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On the Inverses of a Family of Pascal-Like Matrices Defined by Riordan Arrays

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Abstract

We study a number of characteristics of the inverses of the elements of a family of Pascal-like matrices that are defined by Riordan arrays. We give several forms of the bivariate generating function of these inverses, along with four different closed-form expressions for the general element of the inverse. We study the row sums and the diagonal sums of the inverses, and the first column sequence. We exhibit the elements of the first column sequence of the inverse matrix as the moments of a family of orthogonal polynomials, whose coefficient array is again a Riordan array. We also give the Hankel transform of these latter sequences. Other related sequences are also studied.

1 Introduction

It is known [4] that the family of Riordan arrays defined by

$$T(r) = \left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right)$$

is a family of Pascal-like matrices. By this we mean that if $T_{n,k}(r)$ is the (n,k)-th element of the matrix $\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right)$, then we have

$$T_{n,0} = T_{n,n} = 1, \quad T_{n,n-k} = T_{n,k}.$$

We note that as Riordan arrays, these matrices are lower-triangular. The matrix $T(0)$ is Pascal's triangle A007318, defined by $\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$. The matrix $T(1)$, defined by $\left(\frac{1}{1-x}, \frac{x(1+x)}{1-x}\right)$, is the Delannoy triangle A008288. This array begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 5 & 5 & 1 & 0 & 0 & \cdots \\ 1 & 7 & 13 & 7 & 1 & 0 & \cdots \\ 1 & 9 & 25 & 25 & 9 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The (n,k)-th element of $T(1)$ counts the number of lattice paths from $(0,0)$ to (n,k) using the steps $(1,0)$, $(0,1)$, and $(1,1)$. Its central elements $1, 3, 13, 63, \ldots$ A001850 count the number of such paths from $(0,0)$ to (n,n).

The general array $T(r)$ begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & r+2 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2r+3 & 2r+3 & 1 & 0 & 0 & \cdots \\ 1 & 3r+4 & r^2+6r+6 & 3r+4 & 1 & 0 & \cdots \\ 1 & 4r+5 & 3r^2+12r+10 & 3r^2+12r+10 & 4r+5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

and is of significant combinatorial interest, particularly in the area of lattice paths. In this note, we shall be interested in various combinatorial aspects of the inverse matrix $T(r)^{-1}$.

The matrix inverse $T(r)^{-1}$ begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ r+1 & -r-2 & 1 & 0 & 0 & 0 & \cdots \\ -2r^2-3r-1 & 2r^2+5r+3 & -2r-3 & 1 & 0 & 0 & \cdots \\ 5r^3+10r^2+6r+1 & -5r^3-15r^2-14r-4 & 5r^2+11r+6 & -3r-4 & 1 & 0 & \cdots \\ -14r^4-35r^3-30r^2-10r-1 & 14r^4+49r^3+60r^2+30r+5 & -14r^3-39r^2-35r-10 & 9r^2+19r+10 & -4r-5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Among the results concerning $T(r)^{-1}$ that we shall discuss, we give the following examples.

The elements of $T(r)^{-1}$ are polynomials in r, with the (n,k)-th element $S_{n,k}(r)$ of the inverse matrix $S(r) = T(r)^{-1}$ being given by

$$S_{n,k}(r) = \sum_{i=0}^{n} \frac{i+1}{n+1} \sum_{j=0}^{n+1} \binom{n+1}{j} \binom{2n-i-j}{n-i-j} \left((1-r)\binom{0}{i-k} + r\binom{1}{i-k}\right)(-1)^j(-r)^{n-i-j}.$$

We shall give other expressions for $S_{n,k}(r)$ during this note. The bivariate generating function (g.f.) of the triangle $T(r)^{-1}$ may be described by the following continued fraction,

$$\cfrac{1}{1+\cfrac{x-xy}{1+\cfrac{rx}{1+\cfrac{(r+1)x}{1+\cfrac{rx}{1+\cfrac{(r+1)x}{1+\cdots}}}}}}.$$

We shall be particularly interested in the first column elements of $T(r)^{-1}$. The first column sequence $a_n(r) = S_{n,0}(r)$ of $T(r)^{-1}$ can be expressed as

$$a_n(r) = 0^n + (-1)^n \sum_{k=0}^{n-1} \binom{n+k-1}{2k} C_k r^k,$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number A000108, and the Hankel transform of $a_n(r)$ is given by

$$h_n(r) = r^{\binom{2n+1}{n}} (r+1)^{\binom{2n}{n}}.$$

The row sums sequence of $T(r)^{-1}$ is the sequence 0^n with elements $1, 0, 0, 0, \ldots$, while the diagonal sums sequence $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} S_{n-k,k}(r)$ has a g.f. expressible as the continued fraction

$$\cfrac{1}{1+\cfrac{x-x^2}{1+\cfrac{rx}{1+\cfrac{(r+1)x}{1+\cfrac{rx}{1+\cfrac{(r+1)x}{1+\cdots}}}}}},$$

and we conjecture that this sequence has Hankel transform given by

$$h_n = r^{\binom{n}{2}} (r+1)^{\lceil \frac{n^2}{2} \rceil} [x^n] \frac{1+x-(r+2)x^2-rx^3}{1-2(r+2)x^2+r^2x^4}.$$

In addition, various factorizations of the inverse matrix $T(r)^{-1}$ are examined, which lead to many interesting intermediate results.

In the next section, we shall cover the elements of integer sequence theory, the theory of Riordan arrays, and the Lagrange Inversion theorem that are relevant to this note. Readers familiar with these notions may skip this section.

2 Integer sequences, Lagrange Inversion and Riordan arrays

For an integer sequence a_n, that is, an element of $\mathbb{Z}^{\mathbb{N}}$, the power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ is called the *ordinary generating function* or g.f. of the sequence. a_n is thus the coefficient of x^n in this series. We denote this by $a_n = [x^n]f(x)$. For instance, $F_n = [x^n]\frac{x}{1-x-x^2}$ is the n-th Fibonacci number A000045, while $C_n = [x^n]\frac{1-\sqrt{1-4x}}{2x}$ is the n-th Catalan number A000108. The properties and examples of use of the notation $[x^n]$ can be found in [13]. Note that $0^n = [x^n]1 = \delta_{n,0} = \binom{0}{n}$ is the sequence with terms $1, 0, 0, 0, \ldots$.

