}{\sqrt{1-4z}}.$$

This generating function represents the total number of NSEW walks. To find a closed form solution we use

$$\frac{c(z)}{\sqrt{1-4z}}=\left(\frac{1}{\sqrt{1-4z}}-1\right)\frac{1}{2z},$$

the binomial theorem, and convolution to obtain $\ell(n)=\binom{2n+1}{n}$ which is the desired solution. This settles the conjecture made in Step (1) given above. □

Remark 3.1. A closed form solution of $\ell(n)$ can also be derived by using Pascal's triangle or the Pfaff–Saalschutz identity.

An asymptotic estimate for $\ell(n)$ is now given.

Proposition 3.2. $\ell(n)\sim 4^n(2/\sqrt{\pi n})$ *as* $n\to\infty$.

Proof. Observe

$$\binom{2n+1}{n}=\frac{2n+1}{n+1}\binom{2n}{n}$$

and use Stirling's formula, or use Darboux's approximation method (Lueker, 1980). □

Remark 3.2. Also, observe

$$[z^n]\left(\frac{c^2(z)}{1-zc^2(z)}\right)=[z^n]\left(\frac{1}{2z\sqrt{1-4z}}-\frac{1}{2z}\right)=\frac{1}{2}\binom{2n+2}{n+1}\sim\frac{1}{2}\cdot\frac{4^{n+1}}{\sqrt{\pi(n+1)}}$$

is a classical result.

3.1.2. Higher dimensions

Definition 3.2. A higher-dimensional *lattice* walk is a sequence of contiguous and reversible unit steps which traverses an m-dimensional integral lattice $\mathbb{Z}^m$. These walks start at the origin and never pass below the hyperplane $x_1+\cdots+x_{m-1}=0$ for $m>3$.

Moving down the leftmost column of $\mathcal{L}_{i,j}$, matrix relations connecting Catalan, Motzkin, Hex arrays, and other arrays as well are derived. Considering the lattice walk interpretations previously mentioned for these matrices, we observed that left multiplication by P takes NS walks to NSE walks, NSE walks to NSEW walks, and NSEW walks to NSEWF walks. Geometrically, P acts as a *matrix transformation* in the sense that certain subsets of lattice walks of length n move from one dimension to two dimensions to three dimensions and so forth. As illustrated in Fig. 5, we view each multiplication by P as adding a half coordinate axis (first to the positive axis, then to the negative axis).

Continuing with left multiplication by P and moving down the leftmost column, the $(m+1)$th-dimensional lattice walk counting problem arises. The unitary steps, given in Fig. 4, are denoted by N, S, and E_d $(d\geq 1)$ where E_1= E, E_2= W, E_3= F, $\mathrm{E}_4=B,\ldots,\mathrm{E}_{d-1}=T$, and E_d= U. Note that the E_d steps are the different kind of unitary steps different from N and S that are used to form the higher-dimensional walks. Thus, we have the following theorem.

Theorem 3.1. *The number of NSE$_1,\ldots,E_d$ walks ℓ of length n in $\mathbb{Z}^{m+1}$, beginning at the origin and never passing below the hyperplane $x_1+\cdots+x_m=0$, is given by C_0=$(k_0(z), zk_0(z))$ for i=d=0, m=0 and $\ell\in\mathbb{Z}^1$. If $i=d>0$ and $m>0$ then the walks are given by*

$P^dC_0=(k_d(z),zk_d(z))$, if $\ell\in\mathbb{Z}^{m+1}$ s.t. the E_dth step moves along coordinate axis in $\mathbb{Z}^{m+1}$ in a positive direction, and

$P^{d+1}C_0=(k_{d+1}(z),zk_{d+1}(z))$, if $\ell\in\mathbb{Z}^{m+1}$ s.t. the E_dth step moves along coordinate axis in $\mathbb{Z}^{m+1}$ in both positive and negative directions.

