SUMMATION OF CERTAIN RECIPROCAL SERIES RELATED TO THE GENERALIZED FIBONACCI AND LUCAS NUMBERS

Fengzhen Zhao
Department of Applied Mathematics, Dalian University of Technology, 116024 Dalian, China
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1. INTRODUCTION

We are interested in the generalized Fibonacci and Lucas numbers defined by

$$U_n(p,q)=\frac{\alpha^n-\beta^n}{\alpha-\beta},\quad V_n(p,q)=\alpha^n+\beta^n,$$

where

$$\alpha=\frac{p+\sqrt{\Delta}}{2},\ \beta=\frac{p-\sqrt{\Delta}}{2},\ \Delta=p^2-4q,\ p>0, \text{ and } q<0.$$

It is well known that $\{U_n(1,-1)\}$ and $\{V_n(1,-1)\}$ are the classical Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$. There are many publications dealing with summation of reciprocal series related to the classical Fibonacci and Lucas numbers (see, e.g., [2]-[5]). Backstrom [3] obtained

$$\sum_{n=0}^{\infty}\frac{1}{F_{2n+1}+F_s}=\frac{\sqrt{5}S}{2L_s}\quad (s\text{ odd})\tag{1}$$

and André-Jeannin [2] proved that

$$\sum_{n=0}^{\infty}\frac{1}{F_{2n+1}+L_s/\sqrt{5}}=\frac{s}{2F_s}\quad (s\text{ even},\ s\neq 0).\tag{2}$$

Are there results similar to (1) or (2) for the generalized Fibonacci and Lucas numbers? In this paper we will discuss the summation of reciprocal series related to the generalized Fibonacci and Lucas numbers. We will establish a series of identities involving the generalized Fibonacci and Lucas numbers and some identities of [2] and [3] will emerge as special cases of our results. In the final section, following the method introduced by Almkvist, we express four reciprocal series related to the generalized Fibonacci and Lucas numbers in terms of the theta functions and give their estimates. Some of the estimates obtained generalize the results of [1] and [2], respectively.

2. MAIN RESULTS

The following lemmas will be used later on.

Lemma 1: Let t be a real number with $|t|>1$, s and a be positive integers, and b be a nonnegative integer. Then one has that

$$\sum_{n=0}^{\infty}\frac{1}{t^{2an+b}+t^{-2an-b}+t^{as}+t^{-as}}=\frac{1}{t^{as}-t^{-as}}\sum_{n=0}^{s-1}\frac{1}{1+t^{2an+b-as}}\tag{3}$$

and

$$\sum_{n=0}^{\infty}\frac{1}{t^{2an+b}+t^{-2an-b}-(t^{as}+t^{-as})}=\frac{1}{t^{as}-t^{-as}}\sum_{n=0}^{s-1}\frac{1}{1-t^{2an+b-as}}.\tag{4}$$

Proof: Because the proof of (4) is similar to that of (3), we only give the proof of (3). One can readily verify that

$$\frac{1}{t^{2an+b}+t^{-2an-b}+t^{as}+t^{-as}}=\frac{1}{t^{-as}-t^{as}}\left(\frac{1}{1+t^{2an+b+as}}-\frac{1}{1+t^{2an+b-as}}\right)$$

holds for $n>s$. Hence, by the telescoping effect, one has that

$$\sum_{n=0}^{N}\frac{1}{t^{2an+b}+t^{-2an-b}+t^{as}+t^{-as}}=\frac{1}{t^{-as}-t^{as}}\left(\sum_{n=N-s+1}^{N}\frac{1}{1+t^{2an+b+as}}-\sum_{n=0}^{s-1}\frac{1}{1+t^{2an+b-as}}\right)$$

for all $N>s$. Letting $N\to+\infty$, we obtain equality (3) (since $|t|>1$). □

Lemma 2: Let t be a real number with $|t|>1$ and s be a positive integer. Then

$$\sum_{n=0}^{s-1}\frac{1}{1+t^{2n-s}}=\frac{s-1}{2}+\frac{1}{1+t^{-s}}, \tag{5}$$

