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RECIPROCAL SUMS OF SECOND ORDER RECURRENT SEQUENCES

Hong Hu, Zhi-Wei Sun and Jian-Xin Liu

Department of Mathematics, Huaiyin Normal College,
Huaiyin 223001, Jiangsu Province, P. R. China

Department of Mathematics, Nanjing University,
Nanjing 210093, P. R. China. *E-mail*: zwsun@nju.edu.cn

The Fundamental Division, Nanjing Engineering College,
Nanjing 210013, P. R. China

1. Introduction

Let $\mathbb{Z}$ and $\mathbb{R}$ ($\mathbb{C}$) denote the ring of the integers and the field of real (complex) numbers respectively. For a field F we put $F^* = F \setminus \{0\}$. Fix $A \in \mathbb{C}$ and $B \in \mathbb{C}^*$, and let $\mathcal{L}(A,B)$ consist of all those second order recurrent sequences $\{w_n\}_{n\in\mathbb{Z}}$ of complex numbers satisfying the recursion:

$$w_{n+1} = Aw_n - Bw_{n-1} \text{ (i.e. } Bw_{n-1} = Aw_n - w_{n+1}) \quad \text{for } n = 0, \pm 1, \pm 2, \cdots. \tag{1}$$

For sequences in $\mathcal{L}(A,B)$ the corresponding characteristic equation is $x^2 - Ax + B = 0$, whose roots $(A \pm \sqrt{A^2-4B})/2$ are denoted by α and β. If $A \in \mathbb{R}^*$ and $\Delta = A^2 - 4B \geqslant 0$, then we let

$$\alpha = \frac{A - \mathrm{sg}(A)\sqrt{\Delta}}{2} \qquad \text{and} \qquad \beta = \frac{A + \mathrm{sg}(A)\sqrt{\Delta}}{2} \tag{2}$$

where $\mathrm{sg}(A) = 1$ if $A > 0$, and $\mathrm{sg}(A) = -1$ if $A < 0$. In the case $w_1 = \alpha w_0$, it is easy to see that $w_n = \alpha^n w_0$ for any integer n. If $A = 0$, then $w_{2n} = (-B)^n w_0$ and $w_{2n+1} = (-B)^n w_1$ for all $n \in \mathbb{Z}$. The Lucas sequences $\{u_n\}_{n\in\mathbb{Z}}$ and $\{v_n\}_{n\in\mathbb{Z}}$ in $\mathcal{L}(A,B)$ take special values at $n = 0, 1$, namely

$$u_0 = 0,\ u_1 = 1,\ v_0 = 2,\ v_1 = A. \tag{3}$$

It is well known that

$$(\alpha - \beta)u_n = \alpha^n - \beta^n \quad \text{and} \quad v_n = \alpha^n + \beta^n \quad \text{for } n \in \mathbb{Z}. \tag{4}$$

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If $A = 1$ and $B = -1$, then those $F_n = u_n$ and $L_n = v_n$ are called Fibonacci numbers and Lucas numbers respectively.

Let m be a positive integer. In 1974 I.J. Good [2] showed that

$$\sum_{n=0}^{m} \frac{1}{F_{2^n}} = 3 - \frac{F_{2^m-1}}{F_{2^m}}, \quad \text{i.e.} \quad \sum_{n=0}^{m-1} \frac{(-1)^{2^n}}{F_{2^{n+1}}} = -\frac{F_{2^m-1}}{F_{2^m}},$$

V.E. Hoggatt, Jr. and M. Bicknell [4] extended this by evaluating $\sum_{n=0}^{m} F_{k2^n}^{-1}$ where k is a positive integer. In 1977 W.E. Greig [3] was able to determine the sum $\sum_{n=0}^{m} u_{k2^n}^{-1}$ with $B = -1$; in 1995 R.S. Melham and A.G. Shannon [5] gave analogous results in the case $B = 1$. In 1990 R. André-Jeannin [1] calculated

$$\sum_{n=1}^{\infty} \frac{1}{u_{kn}u_{k(n+1)}} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{v_{kn}v_{k(n+1)}}$$

in the case $B = -1$ and $2 \nmid k$, using the Lambert series

$$L(x) = \sum_{n=1}^{\infty} \frac{x^n}{1 - x^n} \quad (|x| < 1);$$

in 1995 Melham and Shannon [5] computed the sums in the case $B = 1$, in terms of α and β.

