The Fibonacci Quarterly 1981 (vol.19,5): 433-437

Proof: Let $(\#b_i \le n)$ denote the number of b_i that are $\le n$. Using the formula $CF(n) = n + f^{\dagger}(n)$ on p. 457 of Lambek and Moser [6], we find

$$n + (\#b_i \le n) = n\text{th positive integer not in the sequence } \{b_n + n - 1\},$$

so that

$$(\#b_i \le a_n) = -a_n + a_n\text{th term of the complement of } \{b_n + n - 1\},$$

whence the nth term of $\{a_n\} \oplus \{b_n\}$, which is clearly $a_n - (\#b_i \le a_n)$, must equal $2a_n - c^*_{a_n}$. Since $\{c_n\}$ is almost arithmetic with slope $v + 1$, $\{c^*_n\}$ is almost arithmetic with slope $1 + 1/v$, by Theorem 6. Then $\{c^*_{a_n}\}$ is almost arithmetic with slope $u(1 + 1/v)$, by Theorem 5. Thus, $\{2a_n - c^*_{a_n}\}$ is almost arithmetic with slope $2u - u(1 + 1/v)$.

Theorem 9: Suppose $\{a_n\}$ and $\{b_n\}$ are almost arithmetic sequences having slopes u and v, respectively. Then

$$\{a_n\} \ominus \{b_n\} = \{b^*_{a_n}\}$$

is an almost arithmetic sequence with slope $uv/(v - 1)$.

Proof: By definition, the nth term of $\{a_n\} \ominus \{b_n\}$ is the a_nth positive integer not one of the b_i, as claimed. As a composite of a complement, this is an almost arithmetic sequence with slope $uv/(v - 1)$, much as in the proof of Theorem 8.

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SUMS OF THE INVERSES OF BINOMIAL COEFFICIENTS

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In this note, we discuss several sums of inverses of binomial coefficients. We evaluate these sums by application of a fundamental recurrence relation in much the same manner as sums of binomial coefficients may be treated. As an application, certain iterated integrals of the logarithm are evaluated.

Let $n \geq k$ be positive integers. One of the basic recurrence relations of binomial coefficients

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

is that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

For the inverse of the binomial coefficient,

$$\binom{n}{k}^{-1} = \frac{(n-k)!k!}{n!},$$

we observe that

$$\begin{aligned}\binom{n}{k}^{-1} &= \frac{(n-k)!(k-1)!}{n!} \cdot (n-(n-k)) \\ &= \frac{((n-1)-(k-1))!(k-1)!}{n!} \cdot n - \frac{((n-(k-1))(n-k)!(k-1)!}{n!} \cdot \frac{(n-k)}{(n-(k-1))}\end{aligned}$$

and so

$$(*) \qquad \binom{n}{k}^{-1} = \binom{n-1}{k-1}^{-1} - \frac{(n-k)}{(n-k+1)}\binom{n}{k-1}^{-1}.$$

This relation is studied from a different viewpoint in [5, Ch. 1, Prob. 5]. For a similar sum formula not to be discussed here, see [4, n. 21].

Using mathematical induction on n and the identity (*), we find

$$\binom{n+m}{m}^{-1} = 1 - \frac{n}{n+1}\sum_{k=1}^{m}\binom{n+k}{k-1}^{-1}$$

for any two positive integers n and m (for the corresponding relation for binomial coefficients, see [2, p. 200]).

<u>*Theorem 1*</u>: Let $I_n = \sum_{k=0}^{n}\binom{n}{k}^{-1}$. Then I_n satisfies the recursion relation

$$I_n = \frac{n+1}{2n}I_{n-1} + 1$$

and

$$I_n = \frac{(n+1)}{2^{n+1}}\sum_{k=1}^{n+1}\frac{2^k}{k}.$$

This corrects a slight error in [3].

