Journal of Mathematical Research & Exposition
Nov., 2011, Vol. 31, No. 6, pp. 1030–1034
DOI:10.3770/j.issn:1000-341X.2011.06.010
Http://jmre.dlut.edu.cn

The Infinite Sum of Reciprocal of the Fibonacci Numbers

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Abstract In this paper, we consider infinite sums of the reciprocals of the Fibonacci numbers. Then applying the floor function to the reciprocals of this sums, we obtain a new identity involving the Fibonacci numbers.

Keywords Fibonacci numbers; reciprocals; infinite sum; elementary method; identity.

Document code A
MR(2010) Subject Classification 11B39
Chinese Library Classification O156.1; O157.1

1. Introduction

The Fibonacci sequence $\{F_n\}$ plays a very important role in the theory and applications of mathematics, and its various properties have been investigated by many authors, see [1–5]. Recently, Ohtsuka and Nakamura [2] derived some new formulas for the reciprocals of the Fibonacci numbers, as follows,

$$\Big\lfloor \Big(\sum_{k=n}^{\infty}\frac{1}{F_k}\Big)^{-1}\Big\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even and } n\geq 2;\\ F_{n-2}-1, & \text{if } n \text{ is odd and } n\geq 1.\end{cases}$$

$$\Big\lfloor \Big(\sum_{k=n}^{\infty}\frac{1}{F_k^2}\Big)^{-1}\Big\rfloor = \begin{cases} F_{n-1}F_n-1, & \text{if } n \text{ is even and } n\geq 2;\\ F_{n-1}F_n, & \text{if } n \text{ is odd and } n\geq 1.\end{cases}$$

Inspired by the work of Ohtsuka and Nakamura, in this paper we consider the computational problem of the following summation

$$\Big\lfloor \Big(\sum_{k=n}^{\infty}\frac{1}{F_{3k}}\Big)^{-1}\Big\rfloor,$$

and give an exact computational formula. That is, we shall prove the following

Theorem *For any positive integer n, we have the identity*

$$\Big\lfloor \Big(\sum_{k=n}^{\infty}\frac{1}{F_{3k}}\Big)^{-1}\Big\rfloor = \begin{cases} F_{3n-1}+F_{3n-4}, & \text{if } n \text{ is even and } n\geq 2;\\ F_{3n-1}+F_{3n-4}-1, & \text{if } n \text{ is odd and } n\geq 3.\end{cases}$$

Received October 13, 2010; Accepted January 12, 2011
Supported by the National Natural Science Foundation of China (Grant No. 11071194).
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2. Proof of Theorem

In this section, we shall prove Theorem directly. First we consider the case that $n = 2m > 0$ is an even number. It is clear that our theorem is equivalent to

$$\frac{1}{F_{6m-1} + F_{6m-4} + 1} < \sum_{k=2m}^{\infty} \frac{1}{F_{3k}} < \frac{1}{F_{6m-1} + F_{6m-4}}.$$

For any integer $k \geq 1$, let $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$. Then from the definition of the Fibonacci numbers $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$ and note that $\alpha \cdot \beta = -1$, we have

$$\begin{aligned}
&\left(\frac{1}{F_{3\cdot(2k)}} + \frac{1}{F_{3\cdot(2k+1)}}\right) - \left(\frac{1}{F_{3\cdot 2k-1} + F_{3\cdot 2k-4}} - \frac{1}{F_{3\cdot 2(k+1)-1} + F_{3\cdot 2(k+1)-4}}\right)\\
&= \frac{F_{6k+3} + F_{6k}}{F_{6k}F_{6k+3}} - \frac{1}{F_{6k-1} + F_{6k-4}} + \frac{1}{F_{6k+5} + F_{6k+2}}\\
&= \frac{F_{6k}F_{6k-1} + F_{6k}F_{6k-4} - F_{6k+3}F_{6k-3}}{F_{6k}F_{6k+3}(F_{6k-1} + F_{6k-4})} + \frac{1}{F_{6k+5} + F_{6k+2}}\\
&= \frac{\frac{1}{5}[-(\alpha^{12k-2} + \beta^{12k-2}) - 17] + F_{6k}F_{6k-4}}{F_{6k}F_{6k+3}(F_{6k-1} + F_{6k-4})} + \frac{1}{F_{6k+5} + F_{6k+2}}\\
&= \frac{1}{5}\frac{\alpha^{12k-4}(1-\alpha^2) + \beta^{12k-4}(1-\beta^2) - 24}{F_{6k}F_{6k+3}(F_{6k-1} + F_{6k-4})} + \frac{1}{F_{6k+5} + F_{6k+2}}\\
&= \frac{1}{5}\frac{-4F_{6k-1} - 4F_{6k-4} + (F_{6k+4} - F_{6k+7}) - 24F_{6k+5} - (24F_{6k+2} + F_{6k-8} - F_{6k-5})}{F_{6k}F_{6k+3}(F_{6k-1} + F_{6k-4})(F_{6k+5}F_{6k+2})}\\
&< \frac{1}{5}\frac{-4F_{6k-1} - 4F_{6k-4} - 24F_{6k+5}}{F_{6k}F_{6k+3}(F_{6k-1} + F_{6k-4})(F_{6k+5}F_{6k+2})} < 0.
\end{aligned}$$