Proof. By Theorems 2.3 and 2.4, matrix P^dC_0 has formation rule $[\mathbf{d},\mathbf{1};\mathbf{1},\mathbf{d},\mathbf{1}]$. Let $\ell^*(n,k)$ denote the number of $\mathrm{NSE}_1,\ldots,E_d$ lattice walks of length n ending at height k in $\mathbb{Z}^{m+1}$. Then by the formation rule, for $n,k\geq 1$ the numbers $\ell^*(n,k)$ are recursively defined by $\ell^*(0,0)=1$ and

$$\begin{aligned}&\ell^*(n,k)=\mathbf{1}\ell^*(n-1,k-1)+\mathbf{d}\ell^*(n-1,k)+\mathbf{1}\ell^*(n-1,k+1),\\&\ell^*(n,0)=\mathbf{d}\ell^*(n-1,0)+\mathbf{1}\ell^*(n-1,1)\quad\text{and}\\&\ell^*(n,k)=0\quad\text{when } k\geq n+1.\end{aligned}$$

We now want to prove the recurrence relations for the walks. To form a walk of length n and height k consider the following cases and refer to Fig. 6. Case (i): If a walk has length $n-1$ and height $k-1$, then on the last step there is **1** choice for height $k-1$ (the N step). In this case, all walks whose last step is N are counted by $\mathbf{1}\ell^*(n-1,k-1)$. Case (ii): If a walk has length $n-1$ and height k, then on the last step there are **d** choices for height k (the $\mathrm{E}_1,\mathrm{E}_2,\ldots,\mathrm{E}_d$ different kind of unitary steps). In this case, all walks whose last step is either E_1, $\mathrm{E}_2,\ldots$, or E_d is counted by $\mathbf{d}\ell^*(n-1,k)$. Case (iii): If a walk has length $n-1$ and height $k+1$, then on the last step there is **1** choice for height $k+1$ (the S step). In this case, all walks whose last step is S are counted by $\mathbf{1}\ell^*(n-1,k+1)$. Combining all of the cases gives all possible ways of forming the walk of length n ending at height k, and the step directions E_1, $\mathrm{E}_2,\ldots$, and E_d do not change the height. Again, see Fig. 6.

Applying the addition principle, recurrence $\ell^*(n,k)$ is proved. By similar reasoning, recurrence $\ell^*(n,0)$ is also proved. The boundary condition $\ell^*(n,k)=0$ is trivial since there are no walks of length n ending at height $n+1$. This proves the formation rule and gives $\mathcal{L}_{i,0}$ a lattice walk interpretation. □

Theorem 3.2. *The total number of* $\mathrm{NSE}_1,\ldots,\mathrm{E}_d$ *walks of length n counted by $\mathcal{L}_{i,0}$ is given by*

$$\sum_{n\geq 0}\ell_i^*(n)z^n=\left(\frac{\sqrt{1-(i-2)z}}{\sqrt{1-(i+2)z}}-1\right)\frac{1}{2z}\quad\textit{where}$$

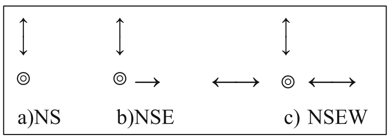

Fig. 5. One and two dimension step directions.

$$
\begin{array}{|llll|}
\hline
 & \downarrow S & & k+1 \\
\mathrm{E}_1 \searrow & & \swarrow\ \mathrm{E}_2 & \\
\vdots & \bullet & \vdots & k \\
\mathrm{E}_{d-1} \nearrow & & \nwarrow\ \mathrm{E}_d & \\
 & \uparrow \mathrm{N} & & k-1 \\
\hline
\end{array}
$$

Fig. 6. Possible step directions at height k.

$$\ell_i^*(n) = \sum_{k \geq 0} (-1)^k (i+2)^{n-k} \binom{n}{k} c_k. \tag{14}$$

Proof. Use the Fundamental Theorems 2.1 and 3.1 to obtain the generating function. Then, recall that $c(z)$ is the Catalan generating function given by Eq. (1) and use

$$k_i^*(z) = \left(\frac{1}{(1-(i+2)z)}\right) c\left(\frac{-z}{(1-(i+2)z)}\right), \tag{15}$$

composition of formal power series, binomial coefficients, and convolution to obtain the final result. □

Some special cases of Theorems 3.1 and 3.2 are as follows. For $i=d=2$ and E_d moves along both positive and negative coordinate axes in $\mathbb{Z}^{m+1}$ we get Sands' problem. The case $i=d=4$ and E_d moves along both positive and negative coordinate axes in $\mathbb{Z}^{m+1}$ was proposed by Deutsch (2000) and solved by Brawner (2001). Another special case of interest is $i=d=1$ and E_d moves along both positive and negative coordinate axes in $\mathbb{Z}^{m+1}$ which gives Motzkin walks (Donaghey and Shapiro, 1977). These Motzkin walks and a generalization called partial t-Motzkin walks are subsequently given below since they are also counted by $\mathcal{L}_{i,0}$.

