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Inverse series relations, formal power series and Blodgett-Gessel's type binomial identities

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Abstract

A pair of simple bivariate inverse series relations are used by embedding machinery to produce several double summation formulae on shifted factorials (or binomial coefficients), including the evaluation due to Blodgett-Gessel [2]. Their q-analogues are established in the second section. Some generalized convolutions are presented through formal power series manipulation.

1. Bivariate inverse series relations

For any given bivariate function $f(x, y)$, define its difference by

$$f_\Delta(x, y) = f(x, y) - f(x-1, y) - f(x, y-1), \tag{1.1a}$$

where it is supposed that $f(x, y) = 0$ if x or/and y is less than zero. Then it is trivial to verify.

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Lemma

Bivariate inverse series relations

$$f_\Delta(m,n)=\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m+n-i-j-2}{m-i,\ n-j}f(i,j), \tag{1.1b}$$

$$f(m,n)=\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m+n-i-j}{m-i,\ n-j}f_\Delta(i,j); \tag{1.1c}$$

where the multinomial coefficient is given by

$$\binom{x}{m,\ n} = \binom{x}{m+n}\binom{m+n}{m,\ n}.$$

Actually, Eq. (1.1b) is a restatement of difference (1.1a). But Eq. (1.1c) would produce innumerable summation formulae for various specified bivariate function $f(x,y)$.

Denote the rising-factorial by

$$(x)_n=x(x+1)\cdots(x+n-1)\ \ (n>0),\ \ (x)_0=1.$$

Some examples may be displayed here.

EXAMPLE 1.1: For

$$A(m,n)=\frac{(1+2x)_{\lfloor\frac{m+n}{2}\rfloor}\,(\frac{1}{2}+x)_{\lceil\frac{m+n}{2}\rceil}}{\lfloor\frac{m}{2}\rfloor!\,\lfloor\frac{n}{2}\rfloor!\,(\frac{1}{2}+x)_{\lceil\frac{m}{2}\rceil}\,(\frac{1}{2}+x)_{\lceil\frac{n}{2}\rceil}}, \tag{1.2a}$$

its difference is given by

$$A_\Delta(m,n)=\begin{cases}\frac{4x}{4x+m+n}\,A(m,n), & m,n-\text{even};\\ 0, & \text{otherwise}.\end{cases} \tag{1.2b}$$

The corresponding summation formula

$$\sum_{i\geq0}\sum_{j\geq0}\binom{m+n-2i-2j}{m-2i,\ n-2j}\frac{(2x)_{i+j}\,(\frac{1}{2}+x)_{i+j}}{i!\,j!\,(\frac{1}{2}+x)_i\,(\frac{1}{2}+x)_j} = \frac{(1+2x)_{\lfloor\frac{m+n}{2}\rfloor}\,(\frac{1}{2}+x)_{\lceil\frac{m+n}{2}\rceil}}{\lfloor\frac{m}{2}\rfloor!\,\lfloor\frac{n}{2}\rfloor!\,(\frac{1}{2}+x)_{\lceil\frac{m}{2}\rceil}\,(\frac{1}{2}+x)_{\lceil\frac{n}{2}\rceil}}, \tag{1.2c}$$

is Gessel's extension of Blodgett's identity [2].

EXAMPLE 1.2: For

$$B(m,n)=\frac{(1+x)_{m+n}}{(1+y)_m\,(1+z)_n}, \tag{1.3a}$$

its difference is given by

$$B_\Delta(m,n)=\frac{x-y\cdot\tau(m)-z\cdot\tau(n)}{x+m+n}\;B(m,n); \tag{1.3b}$$

where the sign function is defined by

$$\tau(x)=\begin{cases}+1, & x>0\\ -1, & x<0\\ \;\;0, & x=0.\end{cases}$$

Then the corresponding summation formula is

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m+n-i-j}{m-i,\;n-j}\frac{(1+x)_{i+j}}{(1+y)_i\,(1+z)_j}\,\frac{x-y\cdot\tau(i)-z\cdot\tau(j)}{x+i+j} \tag{1.3c}$$
$$=\frac{(1+x)_{m+n}}{(1+y)_m\,(1+z)_n}.$$

EXAMPLE 1.3: For

$$C(m,n)=\frac{(1+x)_{m+n}\,(1+y)_{m+n}}{(1+x)_m\,(1+y)_m\,(1+x)_n\,(1+y)_n}, \tag{1.4a}$$

its difference is given by

$$C_\Delta(m,n)=\begin{cases}\frac{2mn-xy}{(x+m+n)\,(y+m+n)}\;C(m,n), & m>0,\; n>0\\ 0, & m \quad\text{or}\;\; n=0\\ 1, & m=n=0.\end{cases} \tag{1.4b}$$

Then the corresponding summation formula is

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\binom{m+n-i-j}{m-i,\;n-j}\frac{(x)_{i+j}\,(y)_{i+j}}{(1+x)_i\,(1+x)_j\,(1+y)_i\,(1+y)_j}\;\frac{2ij-xy}{xy} \tag{1.4c}$$
$$=\frac{(1+x)_{m+n}\,(1+y)_{m+n}}{(1+x)_m\,(1+y)_m\,(1+x)_n\,(1+y)_n}\;-\;\binom{m+n}{m,\;n}.$$