For a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with $f(0) = 0$ and $f'(0) \neq 0$ we define the *reversion* or *compositional inverse* of f to be the power series $\bar{f}(x)$ (also written as $f^{[-1]}(x)$) such that $f(\bar{f}(x)) = x$. We sometimes write $\bar{f} = \text{Rev} f$.

The Hankel transform [10, 11] of a sequence a_n is the sequence h_n where $h_n = |a_{i+j}|_{0 \leq i,j \leq n}$. If the g.f. of a_n is expressible as a continued fraction of the form

$$\cfrac{\mu_0}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cfrac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}},$$

then the Hankel transform of a_n is given by [10] the Heilermann formula

$$h_n = \mu_0^{n+1} \beta_1^n \beta_2^{n-1} \cdots \beta_{n-1}^2 \beta_n.$$

If the g.f. of a_n is expressible as the following type of continued fraction:

$$\cfrac{\mu_0}{1 + \cfrac{\gamma_1 x}{1 + \cfrac{\gamma_2 x}{1 + \cdots}}},$$

then we have

$$h_n = \mu_0^{n+1} (\gamma_1 \gamma_2)^n (\gamma_3 \gamma_4)^{n-1} \cdots (\gamma_{2n-3} \gamma_{2n-2})^2 (\gamma_{2n-1} \gamma_{2n}).$$

The *Riordan group* [6, 14, 17], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = 1 + g_1 x + g_2 x^2 + \cdots$ and $f(x) = f_1 x + f_2 x^2 + \cdots$ where $f_1 \neq 0$ [17]. The corresponding matrix is the matrix whose i-th column is generated by $g(x) f(x)^i$ (the first column being indexed by 0). The matrix corresponding to the pair g, f is denoted by (g, f). The group law, which corresponds to matrix multiplication, is then given by

$$(g, f) \cdot (h, l) = (g, f)(h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is

$$(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f}) \tag{1}$$

where $\bar{f}$ is the compositional inverse of f. We denote by $\mathcal{R}$ the group of Riordan matrices. If $\mathbf{M}$ is the matrix (g, f), and $\mathbf{a} = (a_0, a_1, \ldots)^T$ is an integer sequence with ordinary generating function $\mathcal{A}(x)$, then the sequence $\mathbf{Ma}$ has ordinary generating function $g(x)\mathcal{A}(f(x))$. The (infinite) matrix (g, f) can thus be considered to act on the ring of integer sequences $\mathbb{Z}^{\mathbb{N}}$ by multiplication, where a sequence is regarded as a (infinite) column vector. We can extend this action to the ring of power series $\mathbb{Z}[[x]]$ by

$$(g, f) : \mathcal{A}(x) \mapsto (g, f) \cdot \mathcal{A}(x) = g(x)\mathcal{A}(f(x)).$$

Example 1. The so-called *binomial matrix* B A007318 is the element $(\frac{1}{1-x}, \frac{x}{1-x})$ of the Riordan group. It has general element $\binom{n}{k}$, and hence, as an array, coincides with Pascal's triangle. More generally, B^m is the element $(\frac{1}{1-mx}, \frac{x}{1-mx})$ of the Riordan group, with general term $\binom{n}{k}m^{n-k}$. We find that the inverse B^{-m} of B^m is given by $(\frac{1}{1+mx}, \frac{x}{1+mx})$.

Example 2. The row sums of a Riordan array have g.f. given by $\frac{g(x)}{1-f(x)}$. For instance, the row sums of the binomial matrix have g.f. given by

$$\frac{\frac{1}{1-x}}{1 - \frac{x}{1-x}} = \frac{1}{1-2x}.$$

Thus the row sums of the binomial matrix are

$$[x^n]\frac{1}{1-2x} = 2^n,$$

as expected. The diagonal sums of the Riordan matrix $(g(x), f(x))$ have g.f. given by $\frac{g(x)}{1-xf(x)}$. Thus the g.f. of the diagonal sums of the binomial matrix have g.f. given by

$$\frac{\frac{1}{1-x}}{1 - x(\frac{x}{1-x})} = \frac{1}{1-x-x^2},$$

which is the g.f. of the Fibonacci numbers F_{n+1}, as expected.

The bivariate generating function of the Riordan array $(g(x), f(x))$ is given by

$$\frac{g(x)}{1 - yf(x)}.$$

An important feature of Riordan arrays is that they have a number of sequence characterizations [5, 9]. The simplest of these is as follows.

Proposition 3. *[9, Theorem 2.1, Theorem 2.2] Let $D = [d_{n,k}]$ be an infinite triangular matrix. Then D is a Riordan array if and only if there exist two sequences $A = [a_0, a_1, a_2, \ldots]$ and $Z = [z_0, z_1, z_2, \ldots]$ with $a_0 \neq 0$, $z_0 \neq 0$ such that*

- $d_{n+1,k+1} = \sum_{j=0}^{\infty} a_j d_{n,k+j}, \quad (k, n = 0, 1, \ldots)$
- $d_{n+1,0} = \sum_{j=0}^{\infty} z_j d_{n,j}, \quad (n = 0, 1, \ldots)$.

The coefficients $a_0, a_1, a_2, \ldots$ and $z_0, z_1, z_2, \ldots$ are called the A-sequence and the Z-sequence of the Riordan array $D = (g(x), f(x))$, respectively. Letting $A(x)$ be the generating function of the A-sequence and $Z(x)$ be the generating function of the Z-sequence, we have

$$A(x) = \frac{x}{\bar{f}(x)}, \quad Z(x) = \frac{1}{\bar{f}(x)}\left(1 - \frac{1}{g(\bar{f}(x))}\right). \tag{2}$$

We shall need the following version of Lagrange inversion in the sequel [12, 18].

