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On the Sum of Reciprocals of Numbers Satisfying a Recurrence Relation of Order s

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Abstract

We discuss the partial infinite sum $\sum_{k=n}^{\infty} u_k^{-s}$ for some positive integer n, where u_k satisfies a recurrence relation of order s, $u_n = au_{n-1} + u_{n-2} + \cdots + u_{n-s}$ $(n \geq s)$, with initial values $u_0 \geq 0$, $u_k \in \mathbb{N}$ $(0 \leq k \leq s-1)$, where a and $s(\geq 2)$ are positive integers. If $a = 1$, $s = 2$, and $u_0 = 0$, $u_1 = 1$, then $u_k = F_k$ is the k-th Fibonacci number. Our results include some extensions of Ohtsuka and Nakamura. We also consider continued fraction expansions that include such infinite sums.

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1 Introduction

The so-called *Fibonacci zeta function* is defined by

$$\zeta_F(s) = \sum_{n=1}^{\infty} \frac{1}{F_n^s}\,,$$

where F_n is the n-th Fibonacci number satisfying the recurrence formula

$$F_n = F_{n-1} + F_{n-2} \quad (n \ge 2), \quad F_0 = 0, \quad F_1 = 1\,.$$

Ohtsuka and Nakamura [8] studied the partial infinite sums of reciprocal Fibonacci numbers $\sum_{k\ge n}^{\infty} 1/F_k^s$. They gave an explicit formula for the integer part of $(\sum_{k\ge n}^{\infty} F_k^{-1})^{-1}$ and $(\sum_{k\ge n}^{\infty} F_k^{-2})^{-1}$. Holliday and Komatsu [2] generalized these results to the cases of G_n and H_n, satisfying $G_n = aG_{n-1} + G_{n-2}$ $(n \ge 2)$ with $G_0 = 0$ and $G_1 = 1$, and $H_n = H_{n-1} + H_{n-2}$ $(n \ge 2)$ with $H_0 = c$ and $H_1 = 1$, where $a \ge 1$ and $c \ge 0$ are integers. In this paper we shall not consider the integer part, but the nearest integer function of $(\sum_{k\ge n}^{\infty} u_k^{-1})^{-1}$, where $\{u_k\}_{k\ge 0}$ is a sequence of non-negative integers satisfying a linear recurrence formula of the type

$$u_n = au_{n-1} + u_{n-2} + \cdots + u_{n-s}\,,$$

where a and s (≥ 2) are positive integers. Here, $\|\cdot\|$ denotes the nearest integer[3], namely, $\|x\| = \lfloor x + 1/2 \rfloor$. Our main result is the following:

Theorem 1. *Let $\{u_n\}_{n\ge 0}$ be an integer sequence satisfying the recurrence formula*

$$u_n = au_{n-1} + u_{n-2} + \cdots + u_{n-s} \quad (n \ge s) \tag{1}$$

with initial conditions

$$u_0 \ge 0, \quad u_k \in \mathbb{N} \quad (0 \le k \le s-1)\,, \tag{2}$$

where a and s (≥ 2) are positive integers. Then there is a positive integer n_0 such that

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_k} \right)^{-1} \right\| = u_n - u_{n-1} \quad (n \ge n_0)\,. \tag{3}$$

If $a = 1$, $u_0 = 0$, $u_1 = u_2 = 1$, $u_3 = 2$, $\cdots$, $u_{s-1} = 2^{s-3}$, then the u_n's are generalized Fibonacci numbers (sometimes called "Fibonacci s-step numbers" [1]). If $s = 2$, then $u_k = F_k$ are Fibonacci numbers, while if $s = 3$, then $u_k = T_k$ are Tribonacci numbers.

We need the following two lemmas in order to prove this theorem.

Lemma 2. *Let $a, s \in \mathbb{N}$, $s \ge 2$ and let*

$$f(x) = x^s - ax^{s-1} - x^{s-2} - \cdots - x - 1.$$

Then

[3] In other contexts, this notation is sometimes used for the distance from the nearest integer.