In the present paper we obtain the following theorems which imply all of the above.

Theorem 1. *Let m be a positive integer, and f a function such that $f(n) \in \mathbb{Z}$ and $w_{f(n)} \neq 0$ for all $n = 0, 1, \cdots, m$. Then*

$$\sum_{n=0}^{m-1} \frac{B^{f(n)}u_{\Delta f(n)}}{w_{f(n)}w_{f(n+1)}} = \frac{B^{f(0)}u_{f(m)-f(0)}}{w_{f(0)}w_{f(m)}} \tag{5}$$

where $\Delta f(n) = f(n+1) - f(n)$. If $w_1 \neq \alpha w_0$ then

$$\sum_{n=0}^{m-1} \frac{(-1)^n}{w_{f(n)}} \left(\frac{2\alpha^{f(n)}}{w_1 - \alpha w_0} - \frac{B^{f(n)}u_{\Delta f(n)}}{w_{f(n+1)}} \right) = \frac{1}{w_1 - \alpha w_0} \left(\frac{\alpha^{f(0)}}{w_{f(0)}} - (-1)^m \frac{\alpha^{f(m)}}{w_{f(m)}} \right). \tag{6}$$

Theorem 2. *Suppose that $A, B \in \mathbb{R}^*$ and $\Delta = A^2 - 4B \geqslant 0$. Let $f : \{0, 1, 2, \cdots\} \to \{k \in \mathbb{Z} : w_k \neq 0\}$ be a function such that $\lim_{n\to+\infty} f(n) = +\infty$. If $w_1 \neq \alpha w_0$ then we have*

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{B^{f(n)}u_{\Delta f(n)}}{w_{f(n)}w_{f(n+1)}} &= \frac{\alpha^{f(0)}}{(w_1 - \alpha w_0)w_{f(0)}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{w_{f(n)}} \left(\frac{2\alpha^{f(n)}}{w_1 - \alpha w_0} - \frac{B^{f(n)}u_{\Delta f(n)}}{w_{f(n+1)}} \right). \end{aligned} \tag{7}$$

In the next section we will derive several results from these theorems. Theorems 1 and 2 are proved in Section 3.

2. Consequences of Theorems 1 and 2

Theorem 3. *Let k and l be integers such that $w_{kn+l} \neq 0$ for all $n = 0, 1, 2, \cdots$. Then*

$$u_k \sum_{n=0}^{m-1} \frac{B^{kn}}{w_{kn+l}w_{k(n+1)+l}} = \frac{u_{km}}{w_l w_{km+l}} \quad \text{for all } m = 1, 2, 3, \cdots. \tag{8}$$

If $A, B \in \mathbb{R}^*$, $A^2 \geqslant 4B$, $k > 0$ and $w_1 \neq \alpha w_0$, then

$$\sum_{n=0}^{\infty} \frac{u_k B^{kn+l}}{w_{kn+l}w_{k(n+1)+l}} = \frac{\alpha^l}{(w_1 - \alpha w_0)w_l} \tag{9}$$

and

$$\sum_{n=0}^{\infty} \left(2\frac{(-\alpha^k)^n}{w_{kn+l}} - (w_1 - \alpha w_0)u_k \beta^l \frac{(-B^k)^n}{w_{kn+l}w_{k(n+1)+l}} \right) = \frac{1}{w_l}. \tag{10}$$

Proof. Simply apply Theorems 1 and 2 with $f(n) = kn + l$.

Remark 1. When $B = 1$, $l = k$ and $\{w_n\} = \{u_n\}$ or $\{v_n\}$, Melham and Shannon [5] obtained (8) with the right hand side replaced by a complicated expression in terms of α and β.