2. q-analogues

Recall the notation:

- q-factorial

$$(x;q)_n = \prod_{k=1}^{n}(1-xq^{k-1}) \qquad (n>0), \qquad (x;q)_0 = 1.$$

- Gaussian binomial coefficient

$$\begin{bmatrix} x \\ n \end{bmatrix} = \frac{(q^{x-n+1};q)_n}{(q;q)_n}.$$

- Gaussian multinomial coefficients

$$\begin{bmatrix} x \\ m,\ n \end{bmatrix} = \begin{bmatrix} x \\ m+n \end{bmatrix} \begin{bmatrix} m+n \\ m,\ n \end{bmatrix}.$$

Then one possible q-analogue for the lemma may be formulated as follows: For any given bivariate function $f(x,y)$, define its q-difference by

$$f_\delta(x,y) = f(x,y) - f(x-1,y) - q^x f(x,y-1), \tag{2.1a}$$

where it is supposed either that $f(x,y)=0$ if x or/and y is less than zero. It is routine to check the following reciprocity.

Proposition

Bivariate inverse series relations

$$f_\delta(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n} \begin{bmatrix} m+n-i-j-2 \\ m-i,\ n-j \end{bmatrix} q^{m+n-i-j+m(n-j)}\ f(i,j), \tag{2.1b}$$

$$f(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n} \begin{bmatrix} m+n-i-j \\ m-i,\ n-j \end{bmatrix} q^{i(n-j)}\ f_\delta(i,j). \tag{2.1c}$$

Similarly, this proposition admits the following specifications.

EXAMPLE 2.1: For

$$A(m,n) = x^{-n}\,\frac{(q^2x^2;q^2)_{\lfloor\frac{m+n}{2}\rfloor}\,(qx;q^2)_{\lceil\frac{m+n}{2}\rceil}}{(q^2;q^2)_{\lfloor\frac{m}{2}\rfloor}\,(q^2;q^2)_{\lfloor\frac{n}{2}\rfloor}\,(qx;q^2)_{\lceil\frac{m}{2}\rceil}\,(qx;q^2)_{\lceil\frac{n}{2}\rceil}}, \tag{2.2a}$$

its q-difference is given by

(2.2b)

$$A_\delta(m,n) = \begin{cases} q^m\,x^{-n}\,\frac{1+x\,(-q)^n}{1+x}\,\frac{(x^2;q^2)_{\lceil\frac{m+n}{2}\rceil}\,(qx;q^2)_{\lfloor\frac{m+n}{2}\rfloor}}{(q^2;q^2)_{m/2}\,(q^2;q^2)_{\lfloor\frac{n}{2}\rfloor}\,(qx;q^2)_{m/2}\,(qx;q^2)_{\lceil\frac{n}{2}\rceil}}, & m\text{–even};\\ 0, & m\text{–odd}.\end{cases}$$

The corresponding summation formula

$$\begin{aligned}\text{(2.2c)}\quad &\sum_{i=0}^{\lfloor m/2\rfloor}\sum_{j=0}^{n} x^{n-j}\begin{bmatrix} m+n-2i-j\\ m-2i,\ n-j\end{bmatrix}\frac{(x^2;q^2)_{i+\lceil j/2\rceil}(qx;q^2)_{i+\lfloor j/2\rfloor}}{(q^2;q^2)_i(q^2;q^2)_{\lfloor j/2\rfloor}(qx;q^2)_i(qx;q^2)_{\lceil j/2\rceil}}\\ &\times\, q^{2i(n-j-1)}\,\frac{1+x(-q)^j}{1+x} = \frac{(q^2x^2;q^2)_{\lfloor\frac{m+n}{2}\rfloor}(qx;q^2)_{\lceil\frac{m+n}{2}\rceil}}{(q^2;q^2)_{\lfloor\frac{m}{2}\rfloor}(q^2;q^2)_{\lfloor\frac{n}{2}\rfloor}(qx;q^2)_{\lceil\frac{m}{2}\rceil}(qx;q^2)_{\lceil\frac{n}{2}\rceil}},\end{aligned}$$

is the q-analogue of the extended Blodgett's identity [2].

EXAMPLE 2.2: For

$$B(m,n) = y^{-n}\,\frac{(qx;q)_{m+n}}{(qy;q)_m\,(qz;q)_n}, \tag{2.3a}$$

its q-difference is given by

(2.3b) $B_\delta(m,n)$

$$= \begin{cases}\left\{q^{m+n}\frac{y^{\tau(m)}\,z^{\tau(n)}-x}{1-xq^{m+n}} + (1-\tau(m))\frac{(1-y)\,(1-z\,q^n)}{1-x\,q^{m+n}}\right\}\,B(m,n), & m+n\neq 0\\ 1, & m=n=0.\end{cases}$$

