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ORIGINAL ARTICLE

Matrix inversion using orthogonal polynomials

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Abstract In this note we explicitly evaluate the determinants and inverses of certain matrices that generalize Hilbert matrices by exploiting the relationship between the kernel polynomials of some system of orthogonal polynomials and their associated Hankel matrices.

1. Introduction

In the theory of orthogonal polynomials, it is known that we could calculate the determinants of some Hankel matrices once we know the three term recurrence relation for the associated orthogonal polynomials and vice versa. It is also known

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that the kernel polynomials of the orthogonal polynomials encode important information about the Hankel matrices. In this note we present a method to invert some Hankel matrices associated with orthogonal polynomials. To illustrate this method, we compute some examples using the classical orthogonal polynomials and Wilson orthogonal polynomials. As a consequence of these calculations, we find the explicit determinants and inverses of certain matrices that generalize the well-known Hilbert matrices.

Theorem 1. *Given a probability measure μ on Ω with a support of infinite points, let us consider the Hilbert space of μ-measurable functions*

$$\mathcal{X} := \{f(x) \mid \int_\Omega |f(x)|^2 \, d\mu(x) < \infty\} \tag{1.1}$$

with the inner product defined as

$$(f, g) = \int_\Omega f(x)\overline{g(x)} \, d\mu(x), \quad f, g \in \mathcal{X}. \tag{1.2}$$

Assume that $\{w_n(x)\}_{n=0}^\infty$ is a sequence of linearly independent functions in $\mathcal{X}$ with $w_0(x) = 1$

$$\alpha_{jk} = \int_\Omega w_j(x)\overline{w_k(x)} \, d\mu(x), \quad j, k = 0, 1 \ldots, \tag{1.3}$$

and

$$\Delta_n := \begin{vmatrix} \alpha_{00} & \alpha_{01} & \ldots & \alpha_{0n} \\ \alpha_{10} & \alpha_{11} & \ldots & \alpha_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n0} & \alpha_{n1} & \ldots & \alpha_{nn} \end{vmatrix}. \tag{1.4}$$

For $n = 0, 1, \ldots$, the matrix $\Pi_n = (\alpha_{jk})$ is positive definite, consequently, $\Delta_n > 0$. The nth orthonormal function with positive coefficients in $w_n(x)$ is given by the formula

$$p_n(x) = \frac{1}{\sqrt{\Delta_n \Delta_{n-1}}} \begin{vmatrix} \alpha_{00} & \alpha_{01} & \ldots & \alpha_{0n} \\ \alpha_{10} & \alpha_{11} & \ldots & \alpha_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-10} & \alpha_{n-11} & \ldots & \alpha_{n-1n} \\ w_0(x) & w_1(x) & \ldots & w_n(x) \end{vmatrix} \tag{1.5}$$

for $n = 1, 2, \ldots$ with

$$p_0(x) = w_0(x) = 1. \tag{1.6}$$

Furthermore, the coefficient of $p_n(x)$ in $w_n(x)$ is

$$\gamma_n := \sqrt{\frac{\Delta_{n-1}}{\Delta_n}}, \tag{1.7}$$

and

$$\Delta_n = \prod_{k=1}^{n} \frac{1}{\gamma_n^2}. \tag{1.8}$$

The theorem is well-known in the theory of orthogonal polynomials, you may find it in Ismail (2005), Szeg (1975).

Lemma 2. *Let us assume that $\{p_k(x)\}_{k=0}^{\infty}$ are the orthonormal functions defined in Theorem 1. For each nonnegative integer n, the kernel function*

$$k_n(x, y) := \sum_{k=0}^{n} p_k(x)\overline{p_k(y)}, \tag{1.9}$$

has the reproducing property

$$\int_{\Omega} \pi(x)\overline{k_n(x, y)}\, d\mu(x) = \pi(y) \tag{1.10}$$

for $\pi(x)$ in the linear span of $w_k(x)$ for $k = 0, 1, \ldots, n$. Furthermore, there is only one kernel function with reproducing property (1.10) in the space generated by $w_k(x)$ for $k = 0, 1, \ldots, n$.

Proof. To see (1.10), just expand $\pi(x)$ in $\{p_k(x)\}_{k=0}^{n}$. If there is another kernel $h_n(x, y)$ with the same reproducing property, then for each fixed $y \in \Omega$,

$$\begin{aligned} 0 < \|h_n(\cdot, y) - k_n(\cdot, y)\|^2 &= (h_n(\cdot, y) - k_n(\cdot, y), h_n(\cdot, y) - k_n(\cdot, y)) \\ &= (h_n(\cdot, y) - k_n(\cdot, y), h_n(\cdot, y)) - (h_n(\cdot, y) - k_n(\cdot, y), k_n(\cdot, y)) = 0, \end{aligned} \tag{1.11}$$

which is a contradiction. □

Theorem 3. *For each nonnegative integer n, let $(\beta_{jk})_{0\leqslant j,k\leqslant n}$ be the inverse of $\Pi_n = (\alpha_{jk})_{0\leqslant j,k\leqslant n}$, then*

$$k_n(x, y) = \sum_{j,k=0}^{n} \beta_{jk}\overline{w_j(y)}w_k(x). \tag{1.12}$$

Proof. Let

$$f(x) = \sum_{k=0}^{n} u_k w_k(x), \tag{1.13}$$

then,

$$\left(f(\cdot), \sum_{j,k=0}^{n} \beta_{jk}\overline{w_j(y)}w_k(\cdot)\right) = \sum_{m=0}^{n} u_m(w_m(\cdot), k(\cdot, y))$$
$$= \sum_{m=0}^{n} u_m \sum_{j,k=0}^{n} \overline{\beta_{jk}} w_j(y)(w_m, w_k)$$
$$= \sum_{m=0}^{n} u_m \sum_{j=0}^{n} w_j(y) \sum_{k=0}^{n} \overline{\beta_{jk}\alpha_{km}} = f(y). \tag{1.14}$$

By Lemma 2 we have

$$k_n(x, y) = \sum_{j,k=0}^{n} \beta_{jk}\overline{w_j(y)}w_k(x). \qquad \square \tag{1.15}$$