Theorem 4. *Let $A, B \in \mathbb{R}^*$ and $\Delta = A^2 - 4B > 0$. Then for any positive integer k we have*

$$\sum_{n=1}^{\infty} \frac{(-B^k)^n}{u_{kn}u_{k(n+1)}} = \frac{\alpha^k}{u_k^2} + \mathrm{sg}(A)\frac{\sqrt{\Delta}}{u_k}\left(4L\left(\frac{\alpha^{4k}}{B^{2k}}\right) - 2L\left(\frac{\alpha^{2k}}{B^k}\right)\right) \tag{11}$$

and

$$\sum_{n=1}^{\infty} \frac{(-B^k)^n}{v_{kn}v_{k(n+1)}} = \frac{\mathrm{sg}(A)}{\sqrt{\Delta}}\left(\frac{\alpha^k}{u_{2k}} - \frac{2}{u_k}\left(4L\left(\frac{\alpha^{8k}}{B^{4k}}\right) - 4L\left(\frac{\alpha^{4k}}{B^{2k}}\right) + L\left(\frac{\alpha^{2k}}{B^k}\right)\right)\right). \tag{12}$$

Proof. Clearly $|\alpha| < |\beta|$ and $\beta - \alpha = \mathrm{sg}(A)\sqrt{\Delta}$. Thus $u_n = (\beta^n - \alpha^n)/(\beta - \alpha)$ and $v_n = \alpha^n + \beta^n$ are nonzero for all $n \in \mathbb{Z} \setminus \{0\}$. Obviously $u_1 - \alpha u_0 = 1$ and $v_1 - \alpha v_0 = A - 2\alpha = \beta - \alpha = \mathrm{sg}(A)\sqrt{\Delta}$. Applying Theorem 3 with $l = k$ and $\{w_n\}_{n\in\mathbb{Z}} = \{u_n\}_{n\in\mathbb{Z}}$ or $\{v_n\}_{n\in\mathbb{Z}}$, we then obtain that

$$\sum_{n=1}^{\infty} \left(u_k \frac{(-B^k)^n}{u_{kn}u_{k(n+1)}} - 2\frac{(-\alpha^k)^n}{u_{kn}} \right) = \frac{\alpha^k}{u_k}$$

and

$$\sum_{n=1}^{\infty} \left(u_k \frac{(-B^k)^n}{v_{kn}v_{k(n+1)}} - \frac{2}{\mathrm{sg}(A)\sqrt{\Delta}} \cdot \frac{(-\alpha^k)^n}{v_{kn}} \right) = \frac{\alpha^k/v_k}{\mathrm{sg}(A)\sqrt{\Delta}}.$$

Clearly

$$\sum_{n=1}^{\infty}\frac{(-\alpha^k)^n}{u_{kn}}=\sum_{n=1}^{\infty}(\beta-\alpha)\frac{(-\alpha^k)^n}{\beta^{kn}-\alpha^{kn}}=(\beta-\alpha)\sum_{n=1}^{\infty}\frac{(-1)^n(\alpha/\beta)^{kn}}{1-(\alpha/\beta)^{kn}}$$
$$=(\beta-\alpha)\bigg(2\sum_{\substack{n=1\\2\mid n}}^{\infty}\frac{(\alpha/\beta)^{kn}}{1-(\alpha/\beta)^{kn}}-\sum_{n=1}^{\infty}\frac{(\alpha/\beta)^{kn}}{1-(\alpha/\beta)^{kn}}\bigg)$$
$$=(\beta-\alpha)\bigg(2L\left(\frac{\alpha^{2k}}{\beta^{2k}}\right)-L\left(\frac{\alpha^k}{\beta^k}\right)\bigg)=\mathrm{sg}(A)\sqrt{\Delta}\bigg(2L\left(\frac{\alpha^{4k}}{B^{2k}}\right)-L\left(\frac{\alpha^{2k}}{B^k}\right)\bigg).$$

If $|x|<1$ then

$$\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{1+x^n}=2\sum_{n=1}^{\infty}\frac{x^{2n}}{1+x^{2n}}-\sum_{n=1}^{\infty}\frac{x^n}{1+x^n}$$
$$=2\sum_{n=1}^{\infty}\left(\frac{x^{2n}}{1-x^{2n}}-\frac{2x^{4n}}{1-x^{4n}}\right)-\sum_{n=1}^{\infty}\left(\frac{x^n}{1-x^n}-\frac{2x^{2n}}{1-x^{2n}}\right)$$
$$=2L(x^2)-4L(x^4)-L(x)+2L(x^2)=-4L(x^4)+4L(x^2)-L(x).$$

