EXPLICIT INVERSE OF THE PASCAL MATRIX PLUS ONE

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This paper presents a simple approach to invert the matrix $P_n + I_n$ by applying the Euler polynomials and Bernoulli numbers, where P_n is the Pascal matrix.

1. Introduction

The Pascal matrix has been known since ancient times, and it arises in many different areas of mathematics. However, it has been studied carefully only recently, see [1, 3–5]. For any integer $n > 0$, the $n \times n$ Pascal matrix P_n is defined with the binomial coefficients by

$$P_n(i,j) = \begin{cases} \binom{i-1}{j-1} & \text{if } i \geq j \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

It is known that the $n \times n$ inverse matrix P_n^{-1} is given by

$$P_n(i,j) = \begin{cases} (-1)^{i-j}\binom{i-1}{j-1} & \text{if } i \geq j \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

The Hadamard product $A \circ B$ of two matrices is the matrix obtained by coordinatewise multiplication: $(A \circ B)(i,j) = A(i,j)B(i,j)$. Let Γ_n be the $n \times n$ lower triangular matrices defined by

$$\Gamma_n(i,j) = \begin{cases} (-1)^{i-j} & \text{if } i \geq j \geq 1, \\ 0 & \text{otherwise,} \end{cases} \tag{1.3}$$

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then the inverse of the Pascal matrix can be represented as the Hadamard product $P_n^{-1} = P_n \circ \Gamma_n$. For example, if $n = 5$, then

$$P_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix},$$

$$P_5^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}. \tag{1.4}$$

Now we consider the sum of the Pascal matrix and the identity matrix $P_n + I_n$, where I_n is the $n \times n$ identity matrix. We call $P_n + I_n$ the Pascal matrix plus one simply. An interesting fact is that the inverse of $P_n + I_n$ is related to P_n closely. For instance,

$$P_6 + I_6 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 & 0 \\ 1 & 3 & 3 & 2 & 0 & 0 \\ 1 & 4 & 6 & 4 & 2 & 0 \\ 1 & 5 & 10 & 10 & 5 & 2 \end{pmatrix},$$

$$(P_6 + I_6)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{4} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{8} & 0 & -\frac{3}{4} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{4}{8} & 0 & -\frac{4}{4} & \frac{1}{2} & 0 \\ -\frac{1}{4} & 0 & \frac{10}{8} & 0 & -\frac{5}{4} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{pmatrix} \circ \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{8} & 0 & -\frac{1}{4} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{8} & 0 & -\frac{1}{4} & \frac{1}{2} & 0 \\ -\frac{1}{4} & 0 & \frac{1}{8} & 0 & -\frac{1}{4} & \frac{1}{2} \end{pmatrix}. \tag{1.5}$$

This suggests that there may exist a sequence of constants $\{a_n\}_{n=0}^{\infty}$ such that $(P_n + I_n)^{-1} = P_n \circ \Delta_n$, where the matrix Δ_n is a lower triangular matrix with generic element $\Delta_n(i, j) = a_{i-j}$ when $i \geq j$. Aggarwala and Lamoureux [2] have showed that these constants are values of the Dirichlet eta function evaluated at negative integers, or more generally, certain polylogarithm functions evaluated at the number -1. In this note, we will give a new simple approach to invert the matrix $P_n + I_n$ by applying the Euler polynomials. As a result, we will show that these constants are values of the Euler polynomials evaluated at the number 0.

The Euler polynomials $E_n(x)$ are defined by means of the following generating function (see [7]):

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2e^{tx}}{e^t + 1}, \tag{1.6}$$

since $\sum_{n=0}^{\infty}(E_n(x+1) + E_n(x))(t^n/n!) = \sum_{n=0}^{\infty} E_n(x+1)(t^n/n!) + \sum_{n=0}^{\infty} E_n(x)(t^n/n!) = 2e^{t(x+1)}/(e^t+1) + 2e^{tx}/(e^t+1) = 2e^{tx} = \sum_{n=0}^{\infty} 2x^n(t^n/n!)$. Comparing the coefficients of $t^n/n!$ in this equation, we obtain

$$E_n(x+1) + E_n(x) = 2x^n, \quad n \geq 0. \tag{1.7}$$

The following lemmas are well known and can be found in [9], we give a short proof for the sake of completeness.

Lemma 1.1. *For all* $n \geq 0$,

$$E_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) y^{n-k}, \tag{1.8}$$

$$E_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} E_k(x). \tag{1.9}$$

Proof. $\sum_{n=0}^{\infty} E_n(x+y)(t^n/n!) = 2e^{t(x+y)}/(e^t+1) = (2e^{tx}/(e^t+1))e^{ty} = (\sum_{n=0}^{\infty} E_n(x)(t^n/n!))(\sum_{n=0}^{\infty} y^n(t^n/n!)) = \sum_{n=0}^{\infty}(\sum_{k=0}^{n}\binom{n}{k}E_k(x)y^{n-k})(t^n/n!)$. Comparing the coefficients of $t^n/n!$ in this equation, we obtain (1.8). In particular, when $y = 1$, we get (1.9).