Theorem 4. *Let $\{w_n(x)\}_{n=0}^{\infty}$ be the sequence as in Theorem 1, then the kernel is also given by*

$$k_n(x, y) = -\frac{1}{\Delta_n}\begin{vmatrix} 0 & 1 & \overline{w_1(y)} & \cdots & \overline{w_n(y)} \\ 1 & \alpha_{00} & \alpha_{01} & \dots & \alpha_{0n} \\ w_1(x) & \alpha_{10} & \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w_n(x) & \alpha_{n0} & \alpha_{n1} & \dots & \alpha_{nn} \end{vmatrix} \tag{1.16}$$

for $n = 0, 1, \dots$

Proof. Since

$$k_n(x, y) = \sum_{j,k=0}^{n} \beta_{jk}\overline{w_j(y)}w_k(x), \tag{1.17}$$

where

$$\Pi_n^{-1} = \left(\beta_{jk}\right)_{0 \leqslant i,j \leqslant n}. \tag{1.18}$$

Then,

$$\beta_{jk} = \frac{\Pi_n(k, j)}{\det \Pi_n} = \frac{\Pi_n(k, j)}{\Delta_n}, \tag{1.19}$$

where $\Pi_n(k, j)$ is the (k, j)th co-factor. Therefore,

$$k_n(x, y) = \frac{1}{\Delta_n}\sum_{j,k=0}^{n} \Pi_n(k, j)\overline{w_j(y)}w_k(x). \tag{1.20}$$

It is clear that

$$\sum_{j,k=0}^{n} \Pi_n(k,j)\overline{w_j(y)}w_k(x) = -\begin{vmatrix} 0 & (\overline{\mathbf{W}(\mathbf{y})})^T \\ \mathbf{W}(\mathbf{x}) & \Pi_n \end{vmatrix}, \tag{1.21}$$

by direct expansion, where

$$\mathbf{W}(\mathbf{x}) = \begin{pmatrix} 1 \\ w_1(x) \\ \vdots \\ w_n(x) \end{pmatrix}, \tag{1.22}$$

and

$$(\overline{\mathbf{W}(\mathbf{y})})^T = \left(1, \overline{w_1(y)}, \ldots, \overline{w_n(y)}\right). \tag{1.23}$$

Then

$$k_n(x,y) = -\frac{1}{\Delta_n}\begin{vmatrix} 0 & (\overline{\mathbf{W}(\mathbf{y})})^T \\ \mathbf{W}(\mathbf{x}) & \Pi_n \end{vmatrix}, \tag{1.24}$$

which is (1.16). □

Remarks 5. Theorems 3 and 4 are known for $w_k(x) = x^k$ (see Akhiezer, 1965).

Corollary 6. *Given a sequence $\{w_n(x)\}_{n=0}^{\infty}$ as in Theorem 1. Let us assume that there exists two families of linear functionals $\{u_k\}_{k=0}^{\infty}$ and $\{v_k\}_{k=0}^{\infty}$ over the linear space generated by $\{w_n(x)\}_{n=0}^{\infty}$ with the properties*

$$u_j(w_k) = \delta_{jk}, \tag{1.25}$$

and

$$v_j(\overline{w_k}) = \delta_{jk} \tag{1.26}$$

for $j,k = 0,1,\ldots$ Then, the elements of the inverse matrix $\Pi_n^{-1} = (\beta_{jk})_{0\leqslant j,k\leqslant n}$ of Gram matrix $\Pi_n = (\alpha_{jk})_{0\leqslant j,k\leqslant n}$ are given by the formula

$$\beta_{jk} = \sum_{m=0}^{n} u_k(p_m(x))v_j(\overline{p_m(y)}), \tag{1.27}$$

where $\{p_k(x)\}_{k=0}^{\infty}$ is the associated orthonormal functions defined via Eq. (1.5).

Proof. From Theorem 3, we have

$$\sum_{j,k=0}^{n} \beta_{jk}\overline{w_j(y)}w_k(x) = \sum_{m=0}^{n} \overline{p_m(y)}p_m(x). \tag{1.28}$$

Then we apply the functional u_j and v_k both sides of the above equation, the corollary follows. $\square$

The Barnes G-function is defined as

$$G(z) := (2\pi)^{z/2} e^{-[z(z+1)+\gamma z^2]/2} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^n e^{-z+z^2/(2n)}, \tag{1.29}$$

where

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right). \tag{1.30}$$

The Barnes G-function is an entire function with the property

$$G(z+1) = \Gamma(z)G(z). \tag{1.31}$$

Consequently,

$$\prod_{k=0}^{n} \Gamma(z+k) = \frac{G(z+n+1)}{G(z)}, \tag{1.32}$$

and

$$\prod_{k=0}^{n} (a)_k = \frac{G(a+n+1)}{G(a)\Gamma(a)^{n+1}}. \tag{1.33}$$

It is also known that

$$G(n) = \begin{cases} 0 & n = 0, -1, -2, \ldots, \\ \prod_{i=0}^{n-2} i! & n = 1, 2, \ldots \end{cases} \tag{1.34}$$

2. Examples

2.1. Classical polynomials

2.1.1. The Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$

The Hermite polynomials $\{H_n(x)\}$, $0 \leqslant n \leqslant \infty$ are defined as (Andrews et al., 1999; Ismail, 2005)

$$H_n(x) = (2x)^n {}_2F_0\left(\begin{matrix} -\frac{n}{2}, -\frac{n}{2}+\frac{1}{2} \\ - \end{matrix}; -\frac{1}{x^2} \right) \tag{2.1}$$

for $n \geqslant 0$ and

$$H_{-1}(x) = 0. \tag{2.2}$$

They satisfy the differential difference equation

$$\frac{dH_n(x)}{dx} = 2nH_{n-1}(x) \tag{2.3}$$

for $n = 0, 1 \ldots$

Hermite polynomials are orthogonal with respect to the measure $d\mu(x) = \exp(-x^2)dx$,

$$\int_{\mathbb{R}} H_n(x)H_m(x)\exp(-x^2)dx = 2^n n!\sqrt{\pi}\delta_{mn} \tag{2.4}$$

for $n, m = 0, 1, \ldots$ Thus, the orthonormal polynomials

$$h_n(x) := \frac{H_n(x)}{\sqrt{n!2^n\sqrt{\pi}}} \tag{2.5}$$

have leading coefficients

$$\gamma_n = \sqrt{\frac{2^n}{n!\sqrt{\pi}}}. \tag{2.6}$$

The moments of Hermite measure could be calculated

$$\mu_n = \int_{-\infty}^{\infty} y^n e^{-y^2}\,dy = \frac{1+(-1)^n}{2}\Gamma\left(\frac{n+1}{2}\right). \tag{2.7}$$