$$\sum_{n=0}^{s-1}\frac{1}{1+t^{2n-s+1}}=\frac{s}{2}, \tag{6}$$

$$\sum_{n=0}^{2s}\frac{1}{1-t^{2n-2s-1}}=s+\frac{1}{1-t^{-2s-1}}, \tag{7}$$

and

$$\sum_{n=0}^{2s-1}\frac{1}{1-t^{2n-2s+1}}=s. \tag{8}$$

Proof: We only show that equality (5) is valid. The proofs of (6)-(8) follow the same pattern and therefore are omitted here. First,

$$\sum_{n=0}^{2m-1}\frac{1}{1+t^{2n-2m}}=\frac{1}{1+t^{-2m}}+\frac{1}{2}+\sum_{n=1}^{m-1}\left(\frac{1}{1+t^{2n}}+\frac{1}{1+t^{-2n}}\right)=m-\frac{1}{2}+\frac{1}{1+t^{-2m}}.$$

On the other hand,

$$\sum_{n=0}^{2m}\frac{1}{1+t^{2n-2m-1}}=\frac{1}{1+t^{-2m-1}}+\sum_{n=1}^{m}\left(\frac{1}{1+t^{2n-1}}+\frac{1}{1+t^{-2n+1}}\right)=m+\frac{1}{1+t^{-2m-1}}.$$

Therefore, equality (5) holds. □

The above lemmas are used to find some equalities involving the generalized Fibonacci and Lucas numbers. Using the lemmas, we calculate some reciprocal series related to $\{U_n(p,q)\}$ and $\{V_n(p,q)\}$.

Theorem 1: Assume that a and b are integers with $a\ge 1$ and $b\ge 0$. Then

$$\sum_{n=0}^{\infty}\frac{(-q)^{an+b/2}}{V_{2an+b}(p,q)+(-q)^{an+(b-a)/2}V_a(p,q)}=\frac{(-q)^{a/2}}{\sqrt{\Delta}U_a(p,q)(1+(-\alpha/\beta)^{(b-a)/2})}, \tag{9}$$

$$\sum_{n=0}^{\infty}\frac{(-q)^{an+b/2}}{U_{2an+b}(p,q)+(-q)^{an+(b-a)/2}U_a(p,q)}=\frac{(-q)^{a/2}\sqrt{\Delta}}{V_a(p,q)(1+(-\alpha/\beta)^{(b-a)/2})}, \tag{10}$$

$$\sum_{n=0}^{\infty}\frac{(-q)^{an+b/2}}{\sqrt{\Delta}U_{2an+b}(p,q)+(-q)^{an+(b-a)/2}V_a(p,q)}=\frac{(-q)^{a/2}}{\sqrt{\Delta}U_a(p,q)(1+(-\alpha/\beta)^{(b-a)/2})}, \tag{11}$$

and

$$\sum_{n=0}^{\infty} \frac{(-q)^{an+b/2}}{V_{2an+b}(p,q)+\sqrt{\Delta}(-q)^{an+(b-a)/2}U_a(p,q)} = \frac{(-q)^{a/2}}{V_a(p,q)(1+(-\alpha/\beta)^{(b-a)/2})}, \tag{12}$$

where a is even in (9), (11) and odd in (10), (12), and b is even in (9), (12) and odd in (10), (11), respectively.

Proof: Putting $t=\sqrt{-\alpha/\beta}$ in (3) and noticing that $\alpha\beta=q$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} &\frac{(-q)^{an+b/2}}{\alpha^{2an+b}+(-1)^b\beta^{2an+b}+(-q)^{an+(b-as)/2}(\alpha^{as}+(-1)^{as}\beta^{as})} \\ &= \frac{(-q)^{as/2}}{\alpha^{as}-(-1)^{as}\beta^{as}} \sum_{n=0}^{s-1} \frac{1}{1+(-\alpha/\beta)^{an+(b-as)/2}}.\end{aligned} \tag{13}$$