So for any integer $k \geq 1$, we have

$$\frac{1}{F_{3\cdot(2k)}} + \frac{1}{F_{3\cdot(2k+1)}} < \frac{1}{F_{3\cdot 2k-1} + F_{3\cdot 2k-4}} - \frac{1}{F_{3\cdot 2(k+1)-1} + F_{3\cdot 2(k+1)-4}}. \tag{2}$$

Applying (2) repeatedly we have

$$\begin{aligned}
\sum_{k=2m}^{\infty} \frac{1}{F_{3k}} &= \sum_{k=m}^{\infty} \left(\frac{1}{F_{3\cdot(2k)}} + \frac{1}{F_{3\cdot(2k+1)}}\right)\\
&< \sum_{k=m}^{\infty} \left(\frac{1}{F_{3\cdot 2k-1} + F_{3\cdot 2k-4}} - \frac{1}{F_{3\cdot 2(k+1)-1} + F_{3\cdot 2(k+1)-4}}\right)\\
&= \frac{1}{F_{3\cdot 2m-1} + F_{3\cdot 2m-4}} = \frac{1}{F_{6m-1} + F_{6m-4}}.
\end{aligned} \tag{3}$$

On the other hand, for any integer $k \geq 1$, we have

$$\begin{aligned}
&\frac{1}{F_{3k-1} + F_{3k-4} + 1} - \left(\frac{1}{F_{3k}} + \frac{1}{F_{3(k+1)-1} + F_{3(k+1)-4} + 1}\right)\\
&= \frac{(F_{3k-3} - 1)(F_{3k+2} + F_{3k-1} + 1) - F_{3k}(F_{3k-4} + F_{3k-1} + 1)}{(F_{3k-1} + F_{3k-4} + 1)F_{3k}(F_{3k+2} + F_{3k-1} + 1)}\\
&= \frac{F_{3k-3}(F_{3k+2} + F_{3k-1}) - F_{3k}(F_{3k-4} + F_{3k-1}) + F_{3k-3} - F_{3k+2} - F_{3k-1} - 1 - F_{3k}}{(F_{3k-1} + F_{3k-4} + 1)F_{3k}(F_{3k+2} + F_{3k-1} + 1)}
\end{aligned}$$

$$= \frac{\frac{1}{5}(-1)^{3k-1}(1+\alpha\beta^{-3}\alpha^{-3}\beta-\beta^5-\beta^2-\alpha^5-\alpha^2)+F_{3k-3}-F_{3k+3}-1}{(F_{3k-1}+F_{3k-4}+1)F_{3k}(F_{3k+2}+F_{3k-1}+1)}$$
$$= \frac{\frac{1}{5}(-1)^{3k-1}(1-7-11-3)-\frac{1}{\sqrt{5}}\alpha^{3k-3}(8+4\sqrt{5})+\frac{1}{\sqrt{5}}\beta^{3k-3}(8-4\sqrt{5})-1}{(F_{3k-1}+F_{3k-4}+1)F_{3k}(F_{3k+2}+F_{3k-1}+1)}$$
$$< \frac{-13}{(F_{3k-1}+F_{3k-4}+1)F_{3k}(F_{3k+2}+F_{3k-1}+1)} < 0.$$

Therefore,

$$\frac{1}{F_{3k-1}+F_{3k-4}+1} < \frac{1}{F_{3k}} + \frac{1}{F_{3(k+1)-1}+F_{3(k+1)-4}+1}. \tag{4}$$