Theorem 4. *(Lagrange-Bürmann Inversion). Suppose that a formal power series $w = w(t)$ is implicitly defined by the relation $w = t\phi(w)$, where $\phi(t)$ is a formal power series such that $\phi(0) \neq 0$. Then, for any formal power series $F(t)$,*

$$[t^n]F(w(t)) = \frac{1}{n}[t^{n-1}]F'(t)(\phi(t))^n.$$

A consequence of this is that if $u(x) = \sum_{n \geq 0} a_n x^n$ is a power series with $a_0 = 0$, $a_1 \neq 0$, we have

$$[x^n]F(\mathrm{Rev}(u)) = \frac{1}{n}[x^{n-1}]F'(x)\left(\frac{x}{u}\right)^n.$$

Example 5. We use Lagrange inversion to calculate the inverse of the Riordan array

$$\left(\frac{1+rx}{1-rx^2}, \frac{x(1+rx)}{1-rx^2}\right).$$

This matrix is an element of the Bell subgroup of the Riordan group, consisting of matrices of the form $(g(x), xg(x))$. The inverse of such a matrix is given by $\left(\frac{v(x)}{x}, v(x)\right)$, where $v(x) = \mathrm{Rev}(xg(x))$. In this example,

$$g(x) = \frac{1+rx}{1-rx^2} \quad \& \quad v(x) = \mathrm{Rev}\left(\frac{x(1+rx)}{1-rx^2}\right).$$

The (n,k)-th element of $\left(\frac{1+rx}{1-rx^2}, \frac{x(1+rx)}{1-rx^2}\right)^{-1}$ is given by

$$\begin{aligned}
[x^n]\frac{v(x)}{x}v(x)^k &= [x^{n+1}]v(x)^{k+1} \\
&= [x^{n+1}]\left(\operatorname{Rev}\left(\frac{x(1+rx)}{1-rx^2}\right)\right)^{k+1} \\
&= \frac{1}{n+1}[x^n](k+1)x^k\left(\frac{1-rx^2}{1+rx}\right)^{n+1} \\
&= \frac{k+1}{n+1}[x^{n-k}]\left(\frac{1-rx^2}{1+rx}\right)^{n+1} \\
&= \frac{k+1}{n+1}[x^{n-k}]\sum_{j=0}^{n+1}\binom{n+1}{j}(-r)^j x^{2j}\sum_{i=0}^{\infty}\binom{-(n+1)}{i}r^i x^i \\
&= \frac{k+1}{n+1}[x^{n-k}]\sum_{j=0}^{n+1}\binom{n+1}{j}(-r)^j x^{2j}\sum_{i=0}^{\infty}\binom{n+i}{i}(-1)^i r^i x^i \\
&= \frac{k+1}{n+1}\sum_{j=0}^{n+1}\binom{n+1}{j}\binom{2n-k-2j}{n-k-2j}(-r)^{n-k-j}.
\end{aligned}$$

Finally we note that many of the sequences we shall encounter are documented in the On-Line Encyclopedia of Integer Sequences [15, 16]. Sequences in this database are referred to by their *Annnnnn* number. For instance, Pascal's triangle is A007318.

3 Some preliminaries on $T(r)^{-1}$

In this section, we conduct a preliminary study of the inverse matrix $T(r)^{-1}$. First of all, the theory of Riordan arrays allows us to prove the following result [4].

Proposition 6. *The inverse of the Pascal-like matrix* $T(r) = \left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right)$ *is given by*

$$T(r)^{-1} = (1 - v(x), v(x)),$$

where

$$v(x) = \frac{\sqrt{1+2x(2r+1)+x^2} - x - 1}{2r}.$$

Proof. We solve the equation

$$\frac{v(1+rv)}{1-v} = x.$$

Thus we have

$$rv^2 + (x+1)v - x = 0,$$

which yields

$$v(x)=\frac{\sqrt{1+2x(2r+1)+x^2}-x-1}{2r},$$

the solution with $v(0)=0$. Finally equation (1) yields

$$T(r)^{-1}=\left(\frac{1}{\frac{1}{1-v(x)}},v(x)\right)=(1-v(x),v(x)).$$

□

We note that

$$v(x)=\operatorname{Rev}\left(\frac{x(1+rx)}{1-x}\right).$$

In order to motivate our discussion, we now give some results concerning the inverse matrix.

Corollary 7. *The bivariate generating function $g(x,y;r)$ of $T(r)^{-1}$ is given by*

$$g(x,y;r)=\frac{1-v(x)}{1-yv(x)}=\frac{1+2r+x(1-y)+\sqrt{1+2x(2r+1)+x^2}}{y+r-xy(y-1)}.$$

We now wish to characterize $g(x,y;r)$ as a continued fraction.

Proposition 8. *We have*

$$g(x,y;r)=\cfrac{1}{1+\cfrac{x-xy}{1+\cfrac{rx}{1+\cfrac{(r+1)x}{1+\cfrac{rx}{1+\cfrac{(r+1)x}{1+\cdots}}}}}}.$$

Proof. Let $u(x)$ be the solution of the equation

$$u=\cfrac{1}{1+\cfrac{rx}{1+(r+1)xu}},$$

which satisfies $u(0)=1$. Then the continued fraction above is equal to

$$\frac{1}{1+(x-xy)u(x)}.$$

We find that

$$u(x) = \frac{\sqrt{1+2x(2r+1)+x^2}+x-1}{2x(r+1)}.$$

Substituting and evaluating, we find that

$$\frac{1}{1+(x-xy)u(x)} = g(x,y;r).$$

□

Proposition 9. *We have the following expression for the g.f. $g(x,y;r)$ of $T(r)^{-1}$.*

$$g(x,y;r) = \cfrac{1}{1+x(1-y)-\cfrac{r(1-y)x^2}{1+(2r+1)x-\cfrac{r(r+1)x^2}{1+(2r+1)x-\cfrac{r(r+1)x^2}{1+\cdots}}}}.$$

Proof. An equivalence transformation shows that this is the same as the expression in the previous proposition. Otherwise we can let $u(x)$ be a solution of the equation

$$u = \frac{1}{1+(2r+1)x-r(r+1)x^2u},$$

and then compare $g(x,y;r)$ with $\frac{1}{1+x(1-y)-r(1-y)x^2u(x)}$. □

Corollary 10. *The row sums sequence of $T(r)^{-1}$ is equal to the sequence 0^n with terms $\{1,0,0,0,\ldots\}$.*

Proof. The generating function of the row sums is equal to $g(x,1;r)$. Now $g(x,1;r)=1$ and the result follows. □

We can use the continued fraction form of $g(x,y;r)$ to deduce the Hankel transform of the sequence given by the first column of $T(r)^{-1}$.