Then the corresponding summation formula is

$$\begin{aligned}\text{(2.3c)}\quad &\sum_{i=0}^{m}\sum_{j=0}^{n} y^{n-j}\,q^{i+j+i(n-j)}\begin{bmatrix} m+n-i-j\\ m-i,\ n-j\end{bmatrix}\frac{(x;q)_{i+j}}{(qy;q)_i\,(qz;q)_j}\,\frac{y^{\tau(i)}\,z^{\tau(j)}-x}{1-x}\\ &= \frac{(qx;q)_{m+n}}{(qy;q)_m\,(qz;q)_n} - \frac{(1-y)\,(1-z)}{1-x}\sum_{k=1}^{n} y^{n-k}\begin{bmatrix} m+n-k\\ m\end{bmatrix}\frac{(x;q)_k}{(z;q)_k}.\end{aligned}$$

EXAMPLE 2.3: For

$$C(m,n) = \frac{(qx;q)_{m+n}\,(qy;q)_{m+n}}{(qx;q)_m\,(qy;q)_m\,(qx;q)_n\,(qy;q)_n}, \tag{2.4a}$$

its difference is given by

$$C_\delta(m,n) = \begin{cases} q^m\,\frac{xy(1-q^m)\,(1-q^{2n})-(1-x)\,(1-y)}{(1-xq^{m+n})\,(1-yq^{m+n})}\,C(m,n), & m>0,\ n>0 \\ 0, & m \ \text{ or } \ n=0 \\ 1, & m=n=0. \end{cases} \tag{2.4b}$$

Then the corresponding summation formula is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} q^{i(n-j+1)} \begin{bmatrix} m+n-i-j \\ m-i,\ n-j \end{bmatrix} \frac{(x;q)_{i+j}\,(y;q)_{i+j}}{(qx;q)_i\,(qy;q)_i\,(qx;q)_j\,(qy;q)_j} \times \frac{xy(1-q^i)\,(1-q^{2j})-(1-x)\,(1-y)}{(1-x)\,(1-y)} = \frac{(qx;q)_{m+n}\,(qy;q)_{m+n}}{(qx;q)_m\,(qy;q)_m\,(qx;q)_n\,(qy;q)_n} - \begin{bmatrix} m+n \\ m \end{bmatrix}. \tag{2.4c}$$

This is a q-analogue of Eq. (1.4c), although it is not as compact as desired.

In the same way as demonstrated in Exs. (1.1-1.3) and Exs. (2.1-2.3), the reader is expected to try to invent some more summation formulae.

3. Formal power series

Consider the formal power series defined by

$$f_z(x,y) = \sum_{i,j=0}^{\infty} \binom{-z}{i,\ j} \binom{i+j}{i,\ j} x^i\,y^j.$$

Introducing the redundant variable t and denoting by $[t^n]\,p(t)$ the coefficient of t^n in the power series expansion of $p(t)$, we can perform the following manipulation.

$$\begin{aligned}
f_z(x,y) &= \sum_{i,j=0}^{\infty} \binom{-z}{i+j} \binom{i+j}{i,\, j}^2 x^i y^j \\
&= \sum_{k=0}^{\infty} \binom{-z}{k} y^k [t^0] (1+tx)^k \Big(1+\frac{1}{ty}\Big)^k \\
&= [t^0] \Big\{1+y(1+tx)\Big(1+\frac{1}{ty}\Big)\Big\}^{-z} \\
&= (1+x+y)^{-z} [t^0] \Big(1+\frac{t^{-1}+txy}{1+x+y}\Big)^{-z} \\
&= (1+x+y)^{-z} \sum_{k=0}^{\infty} \frac{\binom{-z}{2n}\binom{2n}{n} x^n y^n}{(1+x+y)^{2n}} \\
&= (1+x+y)^{-z} {}_2F_1 \left[\begin{matrix} \frac{z}{2}, & \frac{1+z}{2} \\ & 1 \end{matrix} ; \frac{4xy}{(1+x+y)^2}\right].
\end{aligned}$$

Some examples may be displayed as follows.

(3.1a) $$f_1(x,y) = \frac{1}{\sqrt{(1+x+y)^2-4xy}}.$$

(3.1b) $$f_2(x,y) = \frac{1+x+y}{\sqrt{\{(1+x+y)^2-4xy\}^3}}.$$

(3.1c) $$f_3(x,y) = (1+x+y)^2 + \frac{2xy}{\sqrt{\{(1+x+y)^2-4xy\}^5}}.$$

Putting

$$f(x,y) = \{(1-x-y)^2-4xy\}^{-1},$$

we can make the expansion further:

$$\begin{aligned}
(1-x-y)^{2\varepsilon} f^{\alpha}(x,y) &= (1-x-y)^{2\varepsilon}\big\{(1-x-y)^2-4xy\big\}^{-\alpha} \\
&= (1-x-y)^{2\varepsilon-2\alpha} \sum_{k=0}^{\infty} \binom{-\alpha}{k} \frac{(-4xy)^k}{(1-x-y)^{2k}} \\
&= \sum_{k=0}^{\infty} \binom{-\alpha}{k} (-4xy)^k \sum_{i,\,j=0}^{\infty} \binom{2\varepsilon-2\alpha-2k}{i,\, j} (-x)^i (-y)^j
\end{aligned}$$

$$= \sum_{m,n=0}^{\infty} x^m y^n \sum_k (-1)^{m+n+k} \binom{-\alpha}{k} \binom{2\varepsilon-2\alpha-2k}{m-k,\, n-k} 4^k$$
$$= \sum_{m,n=0}^{\infty} \frac{(2\alpha-2\varepsilon)_{m+n}}{m!\,n!} x^m y^n \;{}_3F_2\left[\begin{matrix}\alpha, & -m, & -n \\ & \alpha-\varepsilon, & \frac{1}{2}+\alpha-\varepsilon\end{matrix}\right].$$

