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A SYMMETRY PROPERTY OF ALTERNATING SUMS OF PRODUCTS OF RECIPROCALS

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Consider the homogeneous linear recurrence relation

$$G_{n+2} = aG_{n+1} + G_n \quad (n = 0, \pm 1, \pm 2, \ldots) \tag{1}$$

where a is a nonzero real or complex constant. The equation is especially familiar in relation to the theory of simple continued fractions, and in relation to the theory of numbers when a is a natural integer. (For the case where a is a Gaussian integer, see Good [4] and [5].) A solution (G_n) can be regarded as a vector of a countably infinite number of components or elements, and which is completely determined "in both directions" by any two consecutive components. The general solution is a linear combination of any pair of linearly independent solutions. Two solutions are linearly independent under the nonvanishing of the 2-by-2 determinant consisting of two consecutive elements of one solution and the corresponding two elements of the other solution. Perhaps the simplest pair of independent solutions is given by

$$F_n = F_n(\xi) = \frac{\xi^n - \eta^n}{\xi - \eta}, \quad L_n = L_n(\xi) = \xi^n + \eta^n \quad (n = 0, \pm 1, \pm 2, \ldots) \tag{2}$$

where

$$\xi = \frac{a + \sqrt{a^2+4}}{2}, \quad \eta = \frac{a - \sqrt{a^2+4}}{2} \quad (a \neq 2i). \tag{3}$$

Note that $|\xi| > |\eta|$ when a is real and positive; also that $\xi = \eta = i$ if $a = 2i$, and then $F_n = F_n(i)$ must be defined as ni^{n-1} while $L_n(i) = 2i^n$. The numbers ξ and η are the roots of the quadratic equation

$$x^2 - ax - 1 = 0, \tag{4}$$

and, of course,

$$\xi + \eta = a, \quad \xi\eta = -1. \tag{5}$$

In particular, when $a = 1$, in which case ξ is the golden ratio, F_n and L_n reduce to the Fibonacci and Lucas numbers. We write the general solution of (1) as

$$G_n = G_n(\xi) = \lambda F_n(\xi) + \mu L_n(\xi) \tag{6}$$

where λ and μ are not necessarily real.

We shall prove the following symmetry property:

Theorem: We have

$$F_k \sum_{n=1}^{m} \frac{(-1)^n}{G_n G_{n+k}} = F_m \sum_{n=1}^{k} \frac{(-1)^n}{G_n G_{n+m}}, \tag{7}$$

where k and m are nonnegative integers, and where we assume further that all the numbers $G_1, G_2, \ldots, G_{m+k}$ are nonzero.

Comment (i): It follows from equations (2) and (3) that the nonzero condition is certainly true when a is real and (G_n) is either (F_n) or (L_n), that is, when $\lambda = 1$ and $\mu = 0$ or when $\lambda = 0$ and $\mu = 1$.

Comment (ii): The theorem is presumably new even when (G_n) reduces to the ordinary Fibonacci or Lucas sequence, that is, when $a = 1$ and $\lambda = 1$, $\mu = 0$ or $\lambda = 0$, $\mu = 1$.

Comment (iii): If empty sums are regarded as vanishing, the theorem is true but uninformative when k or m is zero. It is also uninformative when $k = m$.

Comment (iv): Even in the simple case $F_n = ni^{n-1}$, $L_n = 2i^n$, the identity (7) is not entirely obvious when G_n is defined by (6).

Corollary: When $|\xi| > |\eta|$ we have

$$F_k \sum_{n=1}^{\infty} \frac{(-1)^n}{G_n G_{n+k}} = \frac{1}{\lambda + \mu\sqrt{(a^2+4)}} \sum_{n=1}^{k} \frac{\eta^n}{G_n}. \tag{8}$$

Comment (v): In the very special case $a = 1$, $k = 1$, $\mu = 0$, (8) reduces to formula (102) of Vajda [8]; and when $a = 1$, $k = 2$, $\mu = 0$, the evaluation of the left side of (8) was proposed as a problem by Clark [2]. The right side of (8) solves a much more general problem.

Proof of the Theorem: Without loss of generality, we assume $m \geq k$. The proof depends on a double induction, beginning with an induction with respect to m. We first note that the result is obvious when $m = k$, so we can proceed at once to the body of the induction. For this we need to show that

$$\sum_{n=1}^{k} \frac{(-1)^n}{G_n}\left(\frac{F_{m+1}}{G_{m+n+1}} - \frac{F_m}{G_{m+n}}\right) = \frac{(-1)^{m+1}F_k}{G_{m+1}G_{m+k+1}}. \tag{9}$$

Now, by means of some straightforward algebra it can be shown from (2), and generalizing the case $h = 1$ of formulas (19b) and (20a) of Vajda [8], that

$$F_{m+1}F_{m+n} - F_m F_{m+n+1} = (-1)^m F_n \tag{10}$$

and that

$$F_{m+1}L_{m+n} - F_m L_{m+n+1} = (-1)^m L_n \tag{11}$$

and hence that

$$F_{m+1}G_{m+n} - F_m G_{m+n+1} = (-1)^m G_n. \tag{12}$$

Therefore, (9) is equivalent to the identity

$$\sum_{n=1}^{k} \frac{(-1)^n}{G_{m+n}G_{m+n+1}} = \frac{F_k}{G_{m+1}G_{m+k+1}}. \tag{13}$$