From (1.7) and (1.9), we obtain

$$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}E_k(x)+\frac{1}{2}E_n(x)=x^n,\quad n\geq 0. \tag{1.10}$$

If we set $x=0$ in (1.8), we get $E_n(y)=\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)y^k$, that is,

$$E_n(x)=\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)x^k,\quad n\geq 0. \tag{1.11}$$

Let $E(x)$ and $X(x)$ be the $n\times 1$ matrices defined by $E(x)=[E_0(x),E_1(x),\dots,E_{n-1}(x)]^T$, $X(x)=[1,x,\dots,x^{n-1}]^T$, and let $\bar{E}_n$ be $n\times n$ lower triangular matrices defined by

$$\bar{E}_n(i,j)=\begin{cases}\binom{i-1}{j-1}E_{i-j}(0) & \text{if } i\geq j\geq 1,\\ 0 & \text{otherwise.}\end{cases} \tag{1.12}$$

Then (1.10), (1.11) can be represented as matrix equations, respectively,

$$\begin{gathered}\frac{1}{2}(P_n+I_n)E(x)=X(x),\\ E(x)=\bar{E}_nX(x).\end{gathered} \tag{1.13}$$

Thus, we have

$$\begin{aligned}&(P_n+I_n)^{-1}\\ &\quad=\frac{1}{2}\bar{E}_n\\ &\quad=\frac{1}{2}\begin{pmatrix}\binom{0}{0}E_0(0) & 0 & 0 & \cdots & 0\\ \binom{1}{0}E_1(0) & \binom{1}{1}E_0(0) & 0 & \cdots & 0\\ \binom{2}{0}E_2(0) & \binom{2}{1}E_1(0) & \binom{2}{2}E_0(0) & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \binom{n-1}{0}E_{n-1}(0) & \binom{n-1}{1}E_{n-2}(0) & \binom{n-1}{2}E_{n-3}(0) & \cdots & \binom{n-1}{n-1}E_0(0)\end{pmatrix}.\end{aligned} \tag{1.14}$$

□

The Bernoulli numbers B_n are defined by (see [7])

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}. \tag{1.15}$$

It is known (see [6, 8]) that the Euler polynomials can be expressed by the Bernoulli numbers as

$$E_n(x) = \frac{1}{n+1} \sum_{k=1}^{n+1} \left(2 - 2^{k+1}\right) \binom{n+1}{k} B_k x^{n+1-k}. \tag{1.16}$$

Putting $x = 0$ in (1.16) gives

$$E_n(0) = \frac{2\left(1 - 2^{n+1}\right) B_{n+1}}{n+1}, \tag{1.17}$$

for all integers $n \geq 0$. Therefore, we obtain an explicit inverse of the Pascal matrix plus one as follows.

Theorem 1.2. *For $n \geq 1$, the $n \times n$ inverse matrix $Q_n = (P_n + I_n)^{-1}$ is given by*

$$Q_n(i,j) = \begin{cases} \dfrac{1}{2} \dbinom{i-1}{j-1} E_{i-j}(0) & \text{if } i \geq j \geq 1, \\ 0 & \text{if } i < j; \end{cases} \tag{1.18}$$

or

$$Q_n(i,j) = \begin{cases} \dbinom{i-1}{j-1} \dfrac{\left(1 - 2^{i-j+1}\right) B_{i-j+1}}{i-j+1} & \text{if } i \geq j \geq 1, \\ 0 & \text{if } i < j. \end{cases} \tag{1.19}$$

In view of the Hadamard product, the inverse matrix $(P_n + I_n)^{-1}$ is the Hadamard product of the Pascal matrix P_n and the matrix Δ_n, where Δ_n is the $n \times n$ lower triangular matrices defined by

$$\Delta_n(i,j) = \begin{cases} \dfrac{1}{2} E_{i-j}(0) & \text{if } i \geq j \geq 1, \\ 0 & \text{if } i < j; \end{cases} \tag{1.20}$$

or

$$\Delta_n(i,j) = \begin{cases} \dfrac{\left(1 - 2^{i-j+1}\right) B_{i-j+1}}{i-j+1} & \text{if } i \geq j \geq 1, \\ 0 & \text{if } i < j. \end{cases} \tag{1.21}$$

The two functions, Euler(n,x) and Bernoulli(n), in the *combinat* library of the computer algebra system *Maple* are very useful in obtaining the matrix Q_n. For example, for $n = 8$, we get

$$Q_8 = (P_8 + I_8)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{8} & 0 & -\frac{3}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{5}{4} & 0 & -\frac{5}{4} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & \frac{5}{2} & 0 & -\frac{3}{2} & \frac{1}{2} & 0 \\ \frac{17}{16} & 0 & -\frac{21}{4} & 0 & \frac{35}{8} & 0 & -\frac{7}{4} & \frac{1}{2} \end{pmatrix}. \tag{1.22}$$

Note that $Q_n(i,j) = 0$ whenever $i < j$ or $i = j+2, j+4, j+6, \ldots$.

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