Let us examine different cases according to the values of a, b, and s. With $s=1$ in (13), then (9) holds if both a and b are even and (10) holds if both a and b are odd. On the other hand, if a is even, b is odd, and $s=1$, then we have (11) from (13). If a is odd, b is even, and $s=1$, then we have (12) from (13). □

Theorem 2: Suppose that a and b are integers with $a\geq 1$, $b\geq 0$, and $a\neq b$. Then

$$\sum_{n=0}^{\infty} \frac{(-q)^{an+b/2}}{V_{2an+b}(p,q)-(-q)^{an+(b-a)/2}V_a(p,q)} = \frac{-(-q)^{a/2}}{\sqrt{\Delta}U_a(p,q)(1-(-\alpha/\beta)^{(b-a)/2})}, \tag{14}$$

$$\sum_{n=0}^{\infty} \frac{(-q)^{an+b/2}}{U_{2an+b}(p,q)-(-q)^{an+(b-a)/2}U_a(p,q)} = \frac{-\sqrt{\Delta}(-q)^{a/2}}{V_a(p,q)(1-(-\alpha/\beta)^{(b-a)/2})}, \tag{15}$$

$$\sum_{n=0}^{\infty} \frac{(-q)^{an+b/2}}{\sqrt{\Delta}\,U_{2an+b}(p,q)-(-q)^{an+(b-a)/2}V_a(p,q)} = \frac{-(-q)^{a/2}}{\sqrt{\Delta}\,U_a(p,q)(1-(-\alpha/\beta)^{(b-a)/2})}, \tag{16}$$

and

$$\sum_{n=0}^{\infty} \frac{(-q)^{an+b/2}}{V_{2an+b}(p,q)-(-q)^{an+(b-a)/2}\sqrt{\Delta}U_a(p,q)} = \frac{-(-q)^{a/2}}{V_a(p,q)(1-(-\alpha/\beta)^{(b-a)/2})}, \tag{17}$$

where a is even in (14), (16) and odd in (15), (17) and b is even in (14), (17) and odd in (15), (16), respectively.

The proof is similar to that of Theorem 1 except that (13) is replaced by

$$\begin{aligned}\sum_{n=0}^{\infty} &\frac{(-q)^{an+b/2}}{\alpha^{2an+b}+(-1)^b\beta^{2an+b}-(-q)^{an+(b-as)/2}(\alpha^{as}+(-1)^{as}\beta^{as})} \\ &= \frac{-(-q)^{as/2}}{\alpha^{as}-(-1)^{as}\beta^{as}} \sum_{n=0}^{s-1} \frac{1}{1-(-\alpha/\beta)^{an+(b-as)/2}}.\end{aligned} \tag{18}$$

Equality (18) is valid by putting $t=\sqrt{-\alpha/\beta}$ in (4).

Theorem 3: Suppose that s is a positive integer. Then

$$\sum_{n=0}^{\infty} \frac{(-q)^n}{V_{2n}(p,q)+(-q)^{n-s/2}V_s(p,q)} = \frac{(-q)^{s/2}}{\sqrt{\Delta}U_s(p,q)}\left(\frac{s-1}{2}+\frac{1}{1+(-\beta/\alpha)^{s/2}}\right) \ (s \text{ even}), \tag{19}$$

$$\sum_{n=0}^{\infty}\frac{(-q)^{n+1/2}}{U_{2n+1}(p,q)+(-q)^{n+(1-s)/2}V_s(p,q)/\sqrt{\Delta}}=\frac{(-q)^{s/2}s}{2U_s(p,q)}\quad (s\text{ even, } s\neq 0), \tag{20}$$

$$\sum_{n=0}^{\infty}\frac{(-q)^{n}}{V_{2n}(p,q)+\sqrt{\Delta}(-q)^{n-s/2}U_s(p,q)}=\frac{(-q)^{s/2}}{V_s(p,q)}\left(\frac{s-1}{2}+\frac{1}{1+(-\beta/\alpha)^{s/2}}\right)\quad (s\text{ odd}), \tag{21}$$

