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Some Properties of Reciprocals of Double Binomial Coefficients*

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Abstract

We investigate the integral representation of infinite sums involving the reciprocals of double binomial coefficients. We also recover some well-known representations of $\zeta(2)$ and other related identities.

Keywords and Phrases: *Double binomial coefficients, Combinatorial identities, Integral representations.*

1. Introduction

This paper is primarily concerned with the summation of the reciprocal of the product of combinatorial coefficients. In particular, we develop integral representations for the series

$$S(a,b,j,k) := \sum_{n=0}^{\infty} \frac{1}{\binom{an+j}{j}\binom{bn+k}{k}},$$

$$Q(a,b,j,k,m) := \sum_{n=0}^{\infty} \frac{\binom{n+m+1}{n}}{\binom{an+j}{j}\binom{bn+k}{k}}$$

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and their alternating series companions.

In the papers [4] and [5], see also the book [3], the author developed a number of integral identities for general sums involving reciprocals of binomial coefficients.

For the sake of completeness we give, without proof, two particular theorems in regard to single binomial coefficients and their integral representations.

Theorem 1. *Let* $a \in \mathbb{R}^+ \backslash \{0\}$ *and* $j > 1$ *then*

$$\begin{aligned} S(a,j) &:= \sum_{n=0}^{\infty} \frac{1}{\binom{an+j}{j}} = j \int_{x=0}^{1} \frac{(1-x)^{j-1}}{1-x^a} dx \\ &= 1 + a \int_{x=0}^{1} \frac{(1-x)^{j-1} x^{a-1}}{(1-x^a)^2} dx \end{aligned}$$

and similarly

$$\begin{aligned} T(a,j) &:= \sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{an+j}{j}} = j \int_{x=0}^{1} \frac{(1-x)^{j-1}}{1+x^a} dx \\ &= 1 - a \int_{x=0}^{1} \frac{(1-x)^{j-1} x^{a-1}}{(1+x^a)^2} dx. \end{aligned}$$

The second integral representation for both $S(a,j)$ and $T(a,j)$ was not given in the papers [4] and [5], however they can be developed in a similar way.

As an example, it can be shown that

$$S(6,5) = \sum_{n=0}^{\infty} \frac{1}{\binom{6n+5}{5}} = \frac{40}{3} \ln 2 - \frac{15}{2} \ln 3$$

and

$$\begin{aligned} T(6,5) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{6n+5}{5}} = \frac{5\pi}{3} \left(\frac{5}{2} - \frac{4}{\sqrt{3}} \right) \\ &= {}_6F_5 \left[\begin{matrix} \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1 \\ \frac{7}{6}, \frac{5}{3}, \frac{3}{2}, \frac{5}{3}, \frac{11}{6} \end{matrix} \middle| -1 \right]. \end{aligned}$$

Similarly, the more general case is embodied in the following theorem.

Theorem 2. *For $m \geq 1$, $a > 0$ and $j > 1$, then*

$$S(a,j,m) := \sum_{n=0}^{\infty} \frac{\binom{n+m-1}{n}}{\binom{an+j}{j}} = j \int_{x=0}^{1} \frac{(1-x)^{j-1}}{(1-x^a)^m} dx$$
$$= 1 + am \int_{x=0}^{1} \frac{(1-x)^{j-1} x^{a-1}}{(1-x^a)^{m+1}} dx.$$

Similarly

$$T(a,j,m) := \sum_{n=0}^{\infty} \frac{(-1)^n \binom{n+m-1}{n}}{\binom{an+j}{j}} = j \int_{x=0}^{1} \frac{(1-x)^{j-1}}{(1+x^a)^{m+1}} dx$$
$$= 1 - am \int_{x=0}^{1} \frac{(1-x)^{j-1} x^{a-1}}{(1+x^a)^{m+1}} dx.$$

An example is:

$$S\left(\frac{3}{2},3,2\right) = \sum_{n=0}^{\infty} \frac{\binom{n+1}{1}}{\binom{\frac{3}{2}n+3}{3}} = 3\ln 3 + \frac{\sqrt{3}}{9}\pi - 2$$

and

$$T\left(\frac{3}{2},3,2\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{n+1}{1}}{\binom{\frac{3}{2}n+3}{3}} = 2 - \frac{2\sqrt{3}}{9}\pi$$
$$= {}_4F_3\left[\begin{array}{c} 2,2,\frac{4}{3},\frac{2}{3} \\ 3,\frac{7}{3},\frac{5}{3} \end{array} \middle| -1\right].$$

Some further results on reciprocals of the central binomial coefficients have recently also been obtained by Sprugnoli [6]. Sprugnoli applies the method of coefficients to generating functions.