<u>*Proof by Induction on n*</u>: For $n = 1$, we have $I_1 = 2$ from the definition and from the formula. We now show that the formula for $n + 1$ follows from the formula for n and the relation (*).

$$I_{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}^{-1} = \binom{n+1}{0}^{-1} + \sum_{k=1}^{n+1}\binom{n+1}{k}^{-1}.$$

Applying (*) to each term of the sum, we have

$$\begin{aligned}I_{n+1} &= 1 + \sum_{k=1}^{n+1}\left(\binom{n}{k-1}^{-1} - \frac{(n+1)-k}{(n+1-k)+1} \cdot \binom{n+1}{k-1}^{-1}\right) \\ &= 1 + I_n - \sum_{k=0}^{n}\frac{n-k}{(n+1)-k}\binom{n+1}{k}^{-1}.\end{aligned}$$

Since

$$\frac{n-k}{(n+1)-k} = 1 - \frac{1}{(n+1)-k},$$

we may rewrite our last expression as two sums:

$$\begin{aligned} I_{n+1} &= 1 + I_n - \sum_{k=0}^{n} \binom{n+1}{k}^{-1} + \sum_{k=0}^{n} \frac{1}{(n+1)-k}\binom{n+1}{k}^{-1} \\ &= 2 + I_n - I_{n+1} + \frac{1}{n+1} I_n \end{aligned}$$

so that

$$I_{n+1} = \frac{n+2}{2(n+1)} I_n + 1$$

and the recursion relation is established. Applying the induction hypothesis for I_n yields

$$I_{n+1} = \frac{n+2}{2(n+1)}\left(\frac{n+1}{2^{n+1}}\sum_{k=1}^{n+1}\frac{2^k}{k}\right) + \frac{n+2}{2^{n+2}}\,\frac{2^{n+2}}{n+2} = \frac{(n+1)+1}{2^{(n+1)+1}}\sum_{k=1}^{(n+1)+1}\frac{2^k}{k},$$

as required.

Theorem 2: For $n \geq 2$, $\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1} = \frac{n}{n-1}$.

Proof by Induction: For $n = 2$, the sum is

$$2\sum_{k=0}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+2}\right)$$

and the terms pairwise cancel. For $n > 2$, we observe that

$$\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1} = \binom{n+0}{0}^{-1} + \sum_{k=1}^{\infty}\binom{n+k}{k}^{-1} = 1 + \sum_{k=0}^{\infty}\binom{n+(k+1)}{k+1}^{-1}.$$

Applying (*) to each term of the sum, we have

$$\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1} = 1 + \sum_{k=0}^{\infty}\left(\binom{n+k}{k}^{-1} - \frac{n}{n+1}\binom{n+(k+1)}{k}^{-1}\right).$$

Assuming $\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1} = \frac{n}{n-1}$ and hence is finite, we obtain

$$\frac{n}{n+1}\sum_{k=0}^{\infty}\binom{(n+1)+k}{k}^{-1} = 1,$$

completing our proof.

Theorem 3: For $n \geq 1$, let $J_n = \sum_{k=0}^{\infty}(-1)^k\binom{n+k}{k}^{-1}$. Then J_n satisfies the recursion relation

$$J_{n+1} = \frac{n+1}{n}(2J_n - 1)$$

and

$$J_n = \frac{n}{2}\left(2^n \ln(2) - \sum_{k=1}^{n-1}\frac{2^k}{n-k}\right).$$

Proof by Induction: For $n = 1$, we have $J_1 = \ln(2)$. For $n > 1$, we follow the method of proof of Theorem 1.

$$\begin{aligned} J_n &= 1 + \sum_{k=0}^{\infty} (-1)^{k+1} \binom{n + (k + 1)}{k + 1}^{-1} \\ &= 1 + \sum_{k=0}^{\infty} (-1)^{k+1} \left(\binom{n + k}{k}^{-1} - \frac{n}{n + 1} \binom{n + (k + 1)}{k}^{-1} \right), \quad \text{by } (*), \\ &= 1 - \sum_{k=0}^{\infty} (-1)^k \binom{n + k}{k}^{-1} + \frac{n}{n + 1} \sum_{k=0}^{\infty} (-1)^k \binom{(n + 1) + k}{k}^{-1} \\ &= 1 - J_n + \frac{n}{n + 1} J_{n+1} \end{aligned}$$

and the recursion relation follows. Thus

$$\begin{aligned} J_{n+1} &= (n + 1)\frac{2}{n}\left(\frac{n}{2}\left(2^n \ln(2) - \sum_{k=1}^{n-1} \frac{2^k}{n - k}\right)\right) - \frac{n + 1}{n} \\ &= \frac{n + 1}{2}\left(2^{n+1} \ln(2) - \sum_{k=1}^{n} \frac{2^k}{(n + 1) - k}\right). \end{aligned}$$

As an application of these last two results, we use them and a theorem of Abel (see [1]) to evaluate an iterated integral of the logarithmic function.