$$\sum_{n=0}^{\infty}\frac{(-q)^{n+1/2}}{U_{2n+1}(p,q)+(-q)^{n+(1-s)/2}U_s(p,q)}=\frac{(-q)^{s/2}\sqrt{\Delta}s}{2V_s(p,q)}\quad (s\text{ odd}), \tag{22}$$

$$\sum_{n=0}^{\infty}\frac{(-q)^{n}}{V_{2n}(p,q)-(-q)^{n-s/2}\sqrt{\Delta}U_s(p,q)}=\frac{(-q)^{s/2}}{V_s(p,q)}\left(\frac{1-s}{2}+\frac{1}{(-\beta/\alpha)^{s/2}-1}\right)\quad (s\text{ odd}), \tag{23}$$

and

$$\sum_{n=0}^{\infty}\frac{(-q)^{n+1/2}}{U_{2n+1}(p,q)-(-q)^{n+(1-s)/2}V_s(p,q)/\sqrt{\Delta}}=\frac{-(-q)^{s/2}s}{2U_s(p,q)}\quad (s\text{ even, } s\neq 0). \tag{24}$$

Proof: Letting $a = 1$ and $b = 0$ in (13), we have

$$\sum_{n=0}^{\infty}\frac{(-q)^{n}}{V_{2n}(p,q)+(-q)^{n-s/2}V_s(p,q)}=\frac{(-q)^{s/2}}{\sqrt{\Delta}U_s(p,q)}\sum_{n=0}^{s-1}\frac{1}{1+(-\alpha/\beta)^{n-s/2}}\quad (s\text{ even}).$$

Due to (5), we obtain equality (19). On the other hand, if $a = b = 1$ in (13), then

$$\sum_{n=0}^{\infty}\frac{(-q)^{n+1/2}}{U_{2n+1}(p,q)+(-q)^{n+(1-s)/2}V_s(p,q)/\sqrt{\Delta}}=\frac{(-q)^{s/2}}{U_s(p,q)}\sum_{n=0}^{s-1}\frac{1}{1+(-\alpha/\beta)^{n+(1-s)/2}}\quad (s\text{ even}).$$

Noticing that (6), we have equality (20).

Similarly, equalities (21) and (22) follow from (13), (5), and (6). Equalities (23) and (24) can be obtained from (18), (7), and (8). □

From the above theorems, we can obtain some results of [2] and [3] according to the values of p and q. For instance, if $p=-q=1$ in (9), we obtain Theorem V of [3]. If $p=-q=1$ in (22), we obtain equality (1). If $p=-q=1$ in (20), we have (2).

3. THE ESTIMATES OF FOUR SERIES

In this section, the summation $\sum_n$ is over all integers n. Using the method introduced by Almkvist [1], we give the estimates of four series related to the generalized Fibonacci and Lucas numbers. Putting $s=0$ in the left-hand side of (20), we have

$$\sum_{n=0}^{\infty}\frac{(-q)^{n+1/2}}{U_{2n+1}(p,q)+2(-q)^{n+1/2}/\sqrt{\Delta}}=\sqrt{\Delta}\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(t^{2n+1}+1)^2},$$

where $t=(-\beta/\alpha)^{1/2}$. By a classical formula (see [1] or [6]), we know that

$$\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(t^{2n+1}+1)^2}=-\frac{\vartheta_3''}{8\pi^2\vartheta_3},$$

where

$$\vartheta_3=\sqrt{-\frac{\pi}{\log t}}\sum_n e^{\pi^2 n^2/\log t}$$

and

$$\vartheta_3'' = \frac{2\pi^2}{\log t}\sqrt{-\frac{\pi}{\log t}}\sum_n\left(1+\frac{2\pi^2 n^2}{\log t}\right)e^{\pi^2 n^2/\log t}.$$

By simple computation, we can obtain

$$\sum_{n=0}^{\infty}\frac{(-q)^{n+1/2}}{U_{2n+1}(p,q)+2(-q)^{n+1/2}/\sqrt{\Delta}} = \frac{\sqrt{\Delta}}{2\log(-\alpha/\beta)} - \frac{4\pi^2\sqrt{\Delta}\sum_{n=1}^{\infty}n^2 e^{-2\pi^2 n^2/\log(-\alpha/\beta)}}{(\log(-\alpha/\beta))^2\left(1+2\sum_{n=1}^{\infty}e^{-2\pi^2 n^2/\log(-\alpha/\beta)}\right)}.$$