(a) $f(x)$ has exactly one positive simple zero $\alpha \in \mathbb{R}$ with $a < \alpha < a+1$;

(b) the remaining $s-1$ zeros of $f(x)$ lie within the unit circle in the complex plane.

Proof. The case where $a = 1$ can be found in [7], so we assume from now on that $a \geq 2$.

By Descarte's rule of signs, we see that $f(x)$ has at most one positive real zero. Since $f(a) < 0$ and $f(a+1) > 0$, its unique positive real zero, say α, satisfies $(2 \leq)\ a < \alpha < a+1$. Since multiple roots are counted separately by Descarte's rule again, part (a) is proved. Observe from part (a) that

$$\text{for real } \ x > \alpha, \text{we have } \ f(x) > 0, \tag{4}$$

$$\text{while for real } \ 0 < x < \alpha, \text{we have } \ f(x) < 0. \tag{5}$$

Next, let

$$g(x) = (x-1)f(x) = x^{s+1} - (a+1)x^s + (a-1)x^{s-1} + 1.$$

Observe further that

$$\text{for real } \ x > \alpha, \text{we have } \ g(x) > 0, \tag{6}$$

$$\text{while for real } \ 1 < x < \alpha, \text{we have } \ g(x) < 0. \tag{7}$$

To prove part (b), we proceed by establishing several claims.

Claim 1. $f(x)$ has no complex zero z_1 with $|z_1| > \alpha$.

Proof of Claim 1. If $0 = f(z_1) = z_1^s - az_1^{s-1} - z_1^{s-2} - \cdots - z_1 - 1$, then

$$|z_1|^s \leq a\,|z_1|^{s-1} + |z_1|^{s-2} + \cdots + |z_1| + 1,$$

which implies that $f\left(|z_1|\right) \leq 0$, contradicting (4).

Claim 2. $f(x)$ has no complex zero z_2 with $1 < |z_2| < \alpha$.

Proof of Claim 2. If $f(z_2) = 0$, then $0 = g(z_2) = z_2^{s+1} - (a+1)z_2^s + (a-1)z_2^{s-1} + 1$ and so

$$(a+1)\,|z_2|^s \leq |z_2|^{s+1} + (a-1)\,|z_2|^{s-1} + 1,$$

i.e., $g\left(|z_2|\right) \geq 0$, contradicting (7).

Claim 3. $f(x)$ has no complex zero $z_3 \neq \alpha$, with either $|z_3| = \alpha$ or $|z_3| = 1$.

Proof of Claim 3. If $f(z_3) = 0$, then

$$0 = g(z_3) = z_3^{s+1} - (a+1)z_3^s + (a-1)z_3^{s-1} + 1 \tag{8}$$

so that

$$(a+1)\,|z_3|^s \leq |z_3|^{s+1} + (a-1)\,|z_3|^{s-1} + 1. \tag{9}$$

If $|z_3| = \alpha$ or $|z_3| = 1$, then $g(|z_3|) = 0$ and so (9) must be an equality. Then the two conditions z_3^{s+1} and $z_3^{s-1} \in \mathbb{R}$, or $z_3^{s+1} = -(a-1)z_3^{s-1}$ follow from two applications of the fact that

$$|z_1 + z_2| = |z_1| + |z_2| \quad \Longleftrightarrow \quad \frac{z_1}{z_2} \in \mathbb{R}_{\geq 0} \quad (\text{for } z_2 \neq 0)$$

and from

$$\left(z_1 + z_2 \in \mathbb{R} \wedge \frac{z_1}{z_2} \in \mathbb{R}\right) \quad \Longrightarrow \quad (z_1, z_2 \in \mathbb{R} \vee z_1 = -z_2)\,.$$