Asymptotic estimates are now given for $\ell_i^*(n)$ and $\ell_i^*(n,0)$. Note, recurrence $\ell_i^*(n,0)$ denotes the $\mathrm{NSE}_1,\dots,\mathrm{E}_d$ lattice walks of length n that return to the x-axis.

Proposition 3.3. *$\ell_i^*(n) \sim (i+2)^n((i+2)/\pi n)^{1/2}$ as $n \to \infty$.*

Proof. Rewrite (15) in the form

$$k_i^*(z) = \left(-\frac{1}{2z}\right) + \frac{\sqrt{1-(i-2)z}}{2z}\left(1-\frac{z}{1/(i+2)}\right)^{-1/2}.$$

Then, applying Darboux's method (simple version) (Lueker, 1980) gives

$$\ell_i^*(n) = \frac{1}{n} \cdot \frac{g_1\left(\frac{1}{(i+2)}\right)\sqrt{n}}{\Gamma(1/2)\left(\frac{1}{i+2}\right)^n} + o\left(\left(\frac{1}{i+2}\right)^{-n} n^{-1/2}\right) \sim (i+2)^n \left(\frac{i+2}{\pi n}\right)^{1/2}.$$

where $g_1(z) = \sqrt{1-(i-2)z}/2z$ and $\Gamma(1/2) = \sqrt{\pi}$ is the gamma function. Also see Sprugnoli (1994). □

Proposition 3.4. *$\ell_i^*(n,0) \sim (i+2)^{n+3/2}(1/\pi n^3)^{3/2}$ as $n \to \infty$.*

Proof. Rewrite $k_i(z)$ (from Eq. (7)), use Darboux's method, and arguments similar to those used to prove Proposition 3.3. □

3.1.3. Partial t-Motzkin walks

Definition 3.3. Motzkin walks are walks in the first quadrant that begin at the origin (0,0), end on the x-axis, and consist of the step set $S=\{(1,1), (1,-1), (1,0)\}$. If the level steps (1,0) come in t colors, then these walks are called t-Motzkin walks. If in addition, they end at (n,k), they are called partial t-Motzkin walks (Getu and Shapiro, 1998; Sulanke, 2000).

The generating function for partial t-Motzkin walks according to size (i.e., number of steps) is

$$m(z) = 1 + tzm(z) + z^2 m^2(z). \tag{16}$$

Indeed, every t-Motzkin walk is either a point or a level step of any of the t colors followed by a t-Motzkin walk or an elevated t-Motzkin walk.

The **A**- and **Z**-sequences of $\mathcal{L}_{i,0}$

$$A_{i,0}(y) = 1 + iy + y^2 \quad \text{and} \quad Z_{i,0}(y) = i + y$$

follow from Theorem 2.4. The recurrence relations implied by these generating functions show that the (n,k)-entry of P^iC_0 is equal to the number of partial t-Motzkin walks that end at the point (n,k). The walks are of length n and end at height k. For instance when $t=d=i$ for $i=1,2$, and 3, P^iC_0 counts 1,2, and 3-colored Motzkin walks of length n and end at height k.

Thus, P^iC_0 is intimately related to the partial t-Motzkin walks. Another way of seeing this is to recall P^iC_0 has formation rule $[\mathbf{i},\mathbf{1};\ \mathbf{1},\mathbf{i},\mathbf{1}]$ and compare $A_{i,0}(y)$ and Eq. (16).