We now develop double integral identities for $S(a,b,j,k)$, $S(a,b,j,k,m)$ and their alternating series companions and, moreover, recover and extend some of the results obtained by Guillera and Sondow [1] such as

$$S(2,2,1,1) = \sum_{n=0}^{\infty} \frac{1}{\binom{2n+1}{1}\binom{2n+1}{1}} = \int_{x=0}^{1}\int_{y=0}^{1} \frac{dx\ dy}{1-x^2y^2} = \frac{\pi^2}{8} = \frac{3}{4}\zeta(2)$$

and

$$T(2,2,1,1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{2n+1}{1}\binom{2n+1}{1}} = \int_{x=0}^{1}\int_{y=0}^{1} \frac{dx\ dy}{1+x^2y^2} = G,$$

where G is Catalan's constant and $\zeta(z)$ is the standard Zeta function.

2. Double Binomial Coefficients

In this section we develop integral identities for products of reciprocals of binomial coefficients.

Theorem 3. *For a and b positive real numbers and $j,k \geq 1$, then*

$$S(a,b,j,k) = \sum_{n=0}^{\infty} \frac{1}{\binom{an+j}{j}\binom{bn+k}{k}} \tag{2.1}$$

$$= 1 + ab\int_{x=0}^{1}\int_{y=0}^{1} \frac{(1-x)^j (1-y)^k x^{a-1} y^{b-1} \left(1+x^a y^b\right)}{\left(1-x^a y^b\right)^3} dx\ dy \tag{2.2}$$

$$= jk\int_0^1\int_0^1 \frac{(1-x)^{j-1}(1-y)^{k-1}}{1-x^a y^b} dx\ dy \tag{2.3}$$

$$= {}_{j+k+1}F_{j+k}\left[\left.\begin{matrix} 1, \frac{1}{a}, \frac{2}{a}, \ldots, \frac{j}{a}, \frac{1}{b}, \frac{2}{b}, \ldots, \frac{k}{b} \\ \frac{a+1}{a}, \frac{a+2}{a}, \ldots, \frac{a+j}{a}, \frac{b+1}{b}, \frac{b+2}{b}, \ldots, \frac{b+k}{b} \end{matrix}\right| 1\right], \tag{2.4}$$

and

$$T(a,b,j,k) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{an+j}{j}\binom{bn+k}{k}} \tag{2.5}$$

$$= 1 - ab\int_{x=0}^{1}\int_{y=0}^{1} \frac{(1-x)^j (1-y)^k x^{a-1} y^{b-1} \left(1-x^a y^b\right)}{\left(1+x^a y^b\right)^3} dx\ dy \tag{2.6}$$

$$= jk\int_0^1\int_0^1 \frac{(1-x)^{j-1}(1-y)^{k-1}}{1+x^a y^b} dx\ dy \tag{2.7}$$

$$= {}_{j+k+1}F_{j+k}\left[\left.\begin{matrix} 1, \frac{1}{a}, \frac{2}{a}, \ldots, \frac{j}{a}, \frac{1}{b}, \frac{2}{b}, \ldots, \frac{k}{b} \\ \frac{a+1}{a}, \frac{a+2}{a}, \ldots, \frac{a+j}{a}, \frac{b+1}{b}, \frac{b+2}{b}, \ldots, \frac{b+k}{b} \end{matrix}\right| -1\right]. \tag{2.8}$$

Proof. Consider (2.1)

$$\begin{aligned} S(a,b,j,k) &= \sum_{n=0}^{\infty} \frac{1}{\binom{an+j}{j}\binom{bn+k}{k}} \\ &= \sum_{n=0}^{\infty} \frac{abn^2\Gamma(an)\Gamma(j+1)\Gamma(bn)\Gamma(k+1)}{\Gamma(an+j)\Gamma(bn+k+1)} \\ &= 1+\sum_{n=1}^{\infty} abn^2 B(an,j+1)\,B(bn,k+1), \end{aligned}$$

where $B(\cdot,\cdot)$ is the classical Beta function.