For $\varepsilon = 0,\ 1/2,\ 1,\ 3/2$, the corresponding formulae may be displayed, in view of the Chu-Vandermonde summation theorem [1, p. 3], as follows:

$$\text{(3.2a)}\qquad \left\{(1-x-y)^2-4xy\right\}^{-\alpha} = \sum_{m,n=0}^{\infty} \frac{(2\alpha)_{m+n}}{m!\,n!} \frac{(\frac{1}{2}+\alpha)_{m+n}}{(\frac{1}{2}+\alpha)_m(\frac{1}{2}+\alpha)_n} x^m y^n,$$

$$\text{(3.2b)}\qquad \frac{1-x-y}{\{(1-x-y)^2-4xy\}^{\alpha+1/2}} = \sum_{m,n=0}^{\infty} \frac{(2\alpha)_{m+n}}{m!\,n!} \frac{(\alpha)_{m+n}}{(\alpha)_m(\alpha)_n} x^m y^n,$$

$$\text{(3.2c)}\qquad \frac{(1-x-y)^2}{\{(1-x-y)^2-4xy\}^{\alpha+1}} = \sum_{m,n=0}^{\infty} \frac{(2\alpha)_{m+n}}{m!\,n!} \frac{(\frac{1}{2}+\alpha)_{m+n}}{(\frac{1}{2}+\alpha)_m(\frac{1}{2}+\alpha)_n} \times \left\{1-\frac{m\,n}{\alpha\,(\frac{1}{2}-\alpha-m-n)}\right\} x^m y^n,$$

$$\text{(3.2d)}\qquad \frac{(1-x-y)^3}{\{(1-x-y)^2-4xy\}^{\alpha+3/2}} = \sum_{m,n=0}^{\infty} \frac{(2\alpha)_{m+n}}{m!\,n!} \frac{(\alpha)_{m+n}}{(\alpha)_m\,(\alpha)_n} \times \left\{1-\frac{m\,n}{(\frac{1}{2}+\alpha)\,(1-\alpha-m-n)}\right\} x^m y^n,$$

$$\text{(3.2e)}\qquad \frac{1-y}{\{(1-x-y)^2-4xy\}^{\alpha+1/2}} = \sum_{m,n=0}^{\infty} \frac{(2\alpha)_{m+n}}{m!\,n!} \frac{(\alpha)_{m+n}}{(\alpha)_m\,(1+\alpha)_n} \times \left\{\frac{2\alpha+m+2n}{2\alpha}\right\} x^m y^n;$$

where the last one follows from the simple series manipulation.

From the generating functions exhibited above, now we can demonstrate some hypergeometric evaluations.

EXAMPLE 3.1: Note that $f(x,y) = (1-v/u)^{-2} f(u,v)$, where $u = x/(1-y)^2$ and $v = xy/(1-y)^2$. From

$$f^{\alpha}(x,y) = (1-y)^{-2\alpha} f^{\alpha}(u,v),$$
$$(1-y)\, f^{\alpha}(x,y) = (1-y)^{1-2\alpha} f^{\alpha}(u,v);$$

we have a pair of Saalschützian summation formulae [1, page 9]

$$\text{(3.3a)}\quad {}_3F_2\left[\begin{matrix} -m, & -n, & \frac{1}{2}-\alpha-m \\ & \frac{1}{2}+\alpha, & 1-2\alpha-2m-n \end{matrix}\right] = \frac{(2\alpha+m)_n\,(\frac{1}{2}+\alpha+m)_n}{(\frac{1}{2}+\alpha)_n\,(2\alpha+2m)_n},$$

$$\text{(3.3b)}\quad {}_3F_2\left[\begin{matrix} -m, & -n, & \frac{1}{2}-\alpha-m \\ & \frac{1}{2}+\alpha, & 2-2\alpha-2m-n \end{matrix}\right] = \frac{(2\alpha+m)_n\,(\frac{1}{2}+\alpha+m)_n}{(\frac{1}{2}+\alpha)_n\,(2\alpha+2m)_n} \times \frac{(-1+2\alpha+m+2n)\,(-1+2\alpha+2m+n)}{(-1+2\alpha+m+n)\,(-1+2\alpha+2m+2n)}.$$