Then the (i,j)th entry of the matrix $\Pi_n = (\alpha_{jk})_{j,k=0}^n$ is

$$\alpha_{ij} = \frac{1+(-1)^{i+j}}{2}\Gamma\left(\frac{i+j+1}{2}\right) \tag{2.8}$$

for $i, j = 0, 1, \ldots, n$, thus, the determinant of the matrix is

$$\det\left(\frac{1+(-1)^{i+j}}{2}\Gamma\left(\frac{i+j+1}{2}\right)\right)_{j,k=0}^n = 2^{-\frac{n(n+1)}{2}}\pi^{\frac{n+1}{2}}\prod_{k=0}^n k!, \tag{2.9}$$

or

$$\det\left(\frac{1+(-1)^{i+j}}{2}\Gamma\left(\frac{i+j+1}{2}\right)\right)_{j,k=0}^n = 2^{-\frac{n(n+1)}{2}}\pi^{\frac{n+1}{2}}G(n+2) \tag{2.10}$$

for $n = 0, 1, \ldots$

In this case we use functionals defined by

$$u_i(p(x)) = v_i(p(x)) = \frac{1}{i!}\left[\frac{d^i p(x)}{dx^i}\right]_{x=0}, \tag{2.11}$$

The (i,j)th entry of the inverse matrix $\Pi_n^{-1} = (\beta_{jk})_{j,k=0}^n$ is

$$\begin{aligned}\beta_{ij} &= \frac{1}{i!j!}\sum_{k=\max(i,j)}^n \frac{1}{k!2^k\sqrt{\pi}}\left[\frac{d^i H_k(x)}{dx^i}\right]_{x=0}\left[\frac{d^j H_k(y)}{dy^j}\right]_{y=0} \\ &= \sum_{k=\max(i,j)}^n \frac{1}{k!2^k\sqrt{\pi}}\left\{2^i\binom{k}{i}H_{k-i}(0)\right\}\left\{2^j\binom{k}{j}H_{k-j}(0)\right\},\end{aligned} \tag{2.12}$$

or

$$\beta_{ij} = \frac{2^{i+j}}{\sqrt{\pi}} \sum_{k=\max(i,j)}^{n} \frac{1}{k!2^k} \left\{ \binom{k}{i} H_{k-i}(0) \right\} \left\{ \binom{k}{j} H_{k-j}(0) \right\}. \tag{2.13}$$

for $i,j = 0, 1, \ldots, n$.

Theorem 7. *For $n = 0, 1, \ldots,$ the matrix*

$$\left(\frac{1+(-1)^{i+j}}{2\sqrt{\pi}} \Gamma\left(\frac{i+j+1}{2}\right) \right)_{0 \leqslant i,j \leqslant n} \tag{2.14}$$

has the determinant

$$det\left(\frac{1+(-1)^{i+j}}{2\sqrt{\pi}} \Gamma\left(\frac{i+j+1}{2}\right) \right)_{i,j=0}^{n} = 2^{-\frac{n(n+1)}{2}} G(n+2), \tag{2.15}$$

and its inverse matrix is

$$\left(\sum_{k=max(i,j)}^{n} \frac{2^{i+j} \left\{ \binom{k}{i} H_{k-i}(0) \right\} \left\{ \binom{k}{j} H_{k-j}(0) \right\}}{k!2^k} \right)_{0 \leqslant i,j \leqslant n}. \tag{2.16}$$

2.1.2. The Laguerre polynomials $\{L_n^{\alpha}(x)\}_{n=0}^{\infty}$ with $\alpha > -1$

The Laguerre polynomials $\{L_n^{\alpha}(x)\}_{n=0}^{\infty}$ may be defined as (Andrews et al., 1999; Ismail, 2005)

$$L_n^{\alpha}(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix} ; x \right) \tag{2.17}$$

for $n \geqslant 0$, and we assume that

$$L_{-1}^{\alpha}(x) = 0. \tag{2.18}$$

We also have

$$\frac{dL_n^{\alpha}(x)}{dx} = -L_{n-1}^{\alpha+1}(x) \tag{2.19}$$

for $n \geqslant 0$. For $\alpha > -1$, we have

$$\int_0^{\infty} L_m^{\alpha}(x) L_n^{\alpha}(x) x^{\alpha} e^{-x}\, dx = \frac{\Gamma(\alpha+n+1)}{n!} \delta_{mn}, \tag{2.20}$$

for $n, m = 0, 1, \ldots$ Thus the orthonormal polynomials are

$$p_n(x) = (-1)^n \sqrt{\frac{n!}{\Gamma(\alpha+n+1)}} L_n^{\alpha}(x), \tag{2.21}$$

and the leading coefficients are

$$\gamma_n = \frac{1}{\sqrt{n!\Gamma(\alpha+n+1)}} \tag{2.22}$$

for $n = 0, 1, \ldots$ The moments of the Laguerre measure are

$$\mu_n = \int_0^\infty x^{n+\alpha} e^{-x}\, dx = \Gamma(\alpha+n+1) \tag{2.23}$$

for $n = 0, 1, \ldots$ Then (i,j)th entry of the matrix $\Pi_n = (\alpha_{jk})_{j,k=0}^n$ is

$$\alpha_{ij} = \Gamma(\alpha+i+j+1) \tag{2.24}$$

for $i, j = 0, 1, \ldots, n$, the determinant of the matrix is

$$\det\left(\Gamma(\alpha+i+j+1)\right)_{j,k=0}^n = \prod_{k=0}^n \{k!\Gamma(\alpha+k+1)\}, \tag{2.25}$$

or

$$\det\left(\Gamma(\alpha+i+j+1)\right)_{j,k=0}^n = \frac{G(n+2)G(\alpha+n+2)}{G(\alpha+1)}. \tag{2.26}$$