Thus

$$\sum_{n=1}^{\infty}\frac{(-\alpha^k)^n}{v_{kn}}=\sum_{n=1}^{\infty}\frac{(-\alpha^k)^n}{\alpha^{kn}+\beta^{kn}}=\sum_{n=1}^{\infty}(-1)^n\frac{(\alpha/\beta)^{kn}}{1+(\alpha/\beta)^{kn}}$$
$$=-4L\left(\frac{\alpha^{4k}}{\beta^{4k}}\right)+4L\left(\frac{\alpha^{2k}}{\beta^{2k}}\right)-L\left(\frac{\alpha^k}{\beta^k}\right)$$
$$=-4L\left(\frac{\alpha^{8k}}{B^{4k}}\right)+4L\left(\frac{\alpha^{4k}}{B^{2k}}\right)-L\left(\frac{\alpha^{2k}}{B^k}\right).$$

Combining the above and noting that $u_kv_k=u_{2k}$, we then obtain the desired (11) and (12).

Remark 2. If $|x|<1$ then

$$L(-x)=\sum_{n=1}^{\infty}\frac{x^{2n}}{1-x^{2n}}-\sum_{n=1}^{\infty}\frac{x^n}{1+x^n}+\sum_{n=1}^{\infty}\frac{x^{2n}}{1+x^{2n}}$$
$$=L(x^2)-(L(x)-2L(x^2))+(L(x^2)-2L(x^4))=-2L(x^4)+4L(x^2)-L(x).$$

Thus Theorem 2 of Andŕe-Jeannin [1] is essentially our (11) and (12) in the special case $B=-1$ and $2\nmid k$.

Theorem 5. *Let $k,l,m\in\mathbb{Z}$ and $l,m>0$. If $w_{\binom{k+n}{l}}\neq 0$ for all $n=0,1,\cdots,m$, then*

$$\sum_{n=0}^{m-1}\frac{B^{\binom{k+n}{l}}u_{\binom{k+n}{l-1}}}{w_{\binom{k+n}{l}}w_{\binom{k+n+1}{l}}}=\frac{B^{\binom{k}{l}}u_{\binom{k+m}{l}-\binom{k}{l}}}{w_{\binom{k}{l}}w_{\binom{k+m}{l}}}. \qquad (13)$$

Proof. Let $f(n) = \binom{k+n}{l}$ for $n \in \mathbb{Z}$. It is well known that

$$\Delta f(n) = \binom{k+n+1}{l} - \binom{k+n}{l} = \binom{k+n}{l-1}.$$

So Theorem 5 follows from Theorem 1.

Remark 3. In the case $k = 0$ and $l = 2$, (13) says that

$$\sum_{n=0}^{m-1} \frac{u_n B^{n(n-1)/2}}{w_{n(n-1)/2} w_{n(n+1)/2}} = \frac{u_{m(m-1)/2}}{w_0 w_{m(m-1)/2}}. \tag{14}$$

Theorem 6. *Let a, k be integers, and m a positive integer. Suppose that $w_{ka^n} \neq 0$ for each $n = 0, 1, \cdots, m-1$. Then*

$$\sum_{n=0}^{m-1} \frac{B^{ka^n} u_{k(a-1)a^n}}{w_{ka^n} w_{ka^{n+1}}} = \frac{B^k u_{k(a^m-1)}}{w_k w_{ka^m}}. \tag{15}$$

Proof. Just put $f(n) = ka^n$ in Theorem 1.