3.2. Lattice walks and the first column

Computing partial row sums of the matrices of the 0th column gives the first column $\mathcal{L}_{i,1}$ of the array $\mathcal{L}_{i,j}$. Lattice walk interpretations are given for the entries of $\mathcal{L}_{i,1}$ (recall Fig. 1 to see the entries). We start with $C_0\hat{E}$, $PC_0\hat{E}$, $P^2C_0\hat{E}$, and $P^3C_0\hat{E}$ and count walks of a given length n and height k where entries of each matrix are indexed n by k $(0 \le k \le n)$. In two dimensions, the entries of $C_0\hat{E}$ and $PC_0\hat{E}$ count, respectively, NS$\tilde{E}$ and NSE$\tilde{W}$ walks. The NS$\tilde{E}$ walks are NSE walks such that all $\tilde{E}$ steps are restricted at height zero. The NSE$\tilde{W}$ walks are NSEW walks such that all $\tilde{W}$ steps are restricted at height zero. In three dimensions, the entries of $P^2C_0\hat{E}$ and $P^3C_0\hat{E}$ count, respectively, NSEW$\tilde{F}$ and NSEWF$\tilde{B}$ walks. The NSEW$\tilde{F}$ walks denote NSEWF walks such that all $\tilde{F}$ steps are restricted at height zero. The NSEWF$\tilde{B}$ walks denote NSEWFB walks such that all $\tilde{B}$ steps are restricted at height zero. A more restrictive subset of higher-dimensional lattice walks called power walks is now defined for column $\mathcal{L}_{i,j}$.

Definition 3.4. Let N and S be steps as previously defined and $\tilde{E}_{d+1} = \tilde{U}$ denote power walk steps where for $d \ge 0$, $\tilde{E}_1 = \tilde{E}$, $\tilde{E}_2 = \tilde{W}$, $\tilde{E}_3 = \tilde{F}$, etc. are additional different kinds of unitary steps restricted to height zero. A *power* walk $\tilde{\ell}$ is a $\mathrm{NSE}_1, \dots, E_d\tilde{U}$ walk of length n.

Example 3.1. $\tilde{W}$ENNESS$\tilde{W}$NEE is a two-dimensional NSE$\tilde{W}$ walk of length 11 and height 1 counted by $PC_0\hat{E}$. Note $\tilde{W}$ is an additional step at height zero that never leaves the x-axis.

A matrix solution of the power walk problem is then given by the following theorem.

Theorem 3.3. *The number of* $\mathrm{NSE}_1, \dots, \mathrm{E}_d\tilde{U}$ *power walks* $\tilde{\ell}$ *of length n in* $\mathbb{Z}^{m+2}$, *beginning at the origin and never pass below the hyperplane* $x_1 + \cdots + x_{m+1} = 0$, *are given by*

$P^{2d}C_0\hat{E} = (k^*_{2d}(z), zk_{2d}(z))$, *if* $\tilde{\ell} \in \mathbb{Z}^{m+2}$ *s.t. the* $\tilde{U} = \tilde{E}_{d+1}$*th step moves along coordinate axis in* $\mathbb{Z}^{m+2}$ *in a positive direction, and*

$P^{2d+1}C_0\hat{E} = (k^*_{2d+1}(z), zk_{2d+1}(z))$, *if* $\tilde{\ell} \in \mathbb{Z}^{m+2}$ *s.t. the* $\tilde{U} = \tilde{E}_{d+1}$*th step moves along coordinate axis in* $\mathbb{Z}^{m+2}$ *in both positive and negative directions.*

Proof. Recall that the generating function $k^*_{2d}(z)$ is defined by Eq. (15). Use Theorems 2.3 and 2.4 and arguments similar to those used to prove Theorem 3.1. □

Theorem 3.4. *The total number of power walks of length n of the ith entry of the first column of* $\mathcal{L}_{i,j}$ *is*

$$\sum_{n \ge 0} \tilde{\ell}_i(n) z^n = \frac{1}{1-(i+2)z},$$

where $\tilde{\ell}_i(n) = (i+2)^n$.

Proof. Use Riordan multiplication and Theorem 3.3. □

Remark 3.3. The asymptotic estimate for $\tilde{\ell}_i(n,0)$, which denote power walks of length n ending at height zero of $\mathcal{L}_{i,1}$, is the same as given by Proposition 3.3 since these walks are also represented by generating function $k^*_i(z)$.