Proposition 11. *The Hankel transform $h_n(r)$ of the first column sequence of $T(r)^{-1}$ is given by*

$$h_n(r) = r^{\binom{n+1}{2}}(r+1)^{\binom{n}{2}}.$$

Proof. The g.f. of the first column of $T(r)^{-1}$ is given by $g(x,0;r)$. Now

$$g(x,0;r) = \cfrac{1}{1+\cfrac{x}{1+\cfrac{rx}{1+\cfrac{(r+1)x}{1+\cfrac{rx}{1+\cfrac{(r+1)x}{1+\cdots}}}}}}.$$

The proposition now follows from Heilermann's formula for the Hankel transform [10]. $\square$

Note that we have the equivalent expression for the g.f. of $a_n(r)$:

$$g(x,0;r) = \cfrac{1}{1+x-\cfrac{rx^2}{1+(2r+1)x-\cfrac{r(r+1)x^2}{1+(2r+1)x-\cfrac{r(r+1)x^2}{1+\cdots}}}}.$$

We can characterize the first column elements as follows:

Proposition 12. *The first column elements a_n of $T(r)^{-1}$ are given by*

$$a_n(r) = 0^n + (-1)^n \sum_{k=0}^{n-1} \binom{n+k-1}{2k} C_k r^k,$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number (A000108).

Proof. From the above, we have

$$a_n(r) = [x^n](1-v(x)) = 0^n - [x^n]v(x).$$

We find it convenient to express this as

$$a_n(r) = 0^n - [x^{n-1}]\frac{v(x)}{x}.$$

As an intermediate step, we calculate $[x^n]\frac{v(x)}{x}$, subject to letting $n \to n-1$ subsequently. Thus we have

$$\begin{aligned}
[x^n]\frac{v(x)}{x} &= [x^n]\frac{1}{x}v(x) \\
&= [x^{n+1}]v(x) \\
&= [x^{n+1}]\text{Rev}\left(\frac{x(1+rx)}{1-x}\right) \\
&= \frac{1}{n+1}[x^n]\left(\frac{1-x}{1+rx}\right)^{n+1} \qquad \text{(by Lagrange inversion)} \\
&= \frac{1}{n+1}[x^n]\sum_{j=0}^{n+1}\binom{n+1}{j}(-1)^j x^j \sum_{i=0}^{\infty}\binom{-(n+1)}{i} r^i x^i
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n+1}[x^n]\sum_{j=0}^{n+1}\binom{n+1}{j}(-1)^j x^j \sum_{i=0}^{\infty}\binom{n+i}{i}(-1)^i r^i x^i \\
&= \frac{1}{n+1}\sum_{j=0}^{n+1}\binom{n+1}{j}(-1)^j\binom{n+n-j}{n-j}(-1)^{n-j}r^{n-j} \\
&= \frac{(-1)^n}{n+1}\sum_{j=0}^{n+1}\binom{n+1}{j}\binom{2n-j}{n-j}r^{n-j} \\
&= (-1)^n\sum_{j=0}^{n}\binom{n}{j}\binom{2n-j}{n-j}\frac{1}{n-j+1}r^{n-j} \\
&= (-1)^n\sum_{j=0}^{n}\binom{n}{j}\binom{n+j}{j}\frac{1}{j+1}r^{j} \\
&= (-1)^n\sum_{j=0}^{n}\binom{n+j}{2j}C_j r^j.
\end{aligned}
$$

Here, we have used the fact that

$$\binom{n+j}{2j}\binom{2j}{j} = \binom{n+j}{j}\binom{n+j-j}{2j-j} = \binom{n+j}{j}\binom{n}{j}.$$

□

The sequences $|a_{n+1}(r)|$ have a particularly interesting interpretation. We have

$$|a_{n+1}(r)| = \sum_{k=0}^{n}\binom{n+k}{2k}r^k.$$

For instance, $|a_{n+1}(1)|$ gives the large Schröder numbers, and we can similarly link the sequences $|a_{n+1}(r)|$ to generalized Schröder paths in the plane. We can arrange these sequences column-by-column, to get

$$
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
1 & 6 & 15 & 28 & 45 & 66 & \cdots \\
1 & 22 & 93 & 244 & 505 & 906 & \cdots \\
1 & 90 & 645 & 2380 & 6345 & 13926 & \cdots \\
1 & 304 & 4791 & 24868 & 85405 & 229326 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

This array of numbers coincides with A103209, the square array of structurally-different guillotine partitions of a k-dimensional box in R^k by n hyperplanes [1].

We give the following results without proof. The reversion of $x(1-v(x))$ is given by the function $\frac{x(1+2r+\sqrt{1+4rx})}{2(1+r-x)}$. If b_n denotes the sequence with g.f. given by

$$\frac{1}{x}\text{Rev}(x(1-v(x))) = \frac{1+2r+\sqrt{1+4rx}}{2(1+r-x)},$$

then we have

$$b_n = \sum_{k=0}^{n} \frac{n-k+0^{2n-k}}{n+0^{n(n-k)}}(-1)^k \binom{n+k-1}{k} r^k, \tag{3}$$

and the Hankel transform of b_n is given by

$$h_n = (-r)^{n^2}.$$

The coefficients in the expression (3) for b_n are given by a signed version of the Catalan triangle A009766 (the reversal of the Riordan array $(1, xc(-x))$).

We can exhibit the first column terms $a_n(r)$ of $T(r)^{-1}$ as the moments of a family of orthogonal polynomials [2, 3]. This is the content of the following proposition.