For any integer $m \geq 1$, using inequality (4) repeatedly, we have

$$\begin{aligned}\frac{1}{F_{3m-1}+F_{3m-4}+1} &< \frac{1}{F_{3m}} + \frac{1}{F_{3(m+1)-1}+F_{3(m+1)-4}+1}\\ &< \frac{1}{F_{3m}} + (\frac{1}{F_{3(m+1)}} + \frac{1}{F_{3(m+2)-1}+F_{3(m+2)-4}+1})\\ &< \frac{1}{F_{3m}} + \frac{1}{F_{3(m+1)}} + (\frac{1}{F_{3(m+2)}} + \frac{1}{F_{3(m+3)-1}+F_{3(m+3)-4}+1})\\ &< \cdots < \frac{1}{F_{3m}} + \frac{1}{F_{3(m+1)}} + \frac{1}{F_{3(m+2)}} + \frac{1}{F_{3(m+3)}} + \frac{1}{F_{3(m+4)}} + \cdots\\ &= \sum_{k=m}^{\infty} \frac{1}{F_{3k}}\end{aligned}$$

or

$$\sum_{k=2m}^{\infty} \frac{1}{F_{3k}} > \frac{1}{F_{6m-1}+F_{6m-4}+1}. \tag{5}$$

Combining inequalities (3) and (5), the inequality (1)

$$\frac{1}{F_{6m-1}+F_{6m-4}+1} < \sum_{k=2m}^{\infty} \frac{1}{F_{3k}} < \frac{1}{F_{6m-1}+F_{6m-4}}$$

has been proved. Now for any odd number $n = 2m+1 \geq 3$, we prove the inequality

$$\frac{1}{F_{6m+2}+F_{6m-1}} < \sum_{k=2m+1}^{\infty} \frac{1}{F_{3k}} < \frac{1}{F_{6m+2}+F_{6m-1}-1}. \tag{6}$$

Since for any integer $k \geq 2$, we have

$$\begin{aligned}&\frac{1}{F_{3k-1}+F_{3k-4}-1} - (\frac{1}{F_{3k}} + \frac{1}{F_{3(k+1)-1}+F_{3(k+1)-4}-1})\\ &= \frac{F_{3k}-(F_{3k-1}+F_{3k-4}-1)}{F_{3k}(F_{3k-1}+F_{3k-4}-1)} - \frac{1}{F_{3(k+1)-1}+F_{3(k+1)-4}-1}\\ &= \frac{(F_{3k-3}+1)(F_{3k-1}+F_{3k+2}-1)-F_{3k}(F_{3k-1}+F_{3k-4}-1)}{F_{3k}(F_{3k-1}+F_{3k-4}-1)(F_{3k+2}+F_{3k-2}-1)}\\ &= \frac{F_{3k-3}(F_{3k+2}+F_{3k-1})-F_{3k}(F_{3k-4}+F_{3k-1})-F_{3k-3}+F_{3k+2}-1+F_{3k}+F_{3k-1}}{F_{3k}(F_{3k-1}+F_{3k-4}-1)(F_{3k+2}+F_{3k-2}-1)}\\ &= \frac{(-1)^{3k-2}\cdot 4+F_{3k+2}+F_{3k-2}+F_{3k}-1}{F_{3k}(F_{3k-1}+F_{3k-4}-1)(F_{3k+2}+F_{3k-2}-1)}\end{aligned}$$

$$> \frac{3}{F_{3k}(F_{3k-1}+F_{3k-4}-1)(F_{3k+2}+F_{3k-2}-1)} > 0.$$

Therefore, for any $k \geq 2$, we have the inequality

$$\frac{1}{F_{3k-1}+F_{3k-4}-1} > \frac{1}{F_{3k}} + \frac{1}{F_{3(k+1)-1}+F_{3(k+1)-4}-1}.$$

Using this inequality repeatedly, we have

$$\begin{aligned}
\frac{1}{F_{3m-1}+F_{3m-4}-1} &> \frac{1}{F_{3m}} + \frac{1}{F_{3(m+1)-1}+F_{3(m+1)-4}-1} \\
&> \frac{1}{F_{3m}} + (\frac{1}{F_{3(m+1)}} + \frac{1}{F_{3(m+2)-1}+F_{3(m+2)-4}-1}) \\
&> \frac{1}{F_{3m}} + \frac{1}{F_{3(m+1)}} + (\frac{1}{F_{3(m+2)}} + \frac{1}{F_{3(m+3)-1}+F_{3(m+3)-4}-1}) \\
&> \cdots > \frac{1}{F_{3m}} + \frac{1}{F_{3(m+1)}} + \frac{1}{F_{3(m+2)}} + \frac{1}{F_{3(m+3)}} + \frac{1}{F_{3(m+4)}} + \cdots \\
&= \sum_{k=m}^{\infty} \frac{1}{F_{3k}}
\end{aligned}$$

or

$$\sum_{k=2m+1}^{\infty} \frac{1}{F_{3k}} < \frac{1}{F_{6m+2}+F_{6m-1}-1}. \tag{7}$$