Remark 4. In the case $a = 2$ and $\{w_n\} = \{u_n\}$, (15) becomes

$$\sum_{n=0}^{m-1} \frac{B^{k2^n}}{u_{k2^{n+1}}} = \frac{B^k u_{k(2^m-1)}}{u_k u_{k2^m}}. \tag{16}$$

This was obtained by Melham and Shannon [5] in the case $B = 1$ and $k > 0$. In the case $a = 3$ and $\{w_n\} = \{v_n\}$, (15) turns out to be

$$\sum_{n=0}^{m-1} \frac{B^{k3^n} u_{k3^n}}{v_{k3^{n+1}}} = \frac{B^k u_{k(3^m-1)}}{v_k v_{k3^m}} \tag{17}$$

since $u_{2h} = u_h v_h$ for $h \in \mathbb{Z}$.

Theorem 7. *Let k be an integer and m a positive integers. If $w_{k(2^n-1)} \neq 0$ for each $n = 0, 1, \cdots, m-1$, then*

$$\sum_{n=0}^{m-1} \frac{B^{k(2^n-1)} u_{k2^n}}{w_{k(2^n-1)} w_{k(2^{n+1}-1)}} = \frac{u_{k(2^m-1)}}{w_0 w_{k(2^m-1)}}. \tag{18}$$

Proof. Just apply Theorem 1 with $f(n) = k(2^n - 1)$.

3. Proofs of Theorems 1 and 2

Lemma 1. *For $k, l, m \in \mathbb{Z}$ we have*

$$w_k u_{l+m} - w_{k+m} u_l = B^l w_{k-l} u_m \tag{19}$$

and

$$w_k \alpha^l - w_l \alpha^k = (w_1 - \alpha w_0) B^l u_{k-l}. \tag{20}$$

Proof. i) Fix $k, l \in \mathbb{Z}$. Observe that

$$\begin{aligned}\begin{pmatrix} w_{k+1} & w_k \\ u_{l+1} & u_l \end{pmatrix} &= \begin{pmatrix} w_k & w_{k-1} \\ u_l & u_{l-1} \end{pmatrix} \begin{pmatrix} A & 1 \\ -B & 0 \end{pmatrix} \\ &= \begin{pmatrix} w_{k-1} & w_{k-2} \\ u_{l-1} & u_{l-2} \end{pmatrix} \begin{pmatrix} A & 1 \\ -B & 0 \end{pmatrix}^2 = \cdots \\ &= \begin{pmatrix} w_{k-l+1} & w_{k-l} \\ u_1 & u_0 \end{pmatrix} \begin{pmatrix} A & 1 \\ -B & 0 \end{pmatrix}^l.\end{aligned}$$

Taking the determinants we then get that

$$\begin{vmatrix} w_{k+1} & w_k \\ u_{l+1} & u_l \end{vmatrix} = \begin{vmatrix} w_{k-l+1} & w_{k-l} \\ 1 & 0 \end{vmatrix} \times \begin{vmatrix} A & 1 \\ -B & 0 \end{vmatrix}^l,$$

i.e.,

$$w_k u_{l+1} - w_{k+1} u_l = B^l w_{k-l}.$$

Thus (19) holds for $m = 0, 1$.

Each side of (19) can be viewed as a sequence in $\mathcal{L}(A, B)$ with respect to the index m. By induction (19) is valid for every $m = 0, 1, 2, \cdots$; also (19) holds for each $m = -1, -2, -3, \cdots$. Therefore (19) holds for any $m \in \mathbb{Z}$.

ii) By induction on l we find that $w_{l+1} - \alpha w_l = (w_1 - \alpha w_0)\beta^l$. Clearly both sides of (20) lie in $\mathcal{L}(A, B)$ with respect to the index k. Note that if $k = l$ then both sides of (20) are zero. As

$$(w_1 - \alpha w_0) B^l = (w_1 - \alpha w_0)\beta^l \alpha^l = (w_{l+1} - \alpha w_l)\alpha^l = \alpha^l w_{l+1} - \alpha^{l+1} w_l$$

(20) also holds for $k = l + 1$. Therefore (20) is always valid. We are done.