In this case we use the functionals defined in (2.11) and let $\Pi_n^{-1} = (\beta_{jk})_{j,k=0}^n$, then,

$$\begin{aligned}\beta_{ij} &= \frac{1}{i!j!} \sum_{k=\max(i,j)}^n \frac{k!}{\Gamma(\alpha+k+1)} \left[\frac{d^i L_k^\alpha(x)}{dx^i}\right]_{x=0} \left[\frac{d^j L_k^\alpha(y)}{dy^j}\right]_{y=0} \\ &= \frac{1}{i!j!} \sum_{k=\max(i,j)}^n \frac{k!}{\Gamma(\alpha+k+1)} \left[(-1)^i L_{k-i}^{\alpha+i}(x)\right]_{x=0} \left[(-1)^j L_{k-j}^{\alpha+j}(y)\right]_{y=0},\end{aligned} \tag{2.27}$$

or

$$\beta_{ij} = \frac{(-1)^{i+j}}{i!j!} \sum_{k=\max(i,j)}^n \frac{k! L_{k-i}^{\alpha+i}(0) L_{k-j}^{\alpha+j}(0)}{\Gamma(\alpha+k+1)} = \frac{(-1)^{i+j} \sum_{k=\max(i,j)}^n \frac{(\alpha+1)_k}{k!} \binom{k}{i} \binom{k}{j}}{(\alpha+1)_i (\alpha+1)_j \Gamma(\alpha+1)} \tag{2.28}$$

for $j, k = 0, 1, \ldots, n$.

Theorem 8. *For $n = 0, 1, \ldots$, the matrix*

$$\left((\alpha+1)_{i+j}\right)_{0 \leqslant i,j \leqslant n} \tag{2.29}$$

has the determinant

$$\det\left((\alpha+1)_{i+j}\right)_{i,j=0}^n = \frac{G(n+2)G(\alpha+n+2)}{G(\alpha+1)\Gamma(\alpha+1)^{n+1}}, \tag{2.30}$$

it is invertible for $-\alpha \notin \mathbb{N}$, *its inverse is given by*

$$\left(\frac{\sum_{k=\max(i,j)}^{n} \frac{(\alpha+1)_k}{k!} \binom{k}{i} \binom{k}{j}}{(-1)^{i+j} (\alpha+1)_i (\alpha+1)_j} \right)_{0 \leqslant i,j \leqslant n}. \tag{2.31}$$

2.2. The ultraspherical polynomials $\{C_n^\lambda(x)\}_{n=0}^\infty$ *with* $\lambda > -\frac{1}{2},\ \lambda \neq 0$

The ultraspherical polynomials (or Genenbauer polynomials) $\{C_n^\lambda(x)\}_{n=0}^\infty$ are defined as a hypergeometric series (Ismail, 2005)

$$C_n^\lambda(x) = \frac{(2\lambda)_n}{n!} {}_2F_1\left(\begin{matrix} -n, 2\lambda + n \\ \lambda + \frac{1}{2} \end{matrix}; \frac{1-x}{2} \right) \tag{2.32}$$

for $n \geqslant 0$, and we assume that

$$C_{-1}^\lambda(x) = 0. \tag{2.33}$$

We have

$$\frac{dC_n^\lambda(x)}{dx} = 2\lambda C_{n-1}^{\lambda+1}(x) \tag{2.34}$$

for $n \geqslant 0$. For $\lambda > -\frac{1}{2}$ and $\lambda \neq 0$, we also have

$$\int_{-1}^{1} C_m^\lambda(x) C_n^\lambda(x) (1-x^2)^{\lambda-\frac{1}{2}} dx = \frac{\pi \Gamma(2\lambda+n)}{2^{2\lambda-1} n! (\lambda+n) [\Gamma(\lambda)]^2} \delta_{mn}, \tag{2.35}$$

for $n, m = 0, 1, \ldots$ Thus the orthonormal polynomials are

$$p_n(x) = \sqrt{\frac{2^{2\lambda-1} n! (\lambda+n) [\Gamma(\lambda)]^2}{\pi \Gamma(2\lambda+n)}} C_n^\lambda(x), \tag{2.36}$$

for $n = 0, 1, \ldots$ and the leading coefficients are

$$\gamma_n = \sqrt{\frac{(\lambda+n) 2^{2\lambda+2n-1} \Gamma(\lambda+n)^2}{\pi n! \Gamma(2\lambda+n)}}. \tag{2.37}$$

The moments of the ultraspherical measure are

$$\mu_n = \int_{-1}^{1} x^n (1-x^2)^{\lambda-\frac{1}{2}} dx = \frac{1+(-1)^n}{2} B\left(\frac{n+1}{2}, \lambda + \frac{1}{2} \right), \tag{2.38}$$

for $n = 0, 1, \ldots$, where $B(p, q)$ is the beta integral

$$B(p,q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx, \quad \Re(p), \Re(q) > 0. \tag{2.39}$$

Then (i,j)th entry of the matrix $\Pi_n = (\alpha_{jk})_{j,k=0}^n$ is

$$\alpha_{ij} = \frac{1+(-1)^{i+j}}{2} B\left(\frac{i+j+1}{2}, \lambda + \frac{1}{2}\right) \tag{2.40}$$

for $i,j = 0,1,\ldots,n$, and the determinant of the matrix is

$$\det\left(\frac{1+(-1)^{i+j}}{2} B\left(\frac{i+j+1}{2}, \lambda + \frac{1}{2}\right)\right) = \prod_{k=0}^{n} \frac{\pi k!\Gamma(2\lambda+k)}{(\lambda+k)2^{2\lambda+2k-1}\Gamma(\lambda+k)^2} \tag{2.41}$$

or

$$\det\left(\frac{1+(-1)^{i+j}}{2} B\left(\frac{i+j+1}{2}, \lambda + \frac{1}{2}\right)\right) = \frac{\pi^{n+1}G(n+2)}{2^{(n+1)(n+2\lambda-1)}(\lambda)_{n+1}} \cdot \frac{G(2\lambda+n+1)G(\lambda)^2}{G(2\lambda)G(\lambda+n+1)^2} \tag{2.42}$$

It is clear that the matrix Π_n is invertible for $2\lambda \neq 0,-1,\ldots$, we use the functionals defined in (2.11) to find the (i,j)th entry of the inverse matrix $\Pi_n^{-1} = (\beta_{jk})_{j,k=0}^n$,

$$\begin{aligned}\beta_{ij} &= \frac{1}{i!j!} \sum_{k=\max(i,j)}^{n} \frac{2^{2\lambda-1}k!(\lambda+k)[\Gamma(\lambda)]^2}{\pi\Gamma(2\lambda+k)} \left[\frac{d^i C_k^\lambda(x)}{dx^i}\right]_{x=0} \left[\frac{d^j C_k^\lambda(y)}{dy^j}\right]_{y=0} \\ &= \frac{2^{2\lambda-1}[\Gamma(\lambda)]^2}{i!j!\pi} \sum_{k=\max(i,j)}^{n} \frac{k!(\lambda+k)}{\Gamma(2\lambda+k)} \left[2^i(\lambda)_i C_{k-i}^{\lambda+i}(x)\right]_{x=0} \left[2^j(\lambda)_j C_{k-j}^{\lambda+j}(x)\right]_{y=0} \\ &= \frac{2^{i+j}(\lambda)_i(\lambda)_j\Gamma(\lambda)}{i!j!\sqrt{\pi}\Gamma(\lambda+\frac{1}{2})} \sum_{k=\max(i,j)}^{n} \frac{k!(\lambda+k)C_{k-i}^{\lambda+i}(0)C_{k-j}^{\lambda+j}(0)}{(2\lambda)_k}\end{aligned} \tag{2.43}$$

for $i,j = 0,1,\ldots,n$.