• If z_3^{s+1} and $z_3^{s-1} \in \mathbb{R}$, then (8) shows that $z_3^s \in \mathbb{R}$, which in turn forces $z_3 \in \mathbb{R}$. Thus, $z_3 = \pm\, \alpha$ or $z_3 = \pm\, 1$. The possibility $z_3 = \alpha$ is ruled out by the hypothesis, and the possibility $z_3 = 1$ is ruled out by (5). To rule out the remaining two possibilities of negative zeros, consider

$$g(-x) = \begin{cases} -x^{s+1} - (a+1)x^s - (a-1)x^{s-1} + 1, & \text{if } s \text{ is even;} \\ x^{s+1} + (a+1)x^s + (a-1)x^{s-1} + 1, & \text{if } s \text{ is odd.} \end{cases} \tag{10}$$

By Descarte's rule of signs applied to $g(-x)$, we deduce that $g(x)$ and so also $f(x)$, has at most one real negative zero if s is even and has no real negative zeros if s is odd. When s is even, since $f(0) = -1,\ f(-1) = a > 0$, should $f(x)$ have a real negative zero, such zero must lie in the interval $(-1, 0)$ and so can neither be $-\alpha$ nor -1.

• If $z_3^{s+1} = -(a-1)z_3^{s-1}$, then (8) gives $z_3^s = \frac{1}{a+1}$. Thus, either

$$2^s \leq a^s < |\alpha|^s = |z_3|^s = \frac{1}{a+1} \leq \frac{1}{3} \quad \text{or} \quad 1 = |z_3|^s = \frac{1}{a+1} \leq \frac{1}{3}.$$

Both possibilities are untenable and Claim 3 is proved.

Part (b) now follows from Claims 1–3.

□

We shall keep the notation of Lemma 2 throughout the rest of the paper.

Lemma 3. *Let $s \geq 2$ and let $\{u_n\}_{n\geq 0}$ be an integer sequence satisfying the recurrence formula (1) and the initial conditions (2). Then there are real numbers $c > 0$, $d > 1$, and $\alpha > a$ such that*

$$u_n = c\alpha^n + \mathcal{O}(d^{-n}) \qquad (n \to \infty)\,. \tag{11}$$

Proof. Let $\alpha_1 = \alpha,\ \alpha_2, \ldots, \alpha_t$ with $|\alpha_j| < 1$ $(2 \leq t \leq s)$ be the distinct roots of $f(x)$, and let r_j for $j = 2, 3, \ldots, t$ denote the multiplicity of the root α_j. Then the expansion formula (11) follows from the shape of u_n, which is given by

$$u_n = c\alpha^n + \sum_{j=2}^{t} P_j(n)\alpha_J^n\,,$$

where

$$P_j(x) \in \mathbb{R}[x], \qquad \deg P_j = r_J - 1, \qquad 1 + r_2 + r_3 + \cdots + r_t = s\,.$$

□

Proof of Theorem 1. Applying Lemma 3 and the expansion formula

$$\frac{1}{1 \pm \epsilon} = 1 \mp \epsilon + \mathcal{O}(\epsilon^2) = 1 + \mathcal{O}(\epsilon) \qquad (\epsilon \to 0)\,,$$

we have

$$\begin{aligned}\frac{1}{u_k} &= \frac{1}{c\alpha^k + \mathcal{O}(d^{-k})} = \frac{1}{c\alpha^k(1 + \mathcal{O}((\alpha d)^{-k}))}\\ &= \frac{1}{c\alpha^k}(1 + \mathcal{O}((\alpha d)^{-k})) = \frac{1}{c\alpha^k} + \mathcal{O}((\alpha^2 d)^{-k})\,,\end{aligned}$$

Since

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{u_k} &= \frac{1}{c}\sum_{k=n}^{\infty}\frac{1}{\alpha^k} + \mathcal{O}\left(\sum_{k=n}^{\infty}(\alpha^2 d)^{-k}\right)\\ &= \frac{\alpha}{c(\alpha-1)}\alpha^{-n} + \mathcal{O}((\alpha^2 d)^{-n})\,,\end{aligned}$$

we obtain

$$\begin{aligned}\left(\sum_{k=n}^{\infty}\frac{1}{u_k}\right)^{-1} &= \frac{\alpha-1}{\alpha}c\alpha^n + \mathcal{O}(d^{-n})\\ &= u_n - u_{n-1} + \mathcal{O}(d^{-n})\,.\end{aligned}$$

Theorem 1 follows by choosing $n \ge n_0$ sufficient large so that the modulus of the last error term becomes less than $1/2$. □

2 Related results

The following results are similarly obtained. Here, n_1, n_2, n_3, n_4 and n_5 are positive integers depending only on a.