Hence,

$$\sum_{n=0}^{\infty}\frac{(-q)^{n+1/2}}{U_{2n+1}(p,q)+2(-q)^{n+1/2}/\sqrt{\Delta}} \approx \frac{\sqrt{\Delta}}{2\log(-\alpha/\beta)} - \frac{4\pi^2\sqrt{\Delta}}{(\log(-\alpha/\beta))^2(2+e^{2\pi^2/\log(-\alpha/\beta)})}.$$

Using a similar method, we can obtain the estimates of some other series. We have that

$$\sum_{n=0}^{\infty}\frac{(-q)^n}{V_{2n}(p,q)+2(-q)^n} = \frac{1}{4}+\sum_{n=1}^{\infty}\frac{t^{2n}}{(t^{2n}+1)^2},$$

$$\sum_{n=0}^{\infty}\frac{(-q)^{n+1/2}}{U_{2n+1}(p,q)-2(-q)^{n+1/2}/\sqrt{\Delta}} = \sqrt{\Delta}\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(t^{2n+1}-1)^2},$$

and

$$\sum_{n=1}^{\infty}\frac{(-q)^n}{V_{2n}(p,q)-2(-q)^n} = \sum_{n=1}^{\infty}\frac{t^{2n}}{(t^{2n}-1)^2},$$

where $t=(-\beta/\alpha)^{1/2}$. From the following facts (see [1] or [6]), i.e.,

$$\frac{\vartheta_2''}{\vartheta_2} = -\pi^2\left(1+8\sum_{n=1}^{\infty}\frac{t^{2n}}{(t^{2n}+1)^2}\right),$$

$$\frac{\vartheta_4''}{\vartheta_4} = 8\pi^2\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(t^{2n+1}-1)^2},$$

and

$$\frac{1}{24\pi^2}\left(\frac{\vartheta_2''}{\vartheta_2}+\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_4''}{\vartheta_4}\right)+\frac{1}{24} = \sum_{n=1}^{\infty}\frac{t^{2n}}{(t^{2n}-1)^2},$$

where

$$\vartheta_2 = \sqrt{\frac{-\pi}{\log t}}\sum_n(-1)^n e^{\pi^2 n^2/\log t}$$

and

$$\vartheta_4 = \sqrt{\frac{-\pi}{\log t}}\sum_n e^{\pi^2(n-1/2)^2/\log t},$$

we have that

$$\sum_{n=0}^{\infty}\frac{(-q)^n}{V_{2n}(p,q)+2(-q)^n} \approx \frac{1}{8}+\frac{1}{2\log(-\alpha/\beta)}+\frac{4\pi^2}{(\log(-\alpha/\beta))^2}\cdot\frac{1}{e^{2\pi^2/\log(-\alpha/\beta)}-2},$$

$$\sum_{n=0}^{\infty}\frac{(-q)^{n+1/2}}{U_{2n+1}(p,q)-2(-q)^{n+1/2}/\sqrt{\Delta}} \approx \frac{\pi^2\sqrt{\Delta}}{2(\log(-\alpha/\beta))^2} - \frac{\sqrt{\Delta}}{2\log(-\alpha/\beta)},$$

and

$$\sum_{n=1}^{\infty}\frac{(-q)^n}{V_{2n}(p,q)-2(-q)^n}\approx\frac{1}{24}-\frac{1}{2\log(-\alpha/\beta)}$$
$$+\frac{4\pi^2}{3(\log(-\alpha/\beta))^2}\left(\frac{1}{e^{2\pi^2/\log(-\alpha/\beta)}+2}-\frac{1}{e^{2\pi^2/\log(-\alpha/\beta)}-2}+\frac{1}{8}\right).$$

Clearly, some of the estimates obtained in this section are the generalizations of [1] and [2], respectively.

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