`Proof of Theorem 1`. Let $d \in \mathbb{Z}$. In view of Lemma 1, for $n = 0, 1, \cdots, m - 1$ we have

$$\begin{aligned}\frac{u_{d+f(n+1)}}{w_{f(n+1)}} - \frac{u_{d+f(n)}}{w_{f(n)}} &= \frac{u_{d+f(n+1)} w_{f(n)} - u_{d+f(n)} w_{f(n+1)}}{w_{f(n)} w_{f(n+1)}} \\ &= \frac{w_{f(n)} u_{d+f(n)+\Delta f(n)} - w_{f(n)+\Delta f(n)} u_{d+f(n)}}{w_{f(n)} w_{f(n+1)}} \\ &= \frac{B^{d+f(n)} w_{-d} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}}.\end{aligned}$$

It follows that

$$\sum_{n=0}^{m-1} \frac{B^{d+f(n)} w_{-d} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}} = \sum_{n=0}^{m-1} \left(\frac{u_{d+f(n+1)}}{w_{f(n+1)}} - \frac{u_{d+f(n)}}{w_{f(n)}} \right) = \frac{u_{d+f(m)}}{w_{f(m)}} - \frac{u_{d+f(0)}}{w_{f(0)}}$$

and that

$$\sum_{n=0}^{m-1} (-1)^{n+1} \frac{B^{d+f(n)} w_{-d} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}} = \sum_{n=0}^{m-1} \left((-1)^{n+1} \frac{u_{d+f(n+1)}}{w_{f(n+1)}} + (-1)^n \frac{u_{d+f(n)}}{w_{f(n)}} \right)$$

$$= 2 \sum_{n=0}^{m-1} (-1)^n \frac{u_{d+f(n)}}{w_{f(n)}} + (-1)^m \frac{u_{d+f(m)}}{w_{f(m)}} - (-1)^0 \frac{u_{d+f(0)}}{w_{f(0)}}.$$

Putting $d = -f(0)$ we then obtain (5) and that

$$\sum_{n=0}^{m-1} (-1)^{n+1} w_{f(0)} \frac{B^{f(n)} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}}$$

$$= 2 \sum_{n=0}^{m-1} (-1)^n \frac{B^{f(0)} u_{f(n)-f(0)}}{w_{f(n)}} + (-1)^m \frac{B^{f(0)} u_{f(m)-f(0)}}{w_{f(m)}}.$$

By Lemma 1, for each $n = 0, 1, \cdots, m$,

$$\alpha^{f(0)} w_{f(n)} - \alpha^{f(n)} w_{f(0)} = (w_1 - \alpha w_0) B^{f(0)} u_{f(n)-f(0)},$$

i.e.,

$$-\frac{B^{f(0)} u_{f(n)-f(0)}}{w_{f(n)}} = \frac{\alpha^{f(n)} w_{f(0)}}{(w_1 - \alpha w_0) w_{f(n)}} - \frac{\alpha^{f(0)}}{w_1 - \alpha w_0}.$$

Thus

$$w_{f(0)} \sum_{n=0}^{m-1} (-1)^n \frac{B^{f(n)} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}} = 2 \sum_{n=0}^{m-1} (-1)^n \left(\frac{w_{f(0)} \alpha^{f(n)}}{(w_1 - \alpha w_0) w_{f(n)}} - \frac{\alpha^{f(0)}}{w_1 - \alpha w_0} \right)$$

$$+ (-1)^m \left(\frac{w_{f(0)} \alpha^{f(m)}}{(w_1 - \alpha w_0) w_{f(m)}} - \frac{\alpha^{f(0)}}{w_1 - \alpha w_0} \right)$$

and hence

$$\sum_{n=0}^{m-1} \frac{(-1)^n}{w_{f(n)}} \left(\frac{2\alpha^{f(n)}}{w_1 - \alpha w_0} - \frac{B^{f(n)} u_{\Delta f(n)}}{w_{f(n+1)}} \right)$$

$$= \frac{2}{w_1 - \alpha w_0} \sum_{n=0}^{m-1} (-1)^n \frac{\alpha^{f(0)}}{w_{f(0)}} + \frac{(-1)^m}{w_1 - \alpha w_0} \left(\frac{\alpha^{f(0)}}{w_{f(0)}} - \frac{\alpha^{f(m)}}{w_{f(m)}} \right)$$

$$= \frac{1}{w_1 - \alpha w_0} \left(\frac{\alpha^{f(0)}}{w_{f(0)}} - (-1)^m \frac{\alpha^{f(m)}}{w_{f(m)}} \right).$$

This proves (6).