3.2.1. Modified partial t-Motzkin walks

The Motzkin analog is now given. The **A**- and **Z**-sequences of $\mathcal{L}_{i,1}$ are as follows:

$$A_{i,1}(y) = 1 + iy + y^2 \quad \text{and} \quad Z_{i,1}(y) = i + 1 + y.$$

Thus the recurrence relations in the matrix $\mathcal{L}_{i,1}$, implied by these generating functions, show that the (n,k)-entry of $\mathcal{L}_{i,1}$ is equal to the number of certain "modified" partial t-Motzkin walks that end at the point (n,k). The modification consists in the fact that at level zero we have $i+1$ types of level steps. For instance, if $t=i$ and $i=1,2,$and 3, then $P^iC_0\hat{E}$ counts 2,3, and 4-colored modified Motzkin walks of length n and end at height k where at level zero there are additional types of level steps. It is easy to see that Example 3.1 is a 2-colored modified Motzkin walk since there are two different types of level steps at height 0.

4. The first moments of $\mathscr{L}_{i,j}$

The average heights of walks given by the first two columns of $\mathcal{L}_{i,j}$ are computed. Recall P^2C_0 counts NSEW walks (see Proposition 3.1) and there are three walks of length 1 and four steps are associated with the weighted row sums (see Eq. (12)). The weighted steps are the number of steps above the line $y=-1$. They are indicated by the double arrows given below in Fig. 7. Although measuring distances above the x-axis is a natural measure, we consider distances above the line $y=-1$. Recall that the first moments are computed according to Eq. (9). Thus by Eq. (11), $\overline{\mu}(1) = \frac{4}{3}$ is the average distance above the line $y=-1$ of all NSEW walks of length 1. See Fig. 7.

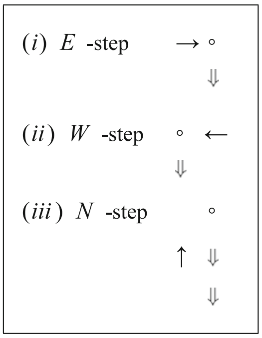

Fig. 7. Steps and moments.

For higher-dimensional walks of length n in $\mathbb{Z}^m$ and $m>3$, the first moments give the distances above certain hyperplanes. Assuming all walks of length n are equally probable and using Eq. (11) yields

$$\overline{\mu}(n)=\frac{\#(\text{distance of all walks})}{\#(\text{number of walks of length } n)}.$$

This ratio gives the average heights of certain walks. The first moments of $\mathcal{L}_{i,j}$ are computed and the average heights of the walks are now derived. Starting with

$$(\overline{\mathcal{L}}_{0,0})(z)_{(mo)}=\frac{1}{1-2z}\leftrightarrow\{2^n\},$$
$$(\overline{\mathcal{L}}_{1,0})(z)_{(mo)}=\frac{1}{1-3z}\leftrightarrow\{3^n\},$$
$$(\overline{\mathcal{L}}_{2,0})(z)_{(mo)}=\frac{1}{1-4z}\leftrightarrow\{4^n\},$$
$$(\overline{\mathcal{L}}_{3,0})(z)_{(mo)}=\frac{1}{1-5z}\leftrightarrow\{5^n\}$$

gives a nice pattern which leads to finding the first moments of $\mathcal{L}_{i,0}$.

Proposition 4.1.

$$(\overline{\mathcal{L}}_{i,0})(z)_{(mo)}=\frac{1}{1-(i+2)z},$$

and the first moments of $\mathcal{L}_{i,0}$ *equal* $(i+2)^n$.

Proof. Use Theorem 2.3 and Lemma 2.1. □

Similarly, the first moments of $\mathcal{L}_{i,1}$ are derived.