When $\alpha = 1/2$, the first one reduces to the well-known Le Jen-shoo's identity

$$\binom{m+n}{m}^2 = \sum_{k\geq 0}\binom{n}{k}^2\binom{m+2n-k}{2n}.$$

Example 3.2: From the functional equations

$$\begin{aligned} f^{\alpha+\gamma}(x,y) &= f^{\alpha}(x,y)\cdot f^{\gamma}(x,y), \\ (1-y)\,f^{\alpha+\gamma+1/2}(x,y) &= f^{\alpha}(x,y)\cdot(1-y)\,f^{\gamma+1/2}(x,y), \\ (1-x-y)\,f^{\alpha+\gamma+1/2}(x,y) &= f^{\alpha}(x,y)\cdot(1-x-y)\,f^{\gamma+1/2}(x,y); \end{aligned}$$

we obtain the following double-convolution formulae

$$\text{(3.4a)}\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\frac{(2\alpha)_{i+j}\,(\frac{1}{2}+\alpha)_{i+j}}{(\frac{1}{2}+\alpha)_i\,(\frac{1}{2}+\alpha)_j} \times \frac{(2\gamma)_{m+n-i-j}\,(\frac{1}{2}+\gamma)_{m+n-i-j}}{(\frac{1}{2}+\gamma)_{m-i}(\frac{1}{2}+\gamma)_{n-j}} = \frac{(2\alpha+2\gamma)_{m+n}\,(\frac{1}{2}+\alpha+\gamma)_{m+n}}{(\frac{1}{2}+\alpha+\gamma)_m\,(\frac{1}{2}+\alpha+\gamma)_n},$$

$$\text{(3.4b)}\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\frac{(2\alpha)_{i+j}\,(\frac{1}{2}+\alpha)_{i+j}}{(\frac{1}{2}+\alpha)_i\,(\frac{1}{2}+\alpha)_j}\cdot\frac{(2\gamma+m+2n-i-2j)}{(2\alpha+2\gamma+m+2n)} \times \frac{2\alpha+2\gamma}{2\gamma}\cdot\frac{(2\gamma)_{m+n-i-j}\,(\gamma)_{m+n-i-j}}{(\gamma)_{m-i}(1+\gamma)_{n-j}} = \frac{(2\alpha+2\gamma)_{m+n}\,(\alpha+\gamma)_{m+n}}{(\alpha+\gamma)_m(1+\alpha+\gamma)_n},$$

$$\text{(3.4c)}\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\frac{(2\alpha)_{i+j}\,(\frac{1}{2}+\alpha)_{i+j}}{(\frac{1}{2}+\alpha)_i\,(\frac{1}{2}+\alpha)_j} \times \frac{(2\gamma)_{m+n-i-j}\,(\gamma)_{m+n-i-j}}{(\gamma)_{m-i}(\gamma)_{n-j}} = \frac{(2\alpha+2\gamma)_{m+n}\,(\alpha+\gamma)_{m+n}}{(\alpha+\gamma)_m(\alpha+\gamma)_n}.$$

Taking $\gamma = 1/2$, we deduce, from the above, the double summations:

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m+n-i-j}{m-i,\, n-j}^2 \frac{(2\alpha)_{i+j}\,(\frac{1}{2}+\alpha)_{i+j}}{i!\,j!\,(\frac{1}{2}+\alpha)_i\,(\frac{1}{2}+\alpha)_j} = \frac{(1+2\alpha)_{m+n}\,(1+\alpha)_{m+n}}{m!\,n!\,(1+\alpha)_m\,(1+\alpha)_n}, \tag{3.5a}$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{2m+2n-2i-2j}{2m-2i,\, 2n-2j} \frac{(2\alpha)_{i+j}\,(\frac{1}{2}+\alpha)_{i+j}}{i!\,j!\,(\frac{1}{2}+\alpha)_i\,(\frac{1}{2}+\alpha)_j} \times \frac{1+m+2n-i-2j}{1+2n-2j}\,\frac{1+2\alpha}{1+2\alpha+m+2n} = \frac{(1+2\alpha)_{m+n}\,(\frac{1}{2}+\alpha)_{m+n}}{m!\,n!\,(\frac{1}{2}+\alpha)_m\,(\frac{3}{2}+\alpha)_n}, \tag{3.5b}$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{2m+2n-2i-2j}{2m-2i,\, 2n-2j} \frac{(2\alpha)_{i+j}\,(\frac{1}{2}+\alpha)_{i+j}}{i!\,j!\,(\frac{1}{2}+\alpha)_i\,(\frac{1}{2}+\alpha)_j} = \frac{(1+2\alpha)_{m+n}\,(\frac{1}{2}+\alpha)_{m+n}}{m!\,n!\,(\frac{1}{2}+\alpha)_m\,(\frac{1}{2}+\alpha)_n}. \tag{3.5c}$$

Among them, the last one has been indicated by Gessel [2]. When $\alpha = 1/2$ further, Eqs. (3.5a - 3.5c) will reduce to the binomial identities due to Blodgett and Jagers [2], respectively.