Lemma 2. *Let $A, B \in \mathbb{R}^*$ and $\Delta = A^2 - 4B \geqslant 0$. Then*

$$\lim_{n\to+\infty} \frac{\alpha^n}{u_n} = 0. \tag{21}$$

and

$$\lim_{n\to+\infty} \frac{w_n}{u_{m+n}} = \frac{w_1 - \alpha w_0}{\beta^m} \quad \text{for any } m \in \mathbb{Z}. \tag{22}$$

Proof. When $\Delta = 0$ (i.e. $\alpha = \beta$), by induction $u_n = n(A/2)^{n-1}$ for all $n \in \mathbb{Z}$, thus $u_n \neq 0$ for $n = \pm1, \pm2, \pm3, \cdots$,

$$\lim_{n\to+\infty} \frac{\alpha^n}{u_n} = \lim_{n\to+\infty} \frac{(A/2)^n}{n(A/2)^{n-1}} = 0$$

and

$$\lim_{n\to+\infty} \frac{u_{m+n}}{u_n} = \lim_{n\to+\infty} \frac{(m+n)(A/2)^{m+n-1}}{n(A/2)^{n-1}} = \left(\frac{A}{2}\right)^m = \beta^m.$$

In the case $\Delta > 0$, $|\alpha| < |\beta|$ and hence $u_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ is zero if and only if $n = 0$. Therefore

$$\lim_{n\to+\infty} \frac{\alpha^n}{u_n} = (\alpha - \beta) \lim_{n\to+\infty} \frac{1}{1 - (\beta/\alpha)^n} = 0$$

Also,

$$\lim_{n\to+\infty} \left(\frac{u_{n+1}}{u_n} - \beta\right) = \lim_{n\to+\infty} \frac{\alpha^{n+1} - \beta^{n+1} - \beta(\alpha^n - \beta^n)}{\alpha^n - \beta^n} = \lim_{n\to+\infty} \frac{(\alpha - \beta)}{1 - (\beta/\alpha)^n} = 0,$$

If $m \in \{0, 1, 2, \cdots\}$, then

$$\lim_{n\to+\infty} \frac{u_{m+n}}{u_n} = \lim_{n\to+\infty} \prod_{0\leqslant k<m} \frac{u_{k+n+1}}{u_{k+n}} = \beta^m$$

and

$$\lim_{n\to+\infty} \frac{u_{n-m}}{u_n} = \lim_{n\to+\infty} \frac{u_n}{u_{m+n}} = \beta^{-m}.$$

In view of the above, (21) always holds and $\lim_{n\to+\infty} u_{m+n}/u_n = \beta^m$ for all $m \in \mathbb{Z}$.

By Lemma 1, $w_1 u_n - w_n u_1 = B w_0 u_{n-1}$ for $n \in \mathbb{Z}$. Thus

$$\lim_{n\to+\infty} \frac{w_n}{u_n} = w_1 - \frac{B w_0}{\lim_{n\to+\infty} u_n/u_{n-1}} = w_1 - \frac{B w_0}{\beta} = w_1 - \alpha w_0$$

and hence (22) is valid.

`Proof of Theorem 2.` Assume that $w_1 \neq \alpha w_0$. In view of Lemma 2,

$$\lim_{m\to+\infty} \frac{B^{f(0)} u_{f(m)-f(0)}}{w_{f(m)}} = B^{f(0)} \frac{\beta^{-f(0)}}{w_1 - \alpha w_0} = \frac{\alpha^{f(0)}}{w_1 - \alpha w_0}$$

and

$$\lim_{m\to+\infty} \frac{\alpha^m}{w_m} = \lim_{m\to+\infty} \frac{\alpha^m}{u_m} \times \lim_{m\to+\infty} \frac{u_m}{w_m} = 0.$$

Applying Theorem 1 we immediately get (7).

Remark 5. On the condition of Theorem 2, if $w_1 = \alpha w_0$ then by checking the proof of Theorem 2 we find that

$$\sum_{n=0}^{\infty} \frac{B^{f(n)} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}} = \infty. \qquad (23)$$

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