Theorem 9. *For all $n = 0,1,\ldots$ the matrix $(\alpha_{ij})_{0\leqslant i,j\leqslant n}$ with entries*

$$\alpha_{ij} = \frac{1+(-1)^{i+j}}{2} B\left(\frac{i+j+1}{2}, \lambda + \frac{1}{2}\right) \tag{2.44}$$

has determinant

$$det\left(\frac{1+(-1)^{i+j}}{2} B\left(\frac{i+j+1}{2}, \lambda + \frac{1}{2}\right)\right) = \prod_{k=0}^{n} \frac{\pi k!\Gamma(2\lambda+k)}{(\lambda+k)2^{2\lambda+2k-1}\Gamma(\lambda+k)^2}. \tag{2.45}$$

When $2\lambda \neq 0,-1,-2,\ldots$, it is invertible and the inverse matrix $(\beta_{ij})_{0\leqslant i,j\leqslant n}$ has entries

$$\beta_{ij} = \frac{2^{i+j}(\lambda)_i(\lambda)_j\Gamma(\lambda)}{i!j!\sqrt{\pi}\Gamma(\lambda+\frac{1}{2})} \sum_{k=max(i,j)}^{n} \frac{k!(\lambda+k)C_{k-i}^{\lambda+i}(0)C_{k-j}^{\lambda+j}(0)}{(2\lambda)_k} \tag{2.46}$$

for $i,j = 0,1,\ldots,n$.

2.2.1. The Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ *with* $\alpha, \beta > -1$

The Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ may be defined as (Andrews et al., 1999; Ismail, 2005)

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n; n+\alpha+\beta+1\\ \alpha+1\end{matrix}; \frac{1-x}{2}\right) \tag{2.47}$$

for $n \geqslant 0$, and

$$P_{-1}^{(\alpha,\beta)}(x) = 0. \tag{2.48}$$

We also have

$$\frac{dP_n^{(\alpha,\beta)}(x)}{dx} = \frac{n+\alpha+\beta+1}{2} P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{2.49}$$

and consequently

$$\frac{d^k P_n^{(\alpha,\beta)}(x)}{dx^k} = \frac{(n+\alpha+\beta+1)_k}{2^k} P_{n-k}^{(\alpha+k,\beta+k)}(x) \tag{2.50}$$

for $k \in \mathbb{N}$. For $\alpha, \beta > -1$, we have

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) w(x) dx = h_n \delta_{mn} \tag{2.51}$$

for $n, m = 0, 1, \ldots$ with

$$w(x) := (1-x)^{\alpha}(1+x)^{\beta}, \tag{2.52}$$

and

$$h_n := \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)n!}. \tag{2.53}$$

Thus, the orthonormal polynomials

$$p_n(x) = \sqrt{\frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)n!}{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}} P_n^{(\alpha,\beta)}(x) \tag{2.54}$$

have leading coefficients

$$\gamma_n = \frac{2^{n+(\alpha+\beta)/2}\Gamma\left(\frac{\alpha+\beta+1}{2}+n\right)\Gamma\left(\frac{\alpha+\beta+2}{2}+n\right)\sqrt{\frac{\alpha+\beta+1}{2}+n}}{\sqrt{n!\pi\Gamma(\alpha+\beta+n+1)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}} \tag{2.55}$$

for $n = 0, 1, \ldots$

We take the polynomial sequence

$$w_n(x) = (x-1)^n, \tag{2.56}$$

and associated functionals

$$u_j(f) = v_j(f) = \frac{1}{j!}\left[\frac{d^j f(x)}{dx^j}\right]_{x=1} \tag{2.57}$$

for $j = 0, 1, \ldots$, then the (i,j)th entry of the matrix $\Pi_n = (\alpha_{jk})_{j,k=0}^{n}$ is

$$\alpha_{ij} = \int_{-1}^{1} (x-1)^{i+j} w(x)dx, \tag{2.58}$$

or

$$\alpha_{ij} = \frac{2^{\alpha+\beta+i+j+1}\Gamma(\alpha+i+j+1)\Gamma(\beta+1)}{(-1)^{i+j}\Gamma(\alpha+\beta+i+j+2)} \tag{2.59}$$

for $i,j = 0, 1, \ldots, n$, and its determinant is given by

$$\det \Pi_n = \frac{G(\alpha+n+1)G(\beta+n+1)G(\alpha+\beta+n+1)}{2^{(n+\alpha+\beta)(n+1)}G(\alpha+1)G(\beta+1)G(\alpha+\beta+1)} \times \frac{\pi^{n+1}G(n+2)G\left(\frac{\alpha+\beta+1}{2}\right)^2 G\left(\frac{\alpha+\beta+2}{2}\right)^2}{\left(\frac{\alpha+\beta+1}{2}\right)_{n+1} G\left(\frac{\alpha+\beta+1}{2}+n+1\right)^2 G\left(\frac{\alpha+\beta+2}{2}+n+1\right)^2}, \tag{2.60}$$