Proposition 13. *The first column terms $a_n(r)$ of the inverse matrix $T(r)^{-1}$ are the moments of the family of orthogonal polynomials with coefficient matrix given by*

$$\left(\frac{(1-rx)^2}{1-(2r+1)x+r(r+1)x^2}, \frac{x}{1-(2r+1)x+r(r+1)x^2}\right).$$

Proof. The Riordan array $\left(\frac{(1-rx)^2}{1-(2r+1)x+r(r+1)x^2}, \frac{x}{1-(2r+1)x+r(r+1)x^2}\right)$ is clearly the coefficient array of a family of orthogonal polynomials. In fact, the production matrix [7, 8] of its inverse is given by

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ r & -(2r+1) & 1 & 0 & 0 & 0 & \cdots \\ 0 & r(r+1) & -(2r+1) & 1 & 0 & 0 & \cdots \\ 0 & 0 & r(r+1) & -(2r+1) & 1 & 0 & \cdots \\ 0 & 0 & 0 & r(r+1) & -(2r+1) & 1 & \cdots \\ 0 & 0 & 0 & 0 & r(r+1) & -(2r+1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is easy to verify that the first column of the inverse matrix coincides with that of $T(r)^{-1}$. □

The family of orthogonal polynomials $P_n(x)$ defined above can be defined by the three-term recurrence

$$P_n(x) = (x+(2r+1))P_{n-1}(x) - r(r+1)P_{n-2}(x),$$

with $P_0(x) = 1$, $P_1(x) = x+1$, and $P_2(x) = x^2 + 2x(r+1) + r + 1$.

The relationship between $T(r)$ and this matrix is made evident by the following result:

Proposition 14. *We have the matrix identity*

$$T(r) = \left(1, \frac{x}{1+rx}\right) \cdot \left(\frac{(1-rx)^2}{1-(2r+1)x+r(r+1)x^2}, \frac{x}{1-(2r+1)x+r(r+1)x^2}\right).$$

Equivalently,

$$T(r)^{-1} = \left(\frac{(1-rx)^2}{1-(2r+1)x+r(r+1)x^2}, \frac{x}{1-(2r+1)x+r(r+1)x^2}\right)^{-1} \cdot \left(1, \frac{x}{1-rx}\right).$$

Proof. Straightforward multiplication of Riordan arrays. □

We have the following result linking the polynomials

$$P_n(y;r) = \sum_{k=0}^{n} S_{n,k}(r) y^k$$

and orthogonal polynomials.

Proposition 15. *The polynomials*

$$P_n(y;r) = \sum_{k=0}^{n} S_{n,k}(r) y^k$$

are the moments of the family of orthogonal polynomials whose coefficient array is given by the Riordan array

$$\left(\frac{1-(y+r)x}{1-(r+1)x}, \frac{x}{1-(2r+1)x+r(r+1)x^2}\right).$$

Proof. The generating function of $P_n(y;r)$ is given by $g(x,y;r)$. A straight-forward calculation shows that the first column of

$$\left(\frac{1-(y+r)x}{1-(r+1)x}, \frac{x}{1-(2r+1)x+r(r+1)x^2}\right)^{-1}$$

has g.f. equal to $g(x,y;r)$. □

We finish this section by looking at a series characterization of the matrix $T(r)^{-1}$. The production matrix of $T(r)^{-1}$ is given by

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ r & -(r+1) & 1 & 0 & 0 & 0 & \cdots \\ -r^2 & r(r+1) & -(r+1) & 1 & 0 & 0 & \cdots \\ r^3 & -r^2(r+1) & r(r+1) & -(r+1) & 1 & 0 & \cdots \\ -r^4 & r^3(r+1) & -r^2(r+1) & r(r+1) & -(r+1) & 1 & \cdots \\ r^5 & -r^4(r+1) & r^3(r+1) & -r^2(r+1) & r(r+1) & -(r+1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

corresponding to the fact that the A-sequence of $T(r)^{-1}$ is given by $[x^n]\frac{1-x}{1+rx}$ and the Z-sequence is $-(-r)^n$.

4 The general term of $T(r)^{-1}$

We shall now look at the technicalities of computing a closed expression for the general (n,k)-th term of the matrix inverse $T(r)^{-1}$. In this section, we find two expressions for this general term. Other expressions will be given in later sections.

A customary way to calculate the general term of a Riordan array $(w(x), v(x))$ is to begin by decomposing the matrix as

$$(w(x), v(x)) = (w(x), x) \cdot (1, v(x)).$$

In the case of $T(r)^{-1}$, we get

$$T(r)^{-1} = (1 - v(x), x) \cdot (1, v(x)).$$

The (n,k)-th element of the Riordan array $(1 - v(x), x)$ is given by a_{n-k} for $k \le n$, and 0 otherwise, where $a_n = a_n(r) = [x^n](1 - v(x))$. This leaves us the task of calculating the general term of $(1, v(x))$. We can do this as follows.

$$\begin{aligned}[x^n]v(x)^k &= \frac{1}{n}[x^{n-1}]kx^{k-1}\left(\frac{1-x}{1+rx}\right)^n \\ &= \frac{k}{n}[x^{n-k}]\left(\frac{1-x}{1+rx}\right)^n \\ &= (-1)^{n-k}\frac{k}{n}\sum_{j=0}^{n}\binom{n}{j}\binom{2n-k-j-1}{n-k-j}r^{n-k-j}\end{aligned}$$

Thus we have the following result.

Proposition 16. *The general (n,k)-th element of the inverse array $S = T(r)^{-1}$ is given by*

$$S_{n,k} = \sum_{i=0}^{n} a_{n-i}(r)\left(\frac{k+0^{k+i}}{i+0^{ki}}\sum_{j=0}^{i}\binom{i}{j}\binom{2i-k-j-1}{i-k-j}(-1)^j(-r)^{i-k-j}\right),$$

where

$$a_n(r) = 0^n + (-1)^n\sum_{k=0}^{n-1}\binom{n+k-1}{2k}C_k r^k.$$

To find a second expression for $S_{n,k}$, we use the following decomposition device. We have

$$T(r)^{-1} = (1 - v(x), v(x)) = (1, v(x)) - (v(x), v(x)).$$

The Riordan array $(1, v(x))$ is a standard Riordan array, equal to

$$(1, v(x)) = \left(1, \frac{x(1+rx)}{1-x}\right)^{-1}.$$

The array $(v(x), v(x))$ is given by the matrix

$$\left(\begin{array}{cc} 0 & \cdots \\ \left(\frac{v(x)}{x}, v(x)\right) & \end{array} \right) = \left(\begin{array}{cc} 0 & \cdots \\ \left(\frac{1+rx}{1-x}, \frac{x(1+rx)}{1-x}\right)^{-1} & \end{array} \right)$$