EXAMPLE 3.3: Consider the expansion

$$\begin{aligned}
\frac{f^{\alpha}(x^2,y^2)}{\{1-(x+y)^2\}^{\varepsilon}} &= \{1-(x+y)^2\}^{-\varepsilon}\ \big\{[1-(x+y)^2]^2+4xy[1-(x+y)^2]\big\}^{-\alpha} \\
&= \sum_{k}\binom{-\alpha}{k}(4xy)^k\,[1-(x+y)^2]^{-\varepsilon-2\alpha-k} \\
&= \sum_{k}\binom{-\alpha}{k}4^k \sum_{i,j\ (i+j-\text{even})}(-1)^{(i+j)/2} \\
&\quad\times\binom{-\varepsilon-2\alpha-k}{(i+j)/2}\binom{i+j}{i,\, j}x^{k+i}\,y^{k+j} \\
&= \sum_{m,n\ (m+n-\text{even})} x^m\,y^n \sum_{k}(-1)^{k+(m+n)/2}\binom{-\alpha}{k}4^k \\
&\quad\times\binom{-\varepsilon-2\alpha-k}{\frac{m+n}{2}-k}\binom{m+n-2k}{m-k,\, n-k}
\end{aligned}$$

$$= \sum_{m,n\ (m+n-\text{even})} x^m\, y^n \binom{m+n}{m,\, n} \frac{(\varepsilon+2\alpha)_{\frac{m+n}{2}}}{(\frac{m+n}{2})!}$$
$$\times\ {}_3F_2\left[\begin{matrix}-m,\ -n, & \alpha\\ \frac{1-m-n}{2}, & \varepsilon+2\alpha\end{matrix}\right].$$

Then a special version of Watson's formulae [1, page 16]

$$\text{(3.6a)}\qquad {}_3F_2\left[\begin{matrix}-m,\ -n, & \alpha\\ \frac{1-m-n}{2}, & 2\alpha\end{matrix}\right] = \begin{cases}\frac{(\frac{1}{2}+\alpha)_{\frac{m+n}{2}}\,(\frac{1}{2})_{\frac{m}{2}}\,(\frac{1}{2})_{\frac{n}{2}}}{(\frac{1}{2})_{\frac{m+n}{2}}\,(\frac{1}{2}+\alpha)_{\frac{m}{2}}\,(\frac{1}{2}+\alpha)_{\frac{n}{2}}}, & m,n-\text{even};\\ 0, & m,n-\text{odd}.\end{cases}$$

is recovered by recalling the expansion from (3.2a),

$$\text{(3.6b)}\qquad f^\alpha(x^2,y^2) = \sum_{m,n=0}^{\infty} \frac{(2\alpha)_{m+n}}{m!\,n!}\,\frac{(\frac{1}{2}+\alpha)_{m+n}}{(\frac{1}{2}+\alpha)_m(\frac{1}{2}+\alpha)_n}\,x^{2m}y^{2n}.$$

From this formula and the decomposition $1 = (k+2\alpha)/2\alpha - k/2\alpha$ on the summation-index, we get another hypergeometric identity

$$\text{(3.7a)}\qquad {}_3F_2\left[\begin{matrix}-m, & -n, & \alpha\\ & \frac{1-m-n}{2}, & 1+2\alpha\end{matrix}\right] = \frac{(\frac{1}{2}+\alpha)_{\frac{m+n}{2}}\,(\frac{1}{2})_{\lceil\frac{m}{2}\rceil}\,(\frac{1}{2})_{\lceil\frac{n}{2}\rceil}}{(\frac{1}{2})_{\frac{m+n}{2}}\,(\frac{1}{2}+\alpha)_{\lceil\frac{m}{2}\rceil}\,(\frac{1}{2}+\alpha)_{\lceil\frac{n}{2}\rceil}},\qquad (m+n-\text{even}),$$

which yields the expansion corresponding to $\varepsilon = 1$

$$\text{(3.7b)}\qquad \frac{f^\alpha(x^2,y^2)}{1-(x-y)^2} = \sum_{m,n\ (m+n-\text{even})} \frac{(1+2\alpha)_{\frac{m+n}{2}}\,(\frac{1}{2}+\alpha)_{\frac{m+n}{2}}}{\lfloor\frac{m}{2}\rfloor!\,\lfloor\frac{n}{2}\rfloor!\,(\frac{1}{2}+\alpha)_{\lceil\frac{m}{2}\rceil}\,(\frac{1}{2}+\alpha)_{\lceil\frac{n}{2}\rceil}}\,x^m y^n.$$

Next perform expansion

$$\frac{f^\alpha(x^2,y^2)}{1-x-y} = \frac{\big\{[1-(x+y)^2]^2 + 4xy[1-(x+y)^2]\big\}^{-\alpha}}{1-x-y}$$
$$= \sum_k \binom{-\alpha}{k} \frac{(4xy)^k}{[1-(x+y)^2]^{1+2\alpha+k}}\,(1+x+y)$$
$$= \sum_k \binom{-\alpha}{k} 4^k \sum_{i,j\ (i+j-\text{even})} (-1)^{(i+j)/2} \binom{-1-2\alpha-k}{(i+j)/2}$$
$$\times \binom{i+j}{i,\, j}\, x^{k+i}\, y^{k+j}\,(1+x+y)$$
$$= \sum_{m,n=0}^{\infty} x^m\, y^n \sum_k (-1)^{k+\lfloor\frac{m+n}{2}\rfloor} \binom{-\alpha}{k}$$
$$\times \binom{-1-2\alpha-k}{\lfloor\frac{m+n}{2}\rfloor-k} \binom{m+n-2k}{m-k,\, n-k} 4^k$$

$$= \sum_{m,n=0}^{\infty} x^m y^n \binom{m+n}{m,\,n} \frac{(1+2\alpha)_{\lfloor\frac{m+n}{2}\rfloor}}{\lfloor\frac{m+n}{2}\rfloor!}$$