its inverse matrix has entries

$$\begin{aligned}\beta_{ij} &= \frac{1}{i!j!}\sum_{k=\max(i,j)}^{n} \frac{(2k+\alpha+\beta+1)\Gamma(\alpha+\beta+k+1)k!}{2^{\alpha+\beta+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \times \left[\frac{d^i P_k^{(\alpha,\beta)}(x)}{dx^i}\right]_{x=1}\left[\frac{d^j P_k^{(\alpha,\beta)}(y)}{dy^j}\right]_{y=1} \\ &= \sum_{k=\max(i,j)}^{n} \frac{(2k+\alpha+\beta+1)\Gamma(\alpha+\beta+k+1)k!}{2^{\alpha+\beta+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \times \left[\frac{(k+\alpha+\beta+1)_i P_{k-i}^{(\alpha+i,\beta+i)}(x)}{i!2^i}\right]_{x=1}\left[\frac{(k+\alpha+\beta+1)_j P_{k-j}^{(\alpha+j,\beta+j)}(y)}{j!2^j}\right]_{y=1},\end{aligned} \tag{2.61}$$

or

$$\beta_{ij} = \sum_{k=\max(i,j)}^{n} \frac{(2k+\alpha+\beta+1)\Gamma(\alpha+\beta+k+1)k!}{\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \times \left\{\frac{\{(k+\alpha+\beta+1)_i P_{k-i}^{(\alpha+i,\beta+i)}(1)\}\{(k+\alpha+\beta+1)_j P_{k-j}^{(\alpha+j,\beta+j)}(1)\}}{i!j!2^{\alpha+\beta+i+j+1}}\right\} \tag{2.62}$$

for $i,j = 0, 1, \ldots, n$.

Since

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!}, \tag{2.63}$$

then

$$\beta_{ij} = \frac{\Gamma(\alpha+\beta+1)(\alpha+\beta+1)_i(\alpha+\beta+1)_j}{2^{\alpha+\beta+i+j+1}(\alpha+1)_i(\alpha+1)_j\Gamma(\alpha+1)\Gamma(\beta+1)} \times \sum_{k=\max(i,j)}^{n} \left\{ \frac{(2k+\alpha+\beta+1)(\alpha+1)_k}{k!(\alpha+\beta+1)_k(\beta+1)_k} \right\} \times \left\{ \binom{k}{i}\binom{k}{j}(\alpha+\beta+i+1)_k(\alpha+\beta+j+1)_k \right\} \tag{2.64}$$

for $i,j = 0,1,\ldots,n$.

Theorem 10. *For $n = 0,1,\ldots$, the determinant of the matrix*

$$\left(\frac{(\alpha+1)_{i+j}}{(\alpha+\beta+2)_{i+j}} \right)_{0\leqslant i,j\leqslant n} \tag{2.65}$$

is

$$det\left(\frac{(\alpha+1)_{i+j}}{(\alpha+\beta+2)_{i+j}} \right)_{i,j=0}^{n} = \frac{G\left(\frac{\alpha+\beta+1}{2}\right)^2 G\left(\frac{\alpha+\beta+2}{2}\right)^2}{G(\alpha+1)G(\beta+1)G(\alpha+\beta+1)} \times \left(\frac{\pi\Gamma(\alpha+\beta+2)}{2^{2n+2\alpha+2\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \right)^{n+1} \times \frac{G(n+2)G(\alpha+n+1)G(\beta+n+1)G(\alpha+\beta+n+1)}{\left(\frac{\alpha+\beta+1}{2}\right)_{n+1} G\left(\frac{\alpha+\beta+1}{2}+n+1\right)^2 G\left(\frac{\alpha+\beta+2}{2}+n+1\right)^2}. \tag{2.66}$$

For $-\alpha, -\beta \notin \mathbb{N}$, the matrix (2.57) is invertible and its inverse matrix $(\gamma_{ij})_{0\leqslant i,j\leqslant n}$ has elements

$$\gamma_{ij} = \frac{(-1)^{i+j}(\alpha+\beta+1)_i(\alpha+\beta+1)_j}{(\alpha+1)_i(\alpha+1)_j(\alpha+\beta+1)} \times \sum_{k=max(i,j)}^{n} \left\{ \frac{(2k+\alpha+\beta+1)(\alpha+1)_k}{k!(\alpha+\beta+1)_k(\beta+1)_k} \right\} \times \left\{ \binom{k}{i}\binom{k}{j}(\alpha+\beta+i+1)_k(\alpha+\beta+j+1)_k \right\}. \tag{2.67}$$

Remarks 11. When $\alpha = \beta = 0$, matrix (2.57) reduces to the famous Hilbert matrix.

2.3. Wilson polynomials

In this subsection, we will compute some examples using a subset of Wilson orthogonal polynomials. Because the polynomials we choose, give similar matrices as in the classical polynomials cases. It suggests that there exists some transformation mapping these polynomials to the corresponding set of classical polynomials.

Given a polynomial $f(x^2)$, the Wilson operator $\mathcal{W}$ is defined as

$$(\mathcal{W}f)(x^2) := \frac{f((x+\frac{i}{2})^2) - f((x-\frac{i}{2})^2)}{2xi}. \tag{2.68}$$

For any fixed complex number a, we let

$$w_n(x^2;a) := (a+ix, a-ix)_n \tag{2.69}$$

for $n = 0, 1, \ldots$ The polynomial sequence $\{w_n(x^2;a)\}_{n\geqslant 0}$ forms a basis for all the polynomials in x^2. Furthermore,

$$(\mathcal{W}w_n(x^2;a)) = nw_{n-1}\left(x^2; a+\frac{1}{2}\right), \tag{2.70}$$

consequently, we have

$$\left[\mathcal{W}^k w_n(x^2;a)\right]_{x^2=x_k^2(a)} = n!\delta_{kn}, \tag{2.71}$$

where

$$x_k^2(a) := -\left(a+\frac{k}{2}\right)^2. \tag{2.72}$$

For the polynomial sequence $\{w_n(x^2;a)\}_{n\geqslant 0}$, we let

$$u_k(f(x^2)) = v_k(f(x^2)) := \frac{1}{k!}\left[(\mathcal{W}^k f)(x^2)\right]_{x^2=x_k^2(a)}. \tag{2.73}$$

