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THE SUM OF INVERSES OF BINOMIAL COEFFICIENTS REVISITED

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The aim of this note is to generalize the work on finite sums of inverses of binomial coefficients that are part of a paper by Andrew Rockett [1] which was published in this Quarterly in 1981. Our work rests on the following lemma.

Lemma 1: For any positive integers n and p, with $p \leq n$:

$$\frac{1}{n+1}\binom{n}{p}^{-1} = \int_0^1 t^p(1-t)^{n-p}dt.$$

Proof: Use the well-known formulas of Euler,

$$\int_0^1 t^{u-1}(1-t)^{v-1}dt = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \quad \text{and} \quad \Gamma(k+1) = k!,$$

that are valid for any positive real numbers u and v, and any positive integer k.

Now we define

$$q_0 = 1 \quad \text{and} \quad q_n(x, y) = \frac{1}{n+1}\sum_{p=0}^{n}\binom{n}{p}^{-1} x^p y^{n-p}$$

for arbitrary nonzero complex numbers x and y. Let $G(z)$ be the generating function

$$G(z) = \sum_{n=0}^{+\infty} z^n q_n(x, y).$$

Theorem 1: For any *real* numbers x, y, and z such that $|z| < \min(|1/x|, |1/y|)$:

$$G(z) = -\frac{\log(1-xz)+\log(1-yz)}{az-bz^2} = \frac{-\log(1-az+bz^2)}{az-bz^2},$$

where $a = x+y$ and $b = xy$ (that is, x and y are the roots of the equation in S: $S^2 - aS + b = 0$).

Proof: We have

$$q_n(x, y) = \int_0^1 \sum_{p=0}^{n}(xt)^p(y(1-t))^{n-p}dt$$

and, therefore, for any z such that $|z| < \min(|1/x|, |1/y|)$,

$$G(z) = \int_0^1 \sum_{n=0}^{+\infty}\left(\sum_{p=0}^{n} z^n(xt)^p(y(1-t))^{n-p}\right)dt$$

because of the uniform convergence (deduced from its *absolute* convergence) of the series over $t \in [0, 1]$. By the Cauchy rule for the product of powers series, this is equivalent to

$$G(z)=\int_0^1\left(\sum_{n=0}^{+\infty}(zxt)^n\right)\left(\sum_{n=0}^{+\infty}(zy(1-t))^n\right)dt=\int_0^1\frac{1}{1-zxt}\cdot\frac{1}{1-zy(1-t)}\,dt,$$

from which we obtain the stated result by elementary techniques of calculus.

The restriction to real values allows us not to manipulate many-valued complex functions such as the log of complex argument, and will not impair our work since its main objective is the *polynomial* identities that follow; these, once proved for the real values of the variables, are also proved for the complex values because the identity for all real values implies the identity of the coefficients of the like powers of the variables on both sides of the relation.

This leads to Theorem 1 of Rockett [1],

$$\sum_{p=0}^{n}\binom{n}{p}^{-1}=\frac{n+1}{2^{n+1}}\sum_{k=1}^{n+1}\frac{2^k}{k}$$

by taking $x=y=1$, because in that case,

$$G(z)=-\frac{2\log(1-z)}{2z-z^2}=-\frac{\log(1-z)}{z-(z^2/2)}=\left(\sum_{n=0}^{+\infty}\frac{z^n}{n+1}\right)\left(\sum_{n=0}^{+\infty}\frac{z^n}{2^n}\right)$$

and the result follows by applying the Cauchy rule for the product of power series and then equating the coefficients of the like powers of z on both sides of the identity. (This formula previously appeared in the literature in 1947 in a paper by Tor B. Staver [2].)

If we take $x=-1$ and $y=1$, we obtain

$$G(z)=-\frac{\log(1-z^2)}{z^2},$$

from which it is easy to derive the closed formula

$$\sum_{p=0}^{2n}(-1)^p\binom{2n}{p}^{-1}=\frac{2n+1}{n+1},$$

already found independently by Tor B. Staver [2] and T. S. Nanjudiah [3].

Theorem 2: If $(U_n(u,v))$ denotes the generalized Fibonacci sequence of the recurrence $r_{n+2}-ur_{n+1}+vr_n=0$, then

$$q_n(x,y)=\int_0^1 U_{n+1}(at,bt)\,dt. \tag{1}$$

Proof: An equivalent statement of Theorem 1 is, for x, y, and z real:

$$G(z)=\int_0^1\frac{1}{1-atz+btz^2}\,dt.$$

But since

$$\frac{1}{1-uz+vz^2}=\sum_{n=0}^{+\infty}z^nU_{n+1}(u,v),$$

we obtain, by integrating term by term with regard to t,

$$G(z) = \sum_{n=0}^{+\infty} z^n \int_0^1 U_{n+1}(at, bt)\, dt,$$

and the stated theorem follows by equating the coefficients of like powers of z on both sides of the identity. Since both sides of relation (1) are polynomials in x and y, this relation is also valid for all the complex values of these variables.