Proposition 4.2.

$$(\overline{\mathcal{L}}_{i,1})(z)_{(mo)}=\left(\frac{\sqrt{1-(i-2)z}}{\sqrt{1-(i+2)z}}+1\right)\frac{1}{2(1-(i+2)z)}.$$

Proof. Use Theorem 2.3 and Lemma 2.1. □

Corollary 4.1. *The first moments of* $\mathcal{L}_{i,1}$ *are*

$$\sum_{\ell\geq 0}\sum_{r\geq 0}(-1)^r(i+2)^{n-1-r}\binom{n-\ell-1}{r}c_r.$$

Proof. Recall that c_r denotes the r th Catalan number given by Eq. (1). Consider $(\overline{\mathcal{L}}_{i,1})(z)_{(mo)}$ (above) as the product of two generating functions. Then, using

$$k_i^*(z)=\frac{k_i(z)}{1-zk_i(z)}=\left(\frac{\sqrt{1-(i-2)z}}{\sqrt{1-(i+2)z}}-1\right)\frac{1}{2z}$$

replacing n with $n-1$ in Eq. (14), and using convolution of formal power series we obtain

$$1+zk_i^*(z)=\left(\frac{\sqrt{1-(i-2)z}}{\sqrt{1-(i+2)z}}+1\right)\frac{1}{2}=1+\sum_{n\geq 0}\left(\sum_{r\geq 0}(-1)^r(i+2)^{n-r}\binom{n}{r}c_r\right)z^{n+1}=1+\sum_{n\geq 1}\ell_i^*(n-1)z^n.$$

Then, the n th coefficients of each factor of $(\overline{\mathcal{L}}_{i,1})(z)_{(mo)}$ are

$$[z^n](1+zk_i^*(z))=\ell_i^*(n-1)\quad\text{and}\quad [z^n]\left(\frac{1}{1-(i+2)z}\right)=(i+2)^n.$$

Finally, use the convolution property to obtain $[z^n]((\overline{\mathcal{L}}_{i,1})(z)_{(mo)})$ which gives the result. □

The first moment generating function of the moment vector $(\mathcal{L}_{i,j})_{mo}$ is now derived.

Theorem 4.1. *For $i,j\geq 0$,*

$$(\overline{\mathcal{L}}_{i,j})(z)_{(mo)}=\frac{k_i(z)}{(1-zk_i(z))^{j+2}}.$$

Proof. Let $\Delta=z/(1-iz)$. Then, by Lemma 2.1, Eq. (1), Theorem 2.3, and simplifying we have

$$(\overline{\mathcal{L}}_{i,j})(z)_{(mo)}=\mathcal{L}_{i,j}\otimes\frac{1}{1-z}=\frac{(1-iz)^{j-1}c(\Delta^2)}{(1-iz-zc(\Delta^2))^j}\cdot\frac{(1-iz)^2}{(1-iz-zc(\Delta^2))^2}=\frac{(1-iz)^{j+1}c(\Delta^2)}{(1-iz-zc(\Delta^2))^{j+2}}=\frac{k_i(z)}{(1-zk_i(z))^{j+2}}.$$

Thus, the result is obtained. □

The expected value of certain combinatorial objects with n properties given by $\mathcal{L}_{i,j}$ is obtained by computing

$$\overline{\mu}(i,j)(n)=\frac{[z^n]\left((\overline{\mathcal{L}}_{i,j})(z)_{(mo)}\right)}{[z^n]\left((\mathcal{L}_{i,j})\otimes\dfrac{1}{1-z}\right)}=\frac{[z^n]\left(\dfrac{k_i(z)}{(1-zk_i(z))^{j+2}}\right)}{[z^n]\left(\dfrac{k_i(z)}{(1-zk_i(z))^{j+1}}\right)}. \tag{17}$$

This equation follows by Eq. (11) and Theorems 2.3 and 4.1. A consequence of Eq. (17) is that this ratio gives a unified approach to finding expected values for combinatorial objects counted by $\mathcal{L}_{i,j}$. The following propositions, which give Catalan results, are special cases of Eq. (17).