Thus our task is equivalent to finding the general terms of the two matrices $\left(1, \frac{x(1+rx)}{1-x}\right)^{-1}$ and $\left(\frac{1+rx}{1-x}, \frac{x(1+rx)}{1-x}\right)^{-1}$. We have already calculated the general term of $\left(1, \frac{x(1+rx)}{1-x}\right)^{-1}$. We now use Lagrange inversion to find the general term of the matrix

$$\left(\frac{1+rx}{1-x}, \frac{x(1+rx)}{1-x}\right)^{-1} = \left(\frac{v(x)}{x}, v(x)\right).$$

We have

$$\begin{aligned} [x^n]\frac{v(x)}{x}v(x)^k &= [x^{n+1}]v(x)^{k+1} \\ &= \frac{1}{n+1}[x^n](k+1)x^k\left(\frac{1-x}{1+rx}\right)^{n+1} \\ &= \frac{k+1}{n+1}[x^{n-k}]\left(\frac{1-x}{1+rx}\right)^{n+1} \\ &= \frac{k+1}{n+1}\sum_{j=0}^{n+1}\binom{n+1}{j}\binom{2n-k-j}{n-k-j}(-1)^j(-r)^{n-k-j}. \end{aligned}$$

This is the (n,k)-th element of the matrix

$$\left(\frac{v(x)}{x}, v(x)\right) = \left(\frac{1+rx}{1-x}, \frac{x(1+rx)}{1-x}\right)^{-1}.$$

Letting $n \to n-1$ in this latter term, subtracting and simplifying, we get the following proposition.

Proposition 17. *The general (n,k)-th term of the inverse matrix $S(r) = T(r)^{-1}$ is given by*

$$S_{n,k} = (-1)^{n-k}\sum_{j=0}^{n}\binom{n}{j}\binom{2n-k-j}{n-k-j}r^{n-k-j}\left\{\frac{k}{2n-k-j} + \frac{(k+1)(n-k-j)}{(2n-k-j)(2n-k-j-1)}\frac{1}{r}\right\}.$$

Note that care must be taken in evaluating terms such as $\frac{k}{2n-k-j}$. In practice, the term $\frac{k+0^k}{2n-k-j+0^{2n-k-j}}$ should be used, with similar terms for other expressions where division by zero or indeterminate expressions such as $\frac{0}{0}$ would otherwise result.

Now that we can calculate $S_{n,k}$, we can calculate the diagonal sums $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} S_{n-k,k}$. This sequence begins

$$1, -1, r+2, -2r^2-4r-3, 5r^3+12r^2+11r+5, \ldots.$$

Proposition 18. *The g.f. of the diagonal sums of the inverse matrix $T(r)^{-1}$ is given by*

$$\frac{2(1+r)}{1+2r+2x-x^2+(1-x)\sqrt{1+2x(2r+1)+x^2}}.$$

Proof. The g.f. is given by $g(x,x;r)$, which takes the form

$$g(x,x;r)=\cfrac{1}{1+\cfrac{x-x^2}{1+\cfrac{rx}{1+\cfrac{(r+1)x}{1+\cfrac{rx}{1+\cfrac{(r+1)x}{1+\cdots}}}}}}.$$

Thus

$$g(x,x;r)=\frac{1}{1+(x-x^2)u(x)},$$

where $u(x)$ satisfies

$$u=\cfrac{1}{1+\cfrac{rx}{1+(r+1)xu}}.$$

□

We can conjecture the following expression for the Hankel transform of the diagonal sums:

$$h_n=r^{\binom{n}{2}}(r+1)^{\lceil\frac{n^2}{2}\rceil}[x^n]\frac{1+x-(r+2)x^2-rx^3}{1-2(r+2)x^2+r^2x^4}.$$

Viewed as a sequence of polynomials in r, the diagonal sum sequence has a coefficient array that begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ -3 & -4 & -2 & 0 & 0 & 0 & \cdots \\ 5 & 11 & 12 & 5 & 0 & 0 & \cdots \\ -8 & -26 & -45 & -40 & -14 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

whose bi-variate generating function (in x and r) is given by $g(x,x;r)$. We note the presence of the signed Fibonacci numbers in the first column (equal to $[x^n]\frac{1}{1+x-x^2}$) and the Catalan numbers in the sub-diagonal.

5 A binomial conjugate

It is interesting to calculate the matrix

$$B^{-1} \cdot T(r)^{-1} \cdot B,$$

where B is the binomial matrix $\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$. We find the following:

$$B^{-1} \cdot T(r)^{-1} \cdot B = \left(\frac{1+x}{1-rx^2}, \frac{x(1+rx)}{1-rx^2}\right)^{-1}.$$

The matrix $\left(\frac{1+x}{1-rx^2}, \frac{x(1+rx)}{1-rx^2}\right)^{-1}$ has the interesting feature that its row sum sequence has Hankel transform equal to

$$h_n = (-r)^{\binom{n}{2}}(1-r)^{\binom{n+1}{2}}.$$

This can be seen from the form of the g.f. of the row sum sequence, which is given by

$$\cfrac{1}{1+\cfrac{(r-1)x^2}{1+2rx-\cfrac{r(r-1)x^2}{1+2rx-\cfrac{r(r-1)x^2}{1+2rx-\cdots}}}}.$$

The row sum sequence of $B^{-1} \cdot T(r)^{-1} \cdot B$ is the moment sequence for the family of orthogonal polynomials that has

$$\left(\frac{1-2rx+(r^2-1)x^2}{1-2rx+r(r-1)x^2}, \frac{x}{1-2rx+r(r-1)x^2}\right)$$

as coefficient array. The bivariate g.f. of the triangle $B^{-1} \cdot T(r)^{-1} \cdot B$ may be expressed in the following continued fraction format:

$$\cfrac{1}{1+x(1-y)+\cfrac{(r-1)x^2y}{1+2rx-\cfrac{r(r-1)x^2}{1+2rx-\cfrac{r(r-1)x^2}{1+\cdots}}}}.$$

For instance, the diagonal sums of this conjugate matrix have g.f. given by

$$\cfrac{1}{1+x(1-x)+\cfrac{(r-1)x^3}{1+2rx-\cfrac{r(r-1)x^2}{1+2rx-\cfrac{r(r-1)x^2}{1+\cdots}}}}.$$