Given the well-known Lucas identity [4]

$$U_{n+1}(u, v) = \sum_{k=0}^{[\frac{n}{2}]} \binom{n-k}{k} u^{n-2k}(-v)^k,$$

we obtain as a direct consequence of Theorem 2 the explicit expression for $q_n(x, y)$ as a function of a and b,

$$q_n(x, y) = p_n(a, b) = \sum_{k=0}^{[\frac{n}{2}]} \frac{1}{n-k+1} \binom{n-k}{k} a^{n-2k}(-b)^k,$$

from which we can deduce the converse relation

$$\frac{\partial}{\partial t}(tp_n(at, bt)) = \frac{\partial}{\partial t}\left(tq_n\left(\frac{at - \sqrt{(at)^2 - 4bt}}{2}, \frac{at + \sqrt{(at)^2 - 4bt}}{2} \right)\right) = U_{n+1}(at, bt).$$

Another consequence of Theorem 2 is that, for any positive integer m, we can compute recursively the development in powers of z of

$${}^{m}G(z) = \frac{G(z)}{(a - bz)^m}.$$

In effect, if we define

$${}^{1}q_n(x, y) = \int_0^1 tU_{n+1}(at, bt)\, dt$$

by the fundamental second-order recurrence satisfied by $(U_n(u, v))$ and Theorem 2, we obtain

$$q_n(x, y) = a\,{}^{1}q_{n-1}(x, y) - b\,{}^{1}q_{n-2}(x, y).$$

From this last relation, one can deduce easily that ${}^{1}G(z)$ is a generating function for $({}^{1}q_n(x, y))$. By recursion on m, and the same kind of reasoning, it can be shown that ${}^{m}G(z)$ is a generating function for $({}^{m}q_n(x, y))$ defined as

$${}^{m}q_n(x, y) = \int_0^1 t^m U_{n+1}(at, bt)\, dt$$

since

$${}^{m-1}q_n(x, y) = a\,{}^{m}q_{n-1}(x, y) - b\,{}^{m}q_{n-2}(x, y).$$

With the previously used Lucas identity, we are able to compute an explicit expression of ${}^{m}q_n(x, y)$.

Another relation deduced from Theorem 2 is

$$2q_n(x, y) - a\,{}^{1}q_{n-1}(x, y) = \int_0^1 V_n(at, bt)\, dt$$

if $(V_n(u,v))$ denotes the generalized Lucas sequence of the recursion $r_{n+2}-ur_{n+1}+vr_n=0$, as a consequence of the well-known relation

$$V_n(u,v)=2U_{n+1}(u,v)-uU_n(u,v).$$

Theorem 3: The sequence $(q_n(x,y))$ satisfies the following third-order recurrence [by writing for convenience, $q_n(x,y)=q_n$]:

$$a(n+2)\,q_{n+1}-\{a^2(n+1)+b(n+2)\}\,q_n+ab(2n+1)\,q_{n-1}-b^2nq_{n-2}=0.$$

Proof: From Theorem 1, we deduce that

$$\begin{aligned}(az-bz^2)\sum_{n=0}^{+\infty}z^nq_n(x,y) &= -\log(1-zx)-\log(1-zy)\\ &= \sum_{n=1}^{+\infty}\frac{1}{n}z^n(x^n+y^n)=\sum_{n=1}^{+\infty}\frac{1}{n}z^nV_n(a,b).\end{aligned}$$

By comparing the coefficients of z^{n+2} on both sides, we obtain

$$\frac{V_{n+2}(a,b)}{n+2}=a\,q_{n+1}-b\,q_n.$$

Then the recursion $V_{n+2}(a,b)-aV_{n+1}(a,b)+bV_n(a,b)=0$ becomes

$$a(n+2)\,q_{n+1}-\{a^2(n+1)+b(n+2)\}\,q_n+ab(2n+1)\,q_{n-1}-b^2nq_{n-2}=0$$

as stated.

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REFERENCES

1. Andrew M. Rockett. "Sums of the Inverses of Binomial Coefficients." *The Fibonacci Quarterly* **19.5** (1981):433-37.
2. Tor B. Staver. "On Summasjion av Potenser av Binomialkœffisientene." *Norsk Matematisk Tidsskrift* **29** (1947):97-103. (In Norwegian.)
3. T. S. Nanjudiah. "On a Formula of A. C. Dixon." *Proc. Amer. Math. Soc.* **9** (1958):308-11.
4. Edouard Lucas. *The Theory of Simply Periodic Numerical Functions.* Tr. from French by Sidney Kravits. Santa Clara, CA: The Fibonacci Association, 1969.

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