Let $f_0(x) = (1 - x)^{-1}$ and, for $n > 0$, let

$$f_n(x) = \int_0^x f_{n-1}(t)\, dt.$$

Recall that integration by parts gives the formula

$$\int x^n \ln(x)\, dx = \frac{x^{n+1}}{n + 1}\ln(x) - \frac{x^{n+1}}{(n + 1)^2} \quad \text{for } n \geq 0.$$

Since $f_1(x) = -\ln(1 - x)$, we see that

$$f_2(x) = \int_0^x - \ln(1 - t)\, dt = (1 - x) \ln(1 - x) - (1 - x) + 1$$

and by induction on n we find

$$f_n(x) = \frac{(-1)^n}{(n - 1)!}(1 - x)^{n-1} \ln(1 - x) + A(n) \cdot (1 - x)^{n-1} + \sum_{k=0}^{n-2} B(n, k) \cdot x^k$$

for $n \geq 2$ and x in the open interval $(-1, 1)$. Here $A(n)$ is given by $A(1) = 0$ and for $n \geq 2$,

$$A(n) = \frac{-1}{n - 1}\left(A(n - 1) + \frac{(-1)^n}{(n - 1)!}\right)$$

and $B(n, k)$ is given by $B(n, 0) = -A(n)$ for $n \geq 1$, while for $n \geq 2$ and $k \geq 1$,

$$B(n, k) = \frac{1}{k}B(n - 1, k - 1).$$

Notice that repeated application of this last relation gives

$$B(n, k) = \frac{1}{k!}B(n - k, 0) \quad \text{for } k \leq n - 2,$$

and so

$$B(n,\ 0) = \frac{(-1)^n}{(n-1)!}\sum_{k=1}^{n-1}\frac{1}{k}.$$

Since

$$\sum_{k=1}^{m}\frac{1}{k} = \sum_{k=1}^{m}\frac{(-1)^{k-1}}{k}\binom{m}{k},$$

we see that each $B(n,\ 0)$ may be regarded as a binomial sum.

On the other hand,

$$f_0(x) = (1-x)^{-1} = \sum_{k=0}^{\infty} x^k$$

and term by term integration of this power series gives

$$f_n(x) = x^n\sum_{k=0}^{\infty}\frac{x^k}{(k+1)\cdot\cdots\cdot(k+n)}.$$

For $n \geq 2$, this series converges at $x = \pm 1$ and is uniformly convergent on the closed interval [-1, 1]. By Abel's theorem for power series, the values of our functions at the endpoints of the interval of convergence are given by the power series

$$\lim_{x\to 1} f_n(x) = \sum_{k=0}^{\infty}\frac{1}{(k+1)\cdot\cdots\cdot(k+n)} = \frac{1}{n!}\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1} = \frac{1}{n!}\cdot\frac{n}{n-1},$$

by out Theorem 2, while our Theorem 3 gives

$$\lim_{x\to -1} f_n(x) = (-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)\cdot\cdots\cdot(k+n)} = \frac{(-1)^n}{n!}\sum_{k=0}^{\infty}(-1)^k\binom{n+k}{k}^{-1} = \frac{(-1)^n}{n!}J_n.$$

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TILING THE PLANE WITH INCONGRUENT REGULAR POLYGONS

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Professor Michael Edelstein asked me how to tile the Euclidean plane with squares of integer side lengths all of which are incongruent. The question can be answered in a way that involves a perfect squared square and a geometric application of the Fibonacci numbers.

A perfect squared square is a square of integer side length which is tiled with more than one (but finitely many) component squares of integer side lengths all of which are incongruent. For more information, see the survey articles [3] and [5]. A perfect squared square is simple if it contains no proper subrectangle