$$\times\ {}_4F_3\left[\begin{matrix} -\lfloor\frac{m+n}{2}\rfloor, & -m, & -n, & \alpha \\ & -\frac{m+n}{2}, & -\frac{m+n-1}{2}, & 1+2\alpha \end{matrix}\right],$$

where the hypergeometric summation can be evaluated as

$$(3.8a)\qquad {}_4F_3\left[\begin{matrix} -\lfloor\frac{m+n}{2}\rfloor, & -m, & -n, & \alpha \\ & -\frac{m+n}{2}, & -\frac{m+n-1}{2}, & 1+2\alpha \end{matrix}\right]$$

$$= \frac{(\frac{1}{2}+\alpha)_{\lceil\frac{m+n}{2}\rceil}\,(\frac{1}{2})_{\lceil\frac{m}{2}\rceil}\,(\frac{1}{2})_{\lceil\frac{n}{2}\rceil}}{(\frac{1}{2})_{\lceil\frac{m+n}{2}\rceil}\,(\frac{1}{2}+\alpha)_{\lceil\frac{m}{2}\rceil}\,(\frac{1}{2}+\alpha)_{\lceil\frac{n}{2}\rceil}}$$

by (3.6a) via decomposing the summation-index

$$1 = \frac{k+2\alpha}{2\alpha} - \frac{k}{2\alpha} \quad \text{when} \quad m+n \quad \text{is even}$$

and

$$1 = \frac{-1-m}{-1-m+k}\,\frac{k+2\alpha}{2\alpha} + \frac{1+2\alpha+m}{2\alpha}\,\frac{k}{-1-m+k} \quad \text{when} \quad m+n \quad \text{is odd.}$$

Thus, we establish the expansion

$$(3.8b)\qquad \frac{f^{\alpha}(x^2,y^2)}{1-x-y} = \sum_{m,n=0}^{\infty} \frac{(1+2\alpha)_{\lfloor\frac{m+n}{2}\rfloor}\,(\frac{1}{2}+\alpha)_{\lceil\frac{m+n}{2}\rceil}}{\lfloor\frac{m}{2}\rfloor!\,\lfloor\frac{n}{2}\rfloor!\,(\frac{1}{2}+\alpha)_{\lceil\frac{m}{2}\rceil}\,(\frac{1}{2}+\alpha)_{\lceil\frac{n}{2}\rceil}}\, x^m y^n.$$

The identity (1.2c) may be revisited through the combination of the expansion and the equation displayed as follows:

$$(1-x-y)^{-1} = \sum_{m,n=0}^{\infty} \binom{m+n}{m,\,n} x^m y^n,$$

$$\frac{f^{\alpha}(x^2,y^2)}{1-x-y} = f^{\alpha}(x^2,y^2)\cdot(1-x-y)^{-1}.$$

Two other relations

$$\frac{f^{\alpha+\gamma}(x^2,y^2)}{1-x-y} = f^{\alpha}(x^2,y^2)\ \cdot\ \frac{f^{\gamma}(x^2,y^2)}{1-x-y},$$

$$\frac{f^{\alpha+\gamma}(x^2,y^2)}{1-(x+y)^2} = \frac{f^{\alpha}(x^2,y^2)}{1+x+y}\cdot\frac{f^{\gamma}(x^2,y^2)}{1-x-y};$$

are equivalent to the convolution formulae

(3.9a)
$$\sum_{i,\,j\geq 0}\binom{\lfloor\frac{m}{2}\rfloor}{i}\binom{\lfloor\frac{n}{2}\rfloor}{j}\frac{(2\alpha)_{i+j}\,(\frac{1}{2}+\alpha)_{i+j}}{(\frac{1}{2}+\alpha)_i\,(\frac{1}{2}+\alpha)_j}$$
$$\times\ \frac{(1+2\gamma)_{\lfloor\frac{m+n}{2}\rfloor-i-j}\,(\frac{1}{2}+\gamma)_{\lceil\frac{m+n}{2}\rceil-i-j}}{(\frac{1}{2}+\gamma)_{\lceil\frac{m}{2}\rceil-i}\,(\frac{1}{2}+\gamma)_{\lceil\frac{n}{2}\rceil-j}}$$
$$=\frac{(1+2\alpha+2\gamma)_{\lfloor\frac{m+n}{2}\rfloor}\,(\frac{1}{2}+\alpha+\gamma)_{\lceil\frac{m+n}{2}\rceil}}{(\frac{1}{2}+\alpha+\gamma)_{\lceil\frac{m}{2}\rceil}\,(\frac{1}{2}+\alpha+\gamma)_{\lceil\frac{n}{2}\rceil}},$$