$$S(a,b,j,k) = 1+ab\sum_{n=1}^{\infty} n^2 \int_{x=0}^{1}\int_{y=0}^{1} x^{an-1}(1-x)^j y^{bn-1}(1-y)^k\,dx\ dy,$$

by an allowable change of integral and sum, we have

$$\begin{aligned} &= 1+ab\int_{x=0}^{1}\int_{y=0}^{1} \frac{(1-x)^j(1-y)^k}{xy}\sum_{n=0}^{\infty} n^2\left(x^a y^b\right)^n dx\ dy \\ &= 1+ab\int_{x=0}^{1}\int_{y=0}^{1} \frac{(1-x)^j(1-y)^k x^{a-1}y^{b-1}\left(1+x^a y^b\right)}{\left(1-x^a y^b\right)^3} dx\ dy \end{aligned}$$

which is the result (2.2). The identity (2.3) can be proved in a similar way, as can also the results (2.6) and (2.7). □

The hypergeometric representation (2.4) and (2.8) can be obtained by the consideration of the ratio of successive terms (2.1) and (2.5) respectively.

Note: By known properties of the hypergeometric function, we may write, from (2.3) and (2.6):

$$\begin{aligned} &{}_{j+k+1}F_{j+k}\left[\left.\begin{array}{c} 1,\frac{1}{a},\frac{2}{a},\dots,\frac{j}{a},\frac{1}{b},\frac{2}{b},\dots,\frac{k}{b} \\ \frac{a+1}{a},\frac{a+2}{a},\dots,\frac{a+j}{a},\frac{b+1}{b},\frac{b+2}{b},\dots,\frac{b+k}{b} \end{array}\right| 1\right] \\ &= {}_{a+b+1}F_{a+b}\left[\left.\begin{array}{c} 1,1,1,\frac{1}{a},\frac{2}{a},\dots,\frac{a-1}{a},\frac{1}{b},\frac{2}{b},\dots,\frac{b-1}{b} \\ \frac{j+1}{a},\frac{j+2}{a},\dots,\frac{j+a}{a},\frac{k+1}{b},\frac{k+2}{b},\dots,\frac{k+b}{b} \end{array}\right| 1\right] \end{aligned}$$

and

$$
{}_{j+k+1}F_{j+k}\left[\left.\begin{array}{c} 1,\frac{1}{a},\frac{2}{a},\ldots,\frac{j}{a},\frac{1}{b},\frac{2}{b},\ldots,\frac{k}{b} \\ \frac{a+1}{a},\frac{a+2}{a},\ldots,\frac{a+j}{a},\frac{b+1}{b},\frac{b+2}{b},\ldots,\frac{b+k}{b} \end{array}\right| -1\right]
$$

$$
= {}_{a+b+1}F_{a+b}\left[\left.\begin{array}{c} 1,1,1,\frac{1}{a},\frac{2}{a},\ldots,\frac{a-1}{a},\frac{1}{b},\frac{2}{b},\ldots,\frac{b-1}{b} \\ \frac{j+1}{a},\frac{j+2}{a},\ldots,\frac{j+a}{a},\frac{k+1}{b},\frac{k+2}{b},\ldots,\frac{k+b}{b} \end{array}\right| -1\right].
$$

Examples

1.

$$
\begin{aligned}
S(1,1,1,1) &= \sum_{n=0}^{\infty}\frac{1}{(n+1)^2} \\
&= \zeta(2) \\
&= \frac{\pi^2}{6} \\
&= \int_0^1\int_0^1 \frac{dx\ dy}{1-xy} \\
&= {}_3F_2\left[\left.\begin{array}{c} 1,1,1 \\ 2,2 \end{array}\right| 1\right] \\
&= 1+\int_0^1\int_0^1 \frac{(1-x)(1-y)(1+xy)}{(1-xy)^3}dx\ dy
\end{aligned}
$$

This is a well known result which can be traced back to Beuker's integrals [3], or otherwise, see Sandor [2]. Note for $S(2,2,1,1)=\frac{3}{4}\zeta(2)$ we recover the result of Guillera and Sondow [1].

2. For non integers a and b values we have

$$
\begin{aligned}
S\left(\frac{1}{4},\frac{1}{2},2,3\right) &= 8\pi^2-\frac{1642}{21} \\
&= 6\int_0^1\int_0^1 \frac{(1-x)(1-y)^2}{1-x^{\frac{1}{4}}y^{\frac{1}{2}}}dx\ dy \\
&= \sum_{n=0}^{\infty}\frac{1}{\binom{\frac{n}{4}+2}{2}\binom{\frac{n}{2}+3}{3}}.
\end{aligned}
$$

3. For the alternating case

$$T(2,2,1,1)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{2n+1}{1}\binom{2n+1}{1}}=\int_{x=0}^{1}\int_{y=0}^{1}\frac{dx\ dy}{1+x^2y^2}=G$$

which recovers Guillera and Sondow's [1] result.