2.3.1. The continuous dual Hahn polynomials $\{S_n(x^2;a,a,a)\}_{n\geqslant 0}$

We consider a special form of dual Hahn polynomials $\{S_n(x^2;a,a,a)\}_{n\geqslant 0}$. For simplicity, we let

$$S_n(x^2;a) := S_n(x^2;a,a,a). \tag{2.74}$$

these polynomials are defined by the hypergeometric series

$$\frac{S_n(x^2;a)}{(2a)_n(2a)_n} := {}_3F_2\left(\begin{matrix}-n, & a+ix, & a-ix \\ & 2a, & 2a\end{matrix}; 1\right), \tag{2.75}$$

for $n = 0, 1, \ldots$ and we assume that

$$S_{-1}(x^2;a) = 0. \tag{2.76}$$

For the sake of simplicity, we only consider the case $a > 0$. It is known that they satisfy the following orthogonality

$$\int_0^\infty S_m(x^2;a)S_n(x^2;a)w_1(x^2;a)dx = 2\pi n!\Gamma(2a+n)^3\delta_{mn}, \tag{2.77}$$

where

$$w_1(x^2;a,b) := \left|\frac{\Gamma(a+ix)^3}{\Gamma(2xi)}\right|^2, \tag{2.78}$$

and the orthonormal polynomials are

$$s_n(x^2) = \frac{(-1)^n S_n(x^2;a)}{\sqrt{2\pi n!\Gamma(2a+n)^3}} \tag{2.79}$$

with leading coefficients

$$\gamma_n = \sqrt{\frac{1}{2\pi n!\Gamma(2a+n)^3}}. \tag{2.80}$$

The polynomials $S_n(x^2;a)$ satisfy the following relation

$$\mathcal{W}S_n(x^2;a) = (-n)S_{n-1}\left(x^2;a+\frac{1}{2}\right), \tag{2.81}$$

consequently, for any $k \in \mathbb{N}$

$$\mathcal{W}^k S_n(x^2;a) = (-n)_k S_{n-k}\left(x^2;a+\frac{k}{2}\right). \tag{2.82}$$

The matrix Π_N associated with polynomial sequence $\{\phi_n(x^2;a)\}_{n\geqslant 0}$ with respect the measure $w_1(x^2;a)$ has entries

$$\begin{aligned}\alpha_{jk} &= \int_0^\infty \phi_j(x^2;a)\phi_k(x^2;a)w_1(x^2;a)dx\\ &= \int_0^\infty \left|\frac{\Gamma(a+ix)\Gamma(a+j+ix)\Gamma(a+k+ix)}{\Gamma(2xi)}\right|^2 dx\\ &= 2\pi\Gamma(2a+j)\Gamma(2a+k)\Gamma(2a+j+k)\end{aligned} \tag{2.83}$$

or

$$\alpha_{jk} = 2\pi\Gamma(2a+j)\Gamma(2a+k)\Gamma(2a+j+k) \tag{2.84}$$

for $j,k = 0,1,\ldots$ The matrix Π_N has determinant

$$\det \Pi_N = (2\pi)^{N+1}\prod_{n=0}^{N}\Gamma(2a+n)^3, \tag{2.85}$$

which is simplified to

$$\det\Big((2a)_{j+k}\Big)_{j,k=0}^{N} = \prod_{n=0}^{N} n!(2a)_n. \tag{2.86}$$

Therefore, for $N = 0,1,\ldots$, we have

$$\det\left(\left(\alpha\right)_{j+k}\right)_{j,k=0}^{N}=\prod_{n=0}^{N}\left[n!(\alpha)_n\right]. \tag{2.87}$$

The entries of $\Pi_N^{-1}=(\beta_{jk})_{0\leqslant j,k\leqslant N}$ are

$$\beta_{jk}=\sum_{n=\max(j,k)}^{N}\frac{(-n)_j(-n)_k\left[\mathcal{W}^j S_n(x^2;a)\right]_{x^2=x_j^2(a)}\left[\mathcal{W}^k S_n(y^2;a)\right]_{y^2=x_k^2(a)}}{2\pi j!k!n!\Gamma(2a+n)^3}. \tag{2.88}$$

Since

$$\left[\mathcal{W}^k S_n(y^2;a)\right]_{y^2=x_k^2(a)}=(-n)_k S_{n-k}\left(x_k^2(a);a+\frac{k}{2}\right), \tag{2.89}$$

and

$$S_{n-k}\left(x_k^2(a);a+\frac{k}{2}\right)=(2a+k)_{n-k}^2, \tag{2.90}$$

Thus, the matrix

$$\left((2a)_{j+k}\right)_{j,k=0}^{N} \tag{2.91}$$

has an inverse matrix

$$\left(\frac{\sum_{n=\max(j,k)}^{N}\frac{(2a)_n}{n!}\binom{n}{j}\binom{n}{k}}{(-1)^{j+k}(2a)_j(2a)_k}\right)_{j,k=0}^{N}, \tag{2.92}$$

they are the same matrices as in Theorem 8.

2.3.2. The Wilson polynomials $W_n(x^2;a,a,a,b)$

In this section, we will look at the inverse of the matrices associated with the Wilson polynomials $\{W_n(x^2;a,a,a,b)\}_{n\geqslant 0}$ respect to the polynomial sequence $\{\phi_n(x^2;a)\}_{n\geqslant 0}$. For the simplicity, we let

$$W_n(x^2;a,b):=W_n(x^2;a,a,a,b) \tag{2.93}$$

for $n=0,1,\ldots$ Then

$$\frac{W_n(x^2;a,b)}{(2a,2a,a+b)_n}:={}_4F_3\left(\begin{matrix}-n,n+3a+b-1,a+ix,a-ix\\ 2a,2a,a+b\end{matrix};1\right), \tag{2.94}$$

for $n=0,1,\ldots$ We always assume that

$$W_{-1}(x^2;a,b)=0. \tag{2.95}$$

In this section, we are only interested in the special case that $a,b>0$. Then, the Wilson polynomials $W_n(x^2):=W_n(x^2;a,b)$ satisfy the following orthogonality

$$\int_0^\infty W_m(x^2)W_n(x^2)w(x^2;a,b)dx = 2\pi n!(n+3a+b-1)_n H_n\delta_{mn}, \tag{2.96}$$

where

$$w(x^2;a,b,c) := \left|\frac{\Gamma(a+ix)^3\Gamma(b+ix)}{\Gamma(2xi)}\right|^2 \tag{2.97}$$

and

$$H_n := \frac{\Gamma(2a+n)^3\Gamma(a+b+n)^3}{\Gamma(3a+b+2n)}. \tag{2.98}$$

Thus, the associated orthonormal polynomials

$$w_n(x^2) = \sqrt{\frac{1}{2\pi n!(n+3a+b-1)_n H_n}}W_n(x^2) \tag{2.99}$$

have leading coefficients

$$\gamma_n = \frac{2\pi n!\Gamma(3a+b+n-1)\Gamma(2a+n)^3\Gamma(a+b+n)^3}{\Gamma(3a+b+2n-1)\Gamma(3a+b+2n)}. \tag{2.100}$$