Theorem 4.

$$\left\|\left(\sum_{k=n}^\infty \frac{1}{u_{2k}}\right)^{-1}\right\| = u_{2n} - u_{2n-2} \quad (n \geq n_1)\,. \tag{12}$$

$$\left\|\left(\sum_{k=n}^\infty \frac{1}{u_{2k-1}}\right)^{-1}\right\| = u_{2n-1} - u_{2n-3} \quad (n \geq n_2)\,. \tag{13}$$

$$\left\|\left(\sum_{k=n}^\infty \frac{(-1)^k}{u_k}\right)^{-1}\right\| = (-1)^n(u_n + u_{n-1}) \quad (n \geq n_3)\,. \tag{14}$$

$$\left\|\left(\sum_{k=n}^\infty \frac{(-1)^k}{u_{2k}}\right)^{-1}\right\| = (-1)^n(u_{2n} + u_{2n-2}) \quad (n \geq n_4)\,. \tag{15}$$

$$\left\|\left(\sum_{k=n}^\infty \frac{(-1)^k}{u_{2k-1}}\right)^{-1}\right\| = (-1)^n(u_{2n-1} + u_{2n-3}) \quad (n \geq n_5)\,. \tag{16}$$

Proof. We shall prove only (14). The other identities are proved similarly. By (11) we get

$$\begin{aligned}
\sum_{k=n}^\infty \frac{(-1)^k}{u_k} &= \sum_{k=n}^\infty \frac{(-1)^k}{c\alpha^k + \mathcal{O}(d^{-k})} \\
&= \sum_{k=n}^\infty \frac{(-1)^k}{c\alpha^k}\left(1 + \mathcal{O}((\alpha d)^{-k})\right) \\
&= \frac{\alpha}{c(-\alpha)^n(\alpha+1)} + \mathcal{O}\left((-\alpha^2 d)^{-n}\right)\,.
\end{aligned}$$

By taking its reciprocal, we have

$$\begin{aligned}
\left(\sum_{k=n}^\infty \frac{(-1)^k}{u_k}\right)^{-1} &= \frac{c(-\alpha)^n(\alpha+1)}{\alpha}\left(1 + \mathcal{O}((\alpha d)^{-n})\right) \\
&= (-1)^n(c\alpha^n + c\alpha^{n-1}) + \mathcal{O}((-d)^{-n}) \\
&= (-1)^n(u_n + u_{n-1}) + \mathcal{O}(d^{-n})\,.
\end{aligned}$$

The identity (14) follows by choosing $n \geq n_3$ sufficiently large so that the modulus of the last error term becomes less than 1/2. □

3 The sum of reciprocal Tribonacci numbers

The so-called Tribonacci numbers T_n ([6, Ch. 46], [9, sequence A000073], [3]) are defined by

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \quad (n \geq 3), \quad T_0 = 0, \quad T_1 = T_2 = 1\,.$$

By setting $a = 1$, $s = 3$ and $u_k = T_k$ ($k \geq 0$) in Theorem 1 and Theorem 4, we get some identities about the partial Tribonacci zeta functions. Numerical evidences imply that identities hold for smaller positive integers n, as indicated in the identities. The detailed explanations for small n can be seen in [5].

Corollary 5.