4.

$$\begin{aligned}T(3,4,3,2)&=\sum_{n=0}^{\infty}\frac{(-1)^n}{\binom{3n+3}{3}\binom{4n+2}{2}}\\&=6\int_0^1\int_0^1\frac{(1-y)(1-x)^2\,dx\ dy}{1+x^3y^4}\\&={}_6F_5\left[\begin{matrix}1,1,\frac{2}{3},\frac{1}{2},\frac{1}{3},\frac{1}{4}\\2,\frac{5}{3},\frac{3}{2},\frac{4}{3},\frac{5}{4}\end{matrix}\middle|-1\right]\\&=\frac{64\sqrt{2}}{15}\ln\left(1+\sqrt{2}\right)+\left(\frac{32\sqrt{2}}{15}-\frac{21\sqrt{3}}{5}+4\right)\pi-\frac{76}{15}\ln 2.\end{aligned}$$

Now consider the following theorem, which is a generalisation of Theorem 3.

Theorem 4. *For a,b and m positive real numbers and $j,k\geq 1$ and $j+k>m$, then*

$$Q(a,b,j,k,m)$$

$$=\sum_{n=0}^{\infty}\frac{\binom{n+m-1}{n}}{\binom{an+j}{j}\binom{bn+k}{k}} \tag{2.9}$$

$$=jk\int_0^1\int_0^1\frac{(1-x)^{j-1}(1-y)^{k-1}}{(1-x^ay^b)^m}dx\ dy \tag{2.10}$$

$$=1+mab\int_0^1\int_0^1\frac{(1-x)^j(1-y)^k x^{a-1}y^{b-1}\left(1+mx^ay^b\right)}{(1-x^ay^b)^{m+2}}dx\ dy \tag{2.11}$$

$$={}_{j+k+1}F_{j+k}\left[\begin{matrix}m,\frac{1}{a},\frac{2}{a},\dots,\frac{j}{a},\frac{1}{b},\frac{2}{b},\dots,\frac{k}{b}\\\frac{a+1}{a},\frac{a+2}{a},\dots,\frac{a+j}{a},\frac{b+1}{b},\frac{b+2}{b},\dots,\frac{b+k}{b}\end{matrix}\middle|1\right], \tag{2.12}$$

and

$$\begin{aligned} &R(a,b,j,k,m) \\ =&\sum_{n=0}^{\infty}\frac{(-1)^n\binom{n+m-1}{n}}{\binom{an+j}{j}\binom{bn+k}{k}} \qquad (2.13)\\ =&jk\int_0^1\int_0^1\frac{(1-x)^{j-1}(1-y)^{k-1}}{(1+x^ay^b)^m}dx\ dy \qquad (2.14)\\ =&1-mab\int_0^1\int_0^1\frac{(1-x)^j(1-y)^k x^{a-1}y^{b-1}\left(1-mx^ay^b\right)}{(1+x^ay^b)^{m+2}}dx\ dy \qquad (2.15)\\ =&\ {}_{j+k+1}F_{j+k}\left[\begin{matrix} m,\frac{1}{a},\frac{2}{a},\dots,\frac{j}{a},\frac{1}{b},\frac{2}{b},\dots,\frac{k}{b} \\ \frac{a+1}{a},\frac{a+2}{a},\dots,\frac{a+j}{a},\frac{b+1}{b},\frac{b+2}{b},\dots,\frac{b+k}{b}\end{matrix}\middle| -1\right]. \qquad (2.16)\end{aligned}$$

Proof. Consider (2.13)

$$\begin{aligned} &R(a,b,j,k,m) \\ =&\sum_{n=0}^{\infty}\frac{(-1)^n\binom{n+m-1}{n}}{\binom{an+j}{j}\binom{bn+k}{k}} \\ =&\sum_{n=0}^{\infty}(-1)^n abn^2\binom{n+m-1}{n}\frac{\Gamma(an)\Gamma(j+1)\Gamma(bn)\Gamma(k+1)}{\Gamma(an+j+1)\Gamma(bn+k+1)} \\ =&1+ab\sum_{n=1}^{\infty}(-1)^n n^2\binom{n+m-1}{n}B(an,j+1)B(bn,k+1) \\ =&1+ab\sum_{n=1}^{\infty}(-1)^n n^2\binom{n+m-1}{n}\int_{x=0}^1\int_{y=0}^1 x^{an-1}(1-x)^j y^{bn-1}(1-y)^k dx\ dy. \end{aligned}$$