We can conjecture that this sequence, which begins

$$1, -1, 2, -r-2, 2r^2+3, -5r^3+2r^2-2r-3, 14r^4-10r^3+5r^2+4, \ldots,$$

has Hankel transform

$$h_n = (r-1)^{\binom{n}{2}} r^{2\lfloor \frac{(n-1)^2}{4} \rfloor} [x^n] \frac{1+x-(r+1)x^2-(r-1)x^3}{1-2(r+1)x^2+(r-1)^2x^4}.$$

As an example, for $r=4$, we get

$$h_n = 3^{\binom{n}{2}} 16^{\lfloor \frac{(n-1)^2}{4} \rfloor}(3+(-1)^n)(3^n+1)/8.$$

The corresponding diagonal sum sequence begins

$$1, 1, 15, 3024, 7651584, 236059951104, 87868169917562880, \ldots.$$

We can easily calculate the general term of the conjugate matrix $B^{-1} \cdot T(r)^{-1} \cdot B$ by matrix multiplication, since

$$B^{-1} \cdot T(r)^{-1} \cdot B = \left((-1)^{n-k}\binom{n}{k}\right) \cdot (S_{n,k}) \cdot \left(\binom{n}{k}\right),$$

but we choose to adopt another route. We have

$$\left(\frac{1+x}{1-rx^2}, \frac{x(1+rx)}{1-rx^2}\right) = \left(\frac{1+x}{1+rx}, x\right) \cdot \left(\frac{1+rx}{1-rx^2}, \frac{x(1+rx)}{1-rx^2}\right),$$

and hence

$$B^{-1} \cdot T(r)^{-1} \cdot B = \left(\frac{1+x}{1-rx^2}, \frac{x(1+rx)}{1-rx^2}\right)^{-1} = \left(\frac{1+rx}{1-rx^2}, \frac{x(1+rx)}{1-rx^2}\right)^{-1} \cdot \left(\frac{1+rx}{1+x}, x\right).$$

Now the general (n,k)-th term of the Riordan array $\left(\frac{1+rx}{1-rx^2}, \frac{x(1+rx)}{1-rx^2}\right)^{-1}$ is given by

$$V_{n,k} = \frac{k+1}{n+1} \sum_{j=0}^{n+1} \binom{n+1}{j} \binom{2n-k-2j}{n-k-2j} (-r)^{n-k-j},$$

while the (n,k)-th term $w_{n,k}$ of the Riordan array $\left(\frac{1+rx}{1+x}, x\right)$ is equal to $(1-r)(-1)^{n-k} + r \cdot 0^{n-k}$, for $k \le n$, and 0 otherwise. Thus the general term of the binomial conjugate $B^{-1} \cdot T(r)^{-1} \cdot B$ of $T(r)^{-1}$ is given by $\sum_{i=0}^n V_{n,i} w_{i,k}$.

The intermediate product $T(r)^{-1} \cdot B$ is also of interest. We find that

$$\begin{aligned} T(r)^{-1} \cdot B &= \left(1, \frac{x(1+(r+1)x)}{1+x}\right)^{-1} \\ &= \left(1, \frac{\sqrt{1+2x(2r+1)+x^2}+x-1}{2(r+1)}\right) \\ &= (1, u(x)), \end{aligned}$$

where

$$u(x) = \text{Rev}\frac{x(1+(r+1)x)}{1+x}.$$

The function $u(x)/x$ is the g.f. of the polynomial sequence (in r) that begins

$$1, -r, 2r^2+r, -5r^3-5r^2-r, 14r^4+21r^3+9r^2+r, \ldots,$$

with

$$\frac{u(x)}{x} = \cfrac{1}{1+rx-\cfrac{r(r+1)x}{1+2(r+1)x-\cfrac{r(r+1)x}{1+2(r+1)x-\cdots}}}.$$

The Hankel transform of this polynomial sequence is thus

$$h_n = (r(r+1))^{\binom{n+1}{2}}.$$

The sequence of polynomials with g.f. $\frac{u(x)}{x}$ has a coefficient array that begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 2 & 0 & 0 & 0 & \cdots \\ 0 & -1 & -5 & -5 & 0 & 0 & \cdots \\ 0 & 1 & 9 & 21 & 14 & 0 & \cdots \\ 0 & -1 & -14 & -56 & -84 & -42 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

which is $(-1)^n$ A086810, itself an augmented version of A033282, whose (n,k)-th element is the number of diagonal dissections of a convex n-gon into $k+1$ regions.

The row sums of $T(r)^{-1} \cdot B$ also deserve attention. They have g.f. given by

$$\frac{1}{1-w(x)} = \frac{3+2r-x+\sqrt{1+2x(2r+1)+x^2}}{2(2+r-2)},$$

and begin

$$1, 1, 1-r, 2r^2-r+1, -5r^3-2r+1, 14r^4+7r^3+6r^2-2r+1, \ldots.$$

The g.f. of the row sums sequence can be expressed as the continued fraction

$$\cfrac{1}{1-x+\cfrac{rx^2}{1+(2r+1)x-\cfrac{r(r+1)x^2}{1+(2r+1)x-\cfrac{r(r+1)x^2}{1+\cdots}}}}.$$

This shows that the Hankel transform of the row sums of $T(r)^{-1} \cdot B$ is given by

$$h_n = (-r)^{\binom{n+1}{2}}(-r-1)^{\binom{n}{2}}.$$

The coefficient array of the row sum polynomials begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 2 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 0 & -5 & 0 & 0 & \cdots \\ 1 & -2 & 6 & 7 & 14 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

where we recognize the Catalan numbers in the sub-diagonal. The reversal of this matrix, which begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & -1 & 1 & 0 & 0 & \cdots \\ 0 & -5 & 0 & -2 & 1 & 0 & \cdots \\ 0 & 14 & 7 & 6 & -2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

is an invertible matrix with the following bivariate generating function:

$$\cfrac{1}{1-xy+\cfrac{x^2y}{1+2x+xy-\cfrac{x^2(1+y)}{1+2x+xy-\cfrac{x^2(1+y)}{1+\cdots}}}}.$$

The row sum sequence s_n of this matrix (corresponding to the row sum sequence of $T(1)^{-1} \cdot B$), which begins

$$1, 1, 0, 2, -6, 26, -114, 526, -2502, \ldots$$

therefore has a g.f. given by

$$\cfrac{1}{1-x+\cfrac{x^2}{1+3x-\cfrac{2x^2}{1+3x-\cfrac{2x^2}{1+\cdots}}}},$$

and a Hankel transform equal to

$$h_n = (-1)^n 2^{\binom{n}{2}}.$$

We note that the once shifted sequence s_{n+1} has g.f. given by

$$\cfrac{1}{1+\cfrac{2x^2}{1+3x-\cfrac{2x^2}{1+3x-\cfrac{2x^2}{1+\cdots}}}},$$

and Hankel transform

$$h_n = (-2)^{\binom{n+1}{2}}.$$

This sequence is $(-1)^n$ A114710, where A114710 represents the number of hill-free Schröder paths of length $2n$ that have no horizontal steps on the x-axis.