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{T_k} \right)^{-1} \right\| = T_n - T_{n-1} \quad (n \geq 1) . \tag{17}$$

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{T_{2k}} \right)^{-1} \right\| = T_{2n} - T_{2n-2} \quad (n \geq 1) . \tag{18}$$

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{T_{2k-1}} \right)^{-1} \right\| = T_{2n-1} - T_{2n-3} \quad (n \geq 2) . \tag{19}$$

$$\left\| \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{T_k} \right)^{-1} \right\| = (-1)^n (T_n + T_{n-1}) \quad (n \geq 2) . \tag{20}$$

$$\left\| \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{T_{2k}} \right)^{-1} \right\| = (-1)^n (T_{2n} + T_{2n-2}) \quad (n \geq 1) . \tag{21}$$

$$\left\| \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{T_{2k-1}} \right)^{-1} \right\| = (-1)^n (T_{2n-1} + T_{2n-3}) \quad (n \geq 2) . \tag{22}$$

4 Continued fraction expansion of generalized m-step zeta functions

The first author [4] studied several continued fraction expansions of some types of Fibonacci zeta functions $\zeta_F(s) := \sum_{n=1}^{\infty} F_n^{-s}$ and Lucas zeta functions in $\zeta_L(s) := \sum_{n=1}^{\infty} L_n^{-s}$, where L_n is the n-th Lucas number defined by

$$L_n = L_{n-1} + L_{n-2} \quad (n \geq 2) \quad L_0 = 2, \quad L_1 = 1.$$

A continued fraction expansion of the generalized m-step zeta functions defined by $\zeta_{u^{(m)}}(s) := \sum_{n=1}^{\infty} u_n^{-s}$, where

$$u_n = a u_{n-1} + u_{n-2} + \cdots + u_{n-m} \quad (n \geq m)$$

with initial positive integral values u_k $(0 \le k \le m-1)$, is given by

$$\zeta_{u^{(m)}}(s) = \cfrac{1}{u_1^s - \cfrac{u_1^{2s}}{u_1^s + u_2^s - \cfrac{u_2^{2s}}{u_2^s + u_3^s - \cfrac{u_3^{2s}}{u_3^s + u_4^s - \ddots - \cfrac{u_{n-1}^{2s}}{u_{n-1}^s + u_n^s - \cdots}}}}}\,.$$

Define A_n (respectively, B_n) as the numerator (respectively, denominator) of the n^{th} convergent of the continued fraction expansion given for $\zeta_{u^{(m)}}(s)$:

$$\frac{A_n}{B_n} = \cfrac{1}{u_1^s - \cfrac{u_1^{2s}}{u_1^s + u_2^s - \cfrac{u_2^{2s}}{u_2^s + u_3^s - \cfrac{u_3^{2s}}{u_3^s + u_4^s - \ddots - \cfrac{u_{n-1}^{2s}}{u_{n-1}^s + u_n^s}}}}}\,.$$

Hence $\{A_\nu\}_{\nu \ge 0}$ and $\{B_\nu\}_{\nu \ge 0}$ satisfy the following recurrence formulas.

$$\begin{array}{llll} A_\nu = (u_{\nu-1}^s + u_\nu^s)A_{\nu-1} - u_{\nu-1}^{2s}A_{\nu-2} & (\nu \ge 2), & A_0 = 0, & A_1 = 1; \\ B_\nu = (u_{\nu-1}^s + u_\nu^s)B_{\nu-1} - u_{\nu-1}^{2s}B_{\nu-2} & (\nu \ge 2), & B_0 = 1, & B_1 = u_1^s \end{array}$$

In fact, A_ν and B_ν can be expressed explicitly as follows.

Lemma 6. *For $n = 1, 2, \ldots$*

$$A_n = (u_1 u_2 \cdots u_n)^s \sum_{\nu=1}^{n} \frac{1}{u_\nu^s}, \qquad B_n = (u_1 u_2 \cdots u_n)^s\,.$$

Proof. By induction we have $B_n = (u_1 u_2 \cdots u_n)^s$. Thus,

$$A_n = B_n \sum_{\nu=1}^{n} \frac{1}{u_\nu^s} = (u_1 u_2 \cdots u_n)^s \sum_{\nu=1}^{n} \frac{1}{u_\nu^s}\,.$$

□

Theorem 1 provides us with interesting information about the nearest integer of the reciprocal of $\zeta_{u^{(m)}}(s) - A_n/B_n$.

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