By an allowable change of integral and sum, we have

$$\begin{aligned} R(a,b,j,k,m) &= 1+ab\int_{x=0}^1\int_{y=0}^1\frac{(1-x)^j(1-y)^k}{xy} \\ &\qquad\times\sum_{n=0}^{\infty}(-1)^n\binom{n+m-1}{n}n^2\left(x^ay^b\right)^n dx\ dy \\ &= 1-mab\int_0^1\int_0^1\frac{(1-x)^j(1-y)^k x^{a-1}y^{b-1}\left(1-mx^ay^b\right)}{(1+x^ay^b)^{m+2}}dx\ dy, \end{aligned}$$

which is the result (2.15).

To arrive at the result (2.14) consider

$$\begin{aligned}R(a,b,j,k,m) &:= jk\sum_{n=0}^{\infty}(-1)^n\binom{n+m-1}{n}\frac{\Gamma(an+1)\Gamma(j)\Gamma(bn+1)\Gamma(k)}{\Gamma(an+j+1)\Gamma(bn+k+1)}\\ &= jk\sum_{n=0}^{\infty}(-1)^n\binom{n+m-1}{n}B(an+1,j)B(bn+1,k)\\ &= jk\int_0^1\int_0^1(1-x)^{j-1}(1-y)^{k-1}\sum_{n=0}^{\infty}\binom{n+m-1}{n}\left(x^ay^b\right)^n dx\ dy\\ &= jk\int_0^1\int_0^1\frac{(1-x)^{j-1}(1-y)^{k-1}}{(1+x^ay^b)^m}dx\ dy.\end{aligned}$$

The results (2.10) and (2.11) can be proved in a similar way. □
In the case when $m=1$, Theorem 4 reduces to Theorem 3.

Examples:

1.

$$\begin{aligned}&Q(4,6,k+1,k+1,m)\\ &=\sum_{n=0}^{\infty}\frac{\binom{n+m-1}{n}}{\binom{4n+k+1}{k+1}\binom{6n+k+1}{k+1}}\\ &=(k+1)^2\int_0^1\int_0^1\frac{(1-x)^k(1-y)^k}{(1+x^4y^6)^m}dx\ dy\\ &= {}_{11}F_{10}\left[\left.\begin{matrix},1,1,\frac{5}{6},\frac{3}{4},\frac{2}{3},\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6}\\ \frac{k+7}{6},\frac{k+6}{6},\frac{k+5}{6},\frac{k+4}{6},\frac{k+3}{6},\frac{k+2}{6},\frac{k+5}{4},\frac{k+4}{4},\frac{k+3}{4},\frac{k+2}{4}\end{matrix}\right|1\right]\\ &=a+\beta\ln 2+\gamma\ln 3+\delta\pi+\rho\pi^2,\end{aligned}$$

for $k=4$ and $m=3$; $\alpha=-\frac{40}{1001}$, $\beta=\frac{44470}{1001}$, $\gamma=-\frac{32805}{1144}$, $\delta=\frac{478125\sqrt{3}}{16016}-\frac{36460}{1001}$, $\rho=-\frac{75}{16}$.

2.

$$\begin{aligned} R(3,2,6,5,3) &:= \sum_{n=0}^{\infty} \frac{(-1)^n \binom{n+2}{2}}{\binom{3n+6}{6}\binom{2n+5}{5}} \\ &= {}_5F_4\left[\left.\begin{matrix} 1,1,\frac{2}{3},\frac{1}{2},\frac{1}{3} \\ 3,\frac{7}{2},\frac{8}{3},\frac{7}{3} \end{matrix}\right| -1\right] \\ &= 30\int_0^1\int_0^1 \frac{(1-x)^5(1-y)^4}{(1+x^3y^2)^3}dx\,dy \\ &= \frac{927505}{3003} + \frac{\pi}{1001}\left(114560 - 88290\sqrt{3}\right) - \frac{270720}{1001}\ln 2. \end{aligned}$$

3. Conclusion

We have provided double integral identities for sums of the reciprocal of double binomial coefficients. In doing so we have generalised and extended some results published previously by other authors.

We can further extend our results to consider binomial coefficients of the form

$$\sum_{n=0}^{\infty} \frac{n^s\binom{n+m-1}{n}}{\binom{an+j}{j}\binom{bn+k}{k}} \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{\binom{n+m-1}{n}}{\binom{an+j}{bn}\binom{cn+k}{dn}}.$$

This will be reported in another forum.

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