Proposition 4.3. *The average distance of all* $\mathrm{NSE}_1,\ldots,\mathrm{E}_d$ *walks* ℓ *of length* n *in* $\mathbb{Z}^{m+1}$ *that are above the hyperplane* $x_1+\cdots+x_m=0$ *is*

$$\overline{\mu}_{(d,0)}(n)=\frac{(d+2)^n}{\sum_{r\geq 0}(-1)^r(d+2)^{n-r}\binom{n}{r}c_r}.$$

Proof. Use Theorem 3.4, Eq. (17), and Proposition 4.1. □

Proposition 4.4. *The average distance of all* $\mathrm{NSE}_1,\ldots,\mathrm{E}_d\tilde{E}_{d+1}$ *walks* $\tilde{\ell}$ *of length* n *in* $\mathbb{Z}^{m+2}$ *that are above the hyperplane* $x_1+\cdots+x_{m+1}=0$ *is*

$$\overline{\mu}_{(d,1)}(n)=\frac{\sum_{\ell\geq 0}\sum_{r\geq 0}(-1)^r(d+2)^{n-1-r}\binom{n-\ell-1}{r}c_r}{(d+2)^n}.$$

Proof. Use Theorem 3.4, Eq. (17), Corollary 4.1, and Proposition 4.2. □

Related topics on moments of Dyck and generalized Motzkin paths are given by Chapman (1999) and Sulanke (2000), respectively. See Nkwanta (1997), Merlini et al. (1996), Sprugnoli (1994) and Getu and Shapiro (1998) for other applications, and Shapiro et al. (1983) for moment computations other than those involving first moments.

5. Other approaches and open problems

We have provided an alternative method called the Riordan method. Although the method is not new, we give a systematic approach to using the Riordan group, recurrence relations, and generating functions. We used the method to count higher-dimensional lattice walks for the first two columns of the infinite two-dimensional array $\mathcal{L}_{i,j}$. Some remarkable characteristics of $\mathcal{L}_{i,j}$ are that this infinite array compresses a wealth of combinatorial information into a generalized Riordan matrix, models higher-dimensional lattice walks, and unifies a large collection of combinatorial arrays, generating functions, and counting sequences. If we look at the first two columns of $\mathcal{L}_{i,j}$ as sequences of matrices $\{\mathcal{L}_{i,0}\}_{i\geq 0}$ and $\{\mathcal{L}_{i,1}\}_{i\geq 0}$, then finding lattice walk interpretations for the columns is equivalent to finding combinatorial interpretations for sequences of integer-valued matrices (or arrays). This generalizes the notion of finding a combinatorial interpretation for specific sequences of integers. More work is of combinatorial interest in this area. We are working on a lattice walk interpretation of the second column $\mathcal{L}_{i,2}$ of the array $\mathcal{L}_{i,j}$. We are also working on another related infinite two-dimensional array that contains column entries that count a less restrictive subset of higher-dimensional lattice walks and unify more combinatorial arrays, generating functions, and counting sequences. There is also a matrix connection between the forthcoming array and $\mathcal{L}_{i,j}$. Thus unified lattice walk interpretations will be subsequently given for more sequences of infinite lower-triangular matrices.

Some open problems related to this paper are listed below.

(1) Find bijections between the generalized Sands walks and other combinatorial objects such as rooted binary trees, interval graphs, edge rooted polyhexes, and/or directed animals.
(2) Find the n th moments of $\mathcal{L}_{i,j}$ and other statistics for the higher-dimensional walks such as variance, standard deviation, and limit distributions.
(3) Find combinatorial interpretations for all columns of $\mathcal{L}_{i,j}$ and interpretations for the corresponding moments $(\overline{\mathcal{L}}_{i,j})_{(mo)}$. In particular finding combinatorial interpretations for the first moments for the cases $j=0$ and 1 would be of interest (see Callan, 1999, for the case when $j=2$).
(4) Find a q-analogue of the entries of $\mathcal{L}_{i,j}$. In particular finding q-analogues for the cases $j=0$ and 1 would be of interest.
(5) Find an asymptotic estimate for $\mathcal{L}_{i,j}$. In particular finding estimates for the cases $j=0$ and 1 would be of interest. See Wilson (2005).
(6) Find connections to linkage folding (Demaine and O'Rourke, 2005), and connections to fractal geometry (Barnsley, 1993).
(7) Find connections to the umbral calculus, see Barnabei et al. (1982) and Roman (1984, pp. 6–31).
(8) Find a non-combinatorial proof for the change of coordinate axes which occurs when moving down $\mathcal{L}_{i,0}$. This may lead to areas of mathematics outside of combinatorics.

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