Calculating the general term of the matrix $T(r)^{-1} \cdot B$ will allow us to give another closed-form expression for the general term of $T(r)^{-1}$. The (n,k)-th element of

$$T(r)^{-1} \cdot B = \left(1, \frac{x(1+(r+1)x}{1+x}\right)$$

is given by

$$\begin{aligned}
[x^n]\left(\operatorname{Rev}\left(\frac{x(1+(r+1)x}{1+x}\right)\right)^n &= \frac{1}{n}[x^{n-1}]kx^{k-1}\left(\frac{1+x}{1+(r+1)x}\right)^n \\
&= \frac{k}{n}[x^{n-k}]\left(\frac{1+x}{1+(r+1)x}\right)^n \\
&= \frac{k}{n}\sum_{k=0}^{n}\binom{n}{j}\binom{2n-k-j-1}{n-k-j}(-r-1)^{n-k-j}.
\end{aligned}$$

Proposition 19. *The general (n,k)-th term $S_{n,k}$ of the inverse matrix $S = T(r)^{-1}$ of the Pascal-like matrix $T(r)$ is given by*

$$S_{n,k} = \sum_{i=0}^{n}\frac{i}{n}\sum_{j=0}^{n}\binom{n}{j}\binom{2n-i-j-1}{n-i-j}\binom{i}{k}(-1)^{i-k}(-r-1)^{n-i-j}.$$

Proof. This follows since

$$T(r)^{-1} = (T(r)^{-1} \cdot B) \cdot B^{-1}.$$

□

As usual, for calculations we use

$$S_{n,k} = \sum_{i=0}^{n} \frac{i + 0^{i+n}}{n + 0^{in}} \sum_{j=0}^{n} \binom{n}{j} \binom{2n-i-j-1}{n-i-j} \binom{i}{k} (-1)^{i-k} (-r-1)^{n-i-j}.$$

We finish by giving one more expression for $S_{n,k}$. Again, a judicious factorization helps to find this. The ease of calculation with Bell matrices suggests the following simple factorization:

$$T(r) = \left(\frac{1}{1+rx}, x\right) \cdot \left(\frac{1+rx}{1-x}, \frac{x(1+rx)}{1-x}\right).$$

Thus

$$T(r)^{-1} = \left(\frac{1+rx}{1-x}, \frac{x(1+rx)}{1-x}\right)^{-1} \cdot (1+rx, x).$$

The general (n,k)-th element of the Bell matrix $\left(\frac{1+rx}{1-x}, \frac{x(1+rx)}{1-x}\right)^{-1}$ can be shown to be

$$\frac{k+1}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{j} \binom{2n-k-j}{n-k-j} (-1)^j (-r)^{n-k-j}.$$

Thus we find the following:

Proposition 20. *The general term $S_{n,k}$ of the inverse matrix $S = T(r)^{-1}$ can be expressed as*

$$S_{n,k} = \sum_{i=0}^{n} \frac{i+1}{n+1} \sum_{j=0}^{n+1} \binom{n+1}{j} \binom{2n-i-j}{n-i-j} \left((1-r)\binom{0}{i-k} + r\binom{1}{i-k}\right) (-1)^j (-r)^{n-i-j}.$$

6 Conclusion

The nature of the inverse of the Pascal-like matrices $T(r) = \left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right)$ has been investigated. In this note, we have confined our study to the nature of the general terms of the inverse, the first column sequence, the row-sum sequence and the diagonal-sum sequence of the inverse matrix. Each investigation has led to interesting results or conjectures. This suggests that the family of triangles $T(r)^{-1}$ merits further study.

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References

[1] E. Ackerman, G. Barequet, R. Y. Pinter, and D. Romik, The number of guillotine partitions in d dimensions, *Inform. Process. Lett.*, **98** (2006), 162–167.

[2] P. Barry and A. Hennessy, Meixner-type results for Riordan arrays and associated integer sequences, *J. Integer Seq.*, **13** (2010), Article 10.9.4.

[3] P. Barry, Riordan arrays, orthogonal polynomials as moments, and Hankel transforms, *J. Integer Seq.*, **14** (2011), Article 11.2.2.

[4] P. Barry, On integer-sequence-based constructions of generalized Pascal triangles, *J. Integer Seq.*, **9** (2006), Article 06.2.4.

[5] G.-S. Cheon, H. Kim, and L. W. Shapiro, Riordan group involution, *Linear Algebra Appl.*, **428** (2008), 941–952.

[6] C. Corsani, D. Merlini, and R. Sprugnoli, Left-inversion of combinatorial sums, *Discrete Math.*, **180** (1998), 107–122.

[7] E. Deutsch, L. Ferrari, and S. Rinaldi, Production matrices, *Adv. in Appl. Math.*, **34** (2005), 101–122.

[8] E. Deutsch, L. Ferrari, and S. Rinaldi, Production matrices and Riordan arrays, *Ann. Comb.*, **13** (2009), 65–85.

[9] T.-X. He and R. Sprugnoli, Sequence characterization of Riordan arrays, *Discrete Math.*, **2009** (2009), 3962–3974.

[10] C. Krattenthaler, Advanced determinant calculus: a complement, *Linear Alg. Appl.*, **411** (2005), 68-166.

[11] J. W. Layman, The Hankel transform and some of its properties, *J. Integer Seq.*, **4** (2001), Article 01