On the other hand, for any integer $k \geq 1$, we have

$$\begin{aligned}
&\frac{1}{F_{3(2k+1)}} + \frac{1}{F_{3(2k+2)}} - (\frac{1}{F_{3(2k+1)-1}+F_{3(2k+1)-4}} - \frac{1}{F_{3[2(k+1)+1]-1}+F_{3[2(k+1)+1]-4}}) \\
&= \frac{(F_{6k+2}+F_{6k-1})(F_{6k+6}+F_{6k+3}) - F_{6k+3}F_{6k+6}}{F_{6k+3}F_{6k+6}(F_{6k-1}+F_{6k+2})} + \frac{1}{F_{6k+5}+F_{6k+8}} \\
&= \frac{F_{6k+2}F_{6k+3}+F_{6k-1}F_{6k+3}-F_{6k}F_{6k+6}}{F_{6k+3}F_{6k+6}(F_{6k-1}+F_{6k+2})} + \frac{1}{F_{6k+5}+F_{6k+8}} \\
&= \frac{1}{5}\frac{(F_{6k+8}+F_{6k+5})(-\alpha^{12k+3}-\beta^{12k+3}+15)+(F_{6k-1}+F_{6k+2})(\alpha^{12k+9}+\beta^{12k+9}+4)}{F_{6k+3}F_{6k+6}(F_{6k-1}+F_{6k+2})(F_{6k+8}+F_{6k+5})} \\
&= \frac{1}{5}\frac{15F_{6k+8}+15F_{6k+5}+F_{6k-5}-F_{6k-2}+4F_{6k+2}+4F_{6k-1}-F_{6k-7}+F_{6k+10}}{F_{6k+3}F_{6k+6}(F_{6k-1}+F_{6k+2})(F_{6k+8}+F_{6k+5})} \\
&> \frac{1}{5}\frac{15F_{6k+8}+15F_{6k+5}+F_{6k-5}+F_{6k+10}}{F_{6k+3}F_{6k+6}(F_{6k-1}+F_{6k+2})(F_{6k+8}+F_{6k+5})} > 0.
\end{aligned}$$

Therefore, for any integer $k \geq 1$, we have

$$\frac{1}{F_{3(2k+1)}} + \frac{1}{F_{3(2k+2)}} > \frac{1}{F_{3(2k+1)-1}+F_{3(2k+1)-4}} - \frac{1}{F_{3[2(k+1)+1]-1}+F_{3[2(k+1)+1]-4}}.$$

Using this inequality repeatedly, we have

$$\begin{aligned}
\sum_{k=2m+1}^{\infty} \frac{1}{F_{3k}} &= \sum_{k=m}^{\infty} (\frac{1}{F_{3(2k+1)}} + \frac{1}{F_{3(2k+2)}}) \\
&> \sum_{k=m}^{\infty} (\frac{1}{F_{3(2k+1)-1}+F_{3(2k+1)-4}} - \frac{1}{F_{3\cdot 2[(k+1)+1]-1}+F_{3\cdot 2[(k+1)+1]-4}})
\end{aligned}$$

$$= \frac{1}{F_{3(2m+1)-1} + F_{3(2m+1)-4}}.$$

Therefore, for any integer $m \geq 1$, we have the inequality

$$\sum_{k=2m+1}^{\infty} \frac{1}{F_{3k}} > \frac{1}{F_{3(2m+1)-1} + F_{3(2m+1)-4}} = \frac{1}{F_{6m+2} + F_{6m-1}}. \tag{8}$$

Combining (7) and (8), the inequality (6)

$$\frac{1}{F_{6m+2} + F_{6m-1}} < \sum_{k=2m+1}^{\infty} \frac{1}{F_{3k}} < \frac{1}{F_{6m+2} + F_{6m-1} - 1}$$

has been proved.

Now Theorem follows from (1) and (6) and the definition of floor function. □

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