(3.9b)
$$\sum_{i,\,j\geq 0}\binom{\lfloor\frac{m}{2}\rfloor}{\lfloor\frac{i}{2}\rfloor,\,\lfloor\frac{(m-i)}{2}\rfloor}\binom{\lfloor\frac{n}{2}\rfloor}{\lfloor\frac{j}{2}\rfloor,\,\lfloor(n-j)/2\rfloor}$$
$$\times\ (-1)^{i+j}\,\frac{(1+2\alpha)_{\lfloor\frac{i+j}{2}\rfloor}\,(\frac{1}{2}+\alpha)_{\lceil\frac{i+j}{2}\rceil}}{(\frac{1}{2}+\alpha)_{\lceil\frac{i}{2}\rceil}\,(\frac{1}{2}+\alpha)_{\lceil\frac{j}{2}\rceil}}$$
$$\times\ \frac{(1+2\gamma)_{\lfloor\frac{m+n-i-j}{2}\rfloor}\,(\frac{1}{2}+\gamma)_{\lceil\frac{m+n-i-j}{2}\rceil}}{(\frac{1}{2}+\gamma)_{\lceil\frac{m-i}{2}\rceil}\,(\frac{1}{2}+\gamma)_{\lceil\frac{n-j}{2}\rceil}}$$
$$=\begin{cases}\frac{(1+2\alpha+2\gamma)_{\frac{m+n}{2}}\,(\frac{1}{2}+\alpha+\gamma)_{\frac{m+n}{2}}}{(\frac{1}{2}+\alpha+\gamma)_{\lceil\frac{m}{2}\rceil}\,(\frac{1}{2}+\alpha+\gamma)_{\lceil\frac{n}{2}\rceil}}, & (m+n-\text{even});\\ 0, & (m+n-\text{odd}).\end{cases}$$

Remark. In an article by Carlitz [3], the bivariate generating functions

(A.1a)
$$\sum_{m,n=0}^{\infty}A_{m,n}(\alpha,\gamma)\,x^m\,y^n\ =\ (1+u)^{\alpha}\,(1+v)^{\gamma},$$

(A.1b)
$$\sum_{m,n=0}^{\infty}\binom{\alpha+am+cn}{m}\binom{\gamma+bm+dn}{n}\,x^m\,y^n$$
$$=\ \frac{(1+u)^{1+\alpha}\,(1+v)^{1+\gamma}}{\Delta};$$

where

$$A_{m,n}(\alpha,\gamma)=\frac{\alpha\gamma+\alpha bm+\gamma cn}{(\alpha+am+cn)\,(\gamma+bm+dn)}\binom{\alpha+am+cn}{m}\binom{\gamma+bm+dn}{n},$$

(A.2a) $\quad x=\ u\,(1+u)^{-a}\,(1+v)^{-b},$

(A.2b) $\quad y=\ v\,(1+u)^{-c}\,(1+v)^{-d},$

(A.2c) $\quad \Delta=\ (1+u-au)\,(1+v-dv)-bcuv\,;$

are established via MacMahon's master theorem, which lead to a pair of convolution identities:

$$\text{(A.3a)}\qquad \sum_{i=0}^{m}\sum_{j=0}^{n} A_{i,j}(\alpha,\gamma)\, A_{m-i,n-j}(\alpha',\gamma') \;=\; A_{m,n}(\alpha+\alpha',\gamma+\gamma'),$$

$$\text{(A.3b)}\qquad \sum_{i=0}^{m}\sum_{j=0}^{n} \binom{\alpha+ai+cj}{i}\binom{\gamma+bi+dj}{j} A_{m-i,n-j}(\alpha',\gamma') = \binom{\alpha+\alpha'+am+cn}{m}\binom{\gamma+\gamma'+bm+dn}{n}.$$

However, the convolution formulae demonstrated in the last section are different essentially from Carlitz' summation, though Eqs. (3.1) fit in the general relations stated in Eqs. (A.1).

Addendum. After having written the first version of this paper, the author was encouraged by V. Strehl to check whether the approach presented here works for Brock numbers.

The original problem proposed by Brock (1960, cf. [5]) is to show that the summations defined by

$$\text{(B.1)}\quad H(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n} \binom{m+n-i-j}{m-i,\, n-j}\binom{i+j}{i,\, j}\binom{m-i+j}{m-i,\, j}\binom{i+n-j}{i,\, n-j}$$

satisfy the recurrence relation

$$\text{(B.2)}\qquad \binom{m+n}{m,\, n}^2 = H(m,n) - H(m-1,n) - H(m,n-1).$$

In fact, an equivalent expression of Brock summation defined by the recurrence can be recovered by Eqs. (1.1b–1.1c)

$$\text{(B.3)}\qquad H(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n} \binom{m+n-i-j}{m-i,\, n-j}\binom{i+j}{i,\, j}^2.$$

In view of the expansion of $(1-x-y)^{2\varepsilon} f^{\alpha}(x,y)$ displayed in the last section, we find that the expression for $H(m,n)$ can be simplified further

$$H(m,n) = [x^m y^n]\left\{\frac{(1-x-y)^{-1}}{\sqrt{(1-x-y)^2-4xy}}\right\}$$

$$= \frac{(1+m+n)!}{m!\,n!}\, {}_3F_2\left[\begin{matrix}-m, & -n, & 1/2\\ & 1, & 3/2\end{matrix}\right],$$

which may be reformulated as follows (cf. [5])

$$\text{(B.4)} \qquad H(m,n) = \frac{(1+m+n)!}{m!\,n!}\sum_k \frac{\binom{m}{k}\binom{n}{k}}{1+2k} = \sum_k \binom{1+m+n}{m-k,\, n-k}\binom{2k}{k}.$$

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