The polynomials $\{W_n(x^2;a,b)\}_{n=0}^\infty$ also satisfy the following relation

$$\mathcal{W}W_n(x^2;a,b) = -n(n+3a+b-1)W_{n-1}\left(x^2;a+\frac{1}{2},b+\frac{1}{2}\right), \tag{2.101}$$

and for any integer $k \in \mathbb{N}$, we have

$$\mathcal{W}^k W_n(x^2;a,b) = (-n)_k(n+3a+b-1)_k W_{n-k}(x^2;a+\frac{k}{2},b+\frac{k}{2}). \tag{2.102}$$

For $a_1, a_2, a_3, a_4 > 0$, The Wilson integral evaluates

$$\frac{1}{2\pi}\int_0^\infty \left|\frac{\prod_{j=1}^4\Gamma(a_j+ix)}{\Gamma(2xi)}\right|^2 dx = \frac{\prod_{1\leqslant j<k\leq 4}\Gamma(a_j+a_k)}{\Gamma(\sum_{j=1}^4 a_j)}. \tag{2.103}$$

Therefore, the matrix $\Pi_N = (\alpha_{jk})_{0\leqslant j,k\leqslant N}$ associated with polynomial sequence $\{\phi_n(x^2;a)\}_{k\geqslant 0}$ with respect the Wilson measure has entries

$$\begin{aligned}
\alpha_{jk} &= \int_0^\infty \phi_j(x^2;a)\phi_k(x^2;a)\left|\frac{\Gamma(a+ix)^3\Gamma(b+ix)}{\Gamma(2xi)}\right|^2 dx \\
&= \int_0^\infty \left|\frac{\Gamma(a+j+ix)\Gamma(a+k+ix)\Gamma(a+ix)\Gamma(b+ix)}{\Gamma(2xi)}\right|^2 dx \\
&= \Gamma(2a+j)\Gamma(a+b+j)\Gamma(2a+k)\Gamma(a+b+k) \\
&\quad\times \frac{2\pi\Gamma(2a+j+k)\Gamma(a+b)}{\Gamma(3a+b+j+k)},
\end{aligned} \tag{2.104}$$

or

$$\frac{\alpha_{jk}}{2\pi\Gamma(a+b)} = \frac{\Gamma(2a+j)\Gamma(a+b+j)\Gamma(2a+k)\Gamma(a+b+k)\Gamma(2a+j+k)}{\Gamma(3a+b+j+k)}. \tag{2.105}$$

for $j,k = 0,1,\ldots,N$. Then we have the determinant evaluation

$$\det\left(\frac{(2a)_{j+k}}{(3a+b)_{j+k}}\right)_{j,k=0}^{N} = \prod_{n=0}^{N} \frac{n!(2a)_n(a+b)_n(3a+b-1)_n}{(3a+b-1)_{2n}(3a+b)_{2n}}, \tag{2.106}$$

which is essentially (2.58).

The entries of $\Pi_N^{-1} = (\beta_{jk})_{0\leqslant j,k\leqslant N}$ are

$$\begin{aligned}\beta_{jk} &= \frac{1}{j!k!}\sum_{n=\max(j,k)}^{N}\left[\mathcal{W}^j w_n(x)\right]_{x^2=x_j^2}\left[\mathcal{W}^k w_n(y)\right]_{y=x_k^2}\\ &= \frac{1}{2\pi j!k!}\sum_{n=\max(j,k)}^{N}\frac{(-n)_j(-n)_k(n+3a+b-1)_j(n+3a+b-1)_k}{n!(n+3a+b-1)_n\Gamma(2a+n)\Gamma(a+b+n)}\\ &\quad\times\frac{W_{n-j}(x_j^2;a+\frac{j}{2},b+\frac{j}{2})W_{n-k}(x_k^2;a+\frac{k}{2},b+\frac{k}{2})\Gamma(3a+b+2n)}{\Gamma(2a+n)^2\Gamma(a+b+n)^2}\end{aligned} \tag{2.107}$$

for $j,k = 0,1,\ldots$ From

$$W_{n-k}\left(x_k^2;a+\frac{k}{2},b+\frac{k}{2}\right) = (2a+k,2a+k,a+b+k)_{n-k}, \tag{2.108}$$

we have

$$\beta_{jk} = \frac{\sum_{n=\max(j,k)}^{N}\frac{(-n)_j(-n)_k(n+3a+b-1)_j(n+3a+b-1)_k\Gamma(2a+n)\Gamma(3a+b+2n)}{n!(n+3a+b-1)_n\Gamma(a+b+n)}}{2\pi j!k!\Gamma^2(2a+j)\Gamma(a+b+j)\Gamma^2(2a+k)\Gamma(a+b+k)}. \tag{2.109}$$

Thus, the matrix $(\alpha_{jk})_{0\leqslant j,k\leqslant N}$ has entries

$$\alpha_{jk} = \frac{2\pi\Gamma(a+b)\Gamma(2a+j)\Gamma(a+b+j)\Gamma(2a+k)\Gamma(a+b+k)\Gamma(2a+j+k)}{\Gamma(3a+b+j+k)}, \tag{2.110}$$

its inverse (β_{jk}) has entries

$$\beta_{jk} = \frac{\sum_{n=\max(j,k)}^{N}\frac{(-n)_j(-n)_k(n+3a+b-1)_j(n+3a+b-1)_k\Gamma(2a+n)\Gamma(3a+b+2n)}{2\pi j!k!n!(n+3a+b-1)_n\Gamma(a+b+n)}}{\Gamma^2(2a+j)\Gamma(a+b+j)\Gamma^2(2a+k)\Gamma(a+b+k)}. \tag{2.111}$$

Thus the matrix

$$\left(\frac{(2a)_{j+k}}{(3a+b)_{j+k}}\right) \tag{2.112}$$

has inverse matrix

$$\left(\frac{(3a+b-1)_j(3a+b-1)_k\sum_{n=\max(j,k)}^{N}\frac{(2a)_n(3a+b+2n-1)(3a+b+j)_n(3a+b+k)_n}{n!(a+b)_n}\binom{n}{j}\binom{n}{k}}{(-1)^{j+k}(2a)_j(2a)_k(3a+b-1)}\right), \tag{2.113}$$

they are essentially the matrices in Theorem 10.

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