To prove this identity, we perform an induction, this time with respect to k, noting first that it is trivially true when $k = 1$. So we now want to prove that

$$\frac{(-1)^{k+1}}{G_{m+k+1}G_{m+k+2}} = \frac{-F_{k+1}}{G_{m+1}G_{m+k+2}} + \frac{F_k}{G_{m+1}G_{m+k+1}}, \tag{14}$$

that is, we want

$$F_{k+1}G_{m+k+1} - F_k G_{m+k+2} = (-1)^k G_{m+1}. \tag{15}$$

But this identity is equivalent to 12) with a change of notation in the subscripts. Hence, in turn, we have proved (14), (13), (9), and (7), the statement of the theorem.

We could reverse the steps of the argument to prove each statement in turn, but the order used here shows the motivation at each step and also shows the way that the *proof* was discovered. It is more difficult to describe, or even to recall, how the theorem itself was discovered except that naturally it depended in part on guesswork and on numerical experimentation. (Many nonmathematicians don't know that pure mathematics is an experimental science.) For an alternative proof, see the Appendix.

Corresponding Trigonometrical Identities

Corresponding to many identities involving ordinary Fibonacci and Lucas numbers, there are "parental" (more general) identities obtained by replacing the golden ratio, and minus its reciprocal, by ξ and by $\eta = -1/\xi$, respectively. (See Lucas [7] and [3].) Our theorem and corollary have exemplified this procedure. We can then come down to "siblingal" formulas by giving ξ special values. As mentioned earlier, the results are number theoretic when $\xi + \eta$ is a natural or Gaussian integer. But if we let $\xi = ie^{ix}$, $\eta = ie^{-ix}$, where x is real, we obtain trigonometrical identities (Lucas [7]), for in this case we have

$$F_n(ie^{ix}) = i^{n-1} \sin nx / \sin x \tag{16}$$

when x is not a multiple of π, and

$$L_n(ie^{ix}) = 2i^n \cos nx. \tag{17}$$

The trigonometrical "siblings," so to speak, of the "Fibonacci" and "Lucas" cases of (7) are

$$\sin kx \sum_{n=1}^{m} \operatorname{cosec} nx \operatorname{cosec}(n+k)x = \sin mx \sum_{n=1}^{k} \operatorname{cosec} nx \operatorname{cosec}(n+m)x \tag{18}$$

and

$$\sin kx \sum_{n=1}^{m} \sec nx \sec(n+k)x = \sin mx \sum_{n=1}^{k} \sec nx \sec(n+m)x \tag{19}$$

where k and m are positive integers and k, m, and x are such that no infinities occur. No infinite terms will occur if x is not a rational multiple of π but the series on the left and the sequence on the right won't converge when $m \to \infty$ because arbitrarily large terms will occur. (The summations, for finite k and m, can be numerically highly ill-conditioned.)

"Parents" and trigonometrical "siblings" can be written down corresponding to the vast majority of the identities on pages 176-183 of Vajda [8] where $\sqrt{5}$ is to be generalized to $\xi - \eta$. Some of these trigonometrical identities are familiar. Conversely, parents and Fibonaccian siblings can be obtained for many of the trigonometrical identities in, say, Hobson [6]. To carry out this program in detail would be straightforward but would occupy a lot of space.

Again, trigonometrical identities can be derived from identities given by Bruckman and Good [1], in addition to the Fibonaccian identities given there.

APPENDIX

L. A. G. Dresel, on trying out the reverse argument, found the following more direct way of proving the identity (7).

On putting $m = t-1$ in (12) and dividing by $G_n G_{n+t-1} G_{n+t}$, we have

$$\frac{F_t}{G_n G_{n+t}} - \frac{F_{t-1}}{G_n G_{n+t-1}} = \frac{(-1)^{t-1}}{G_{n+t} G_{n+t-1}}. \tag{A.1}$$

Summing for $t = 1$ to k, we find that almost all of the terms on the left cancel in pairs, and since $F_0 = 0$ we have

$$\frac{F_k}{G_n G_{n+k}} = \sum_{t=1}^{k} \frac{(-1)^{t-1}}{G_{n+t} G_{n+t-1}}. \tag{A.2}$$

[This is the same as (13), with a change of notation in the subscripts, but is now *proved*.]

Multiplying by $(-1)^n$ and summing for $n = 1$ to m gives

$$F_k \sum_{n=1}^{m} \frac{(-1)^n}{G_n G_{n+k}} = \sum_{n=1}^{m} \sum_{t=1}^{k} \frac{(-1)^{n+t-1}}{G_{n+t} G_{n+t-1}}. \tag{A.3}$$

Similarly, interchanging the roles of k and m, we have

$$F_m \sum_{n=1}^{k} \frac{(-1)^n}{G_n G_{n+m}} = \sum_{n=1}^{k} \sum_{t=1}^{m} \frac{(-1)^{n+t-1}}{G_{n+t} G_{n+t-1}}. \tag{A.4}$$

But the double summations on the right of (A.3) and (A.4) are equal, as the summand is symmetrical in n and t and the order of summation is immaterial. Hence the left sides are equal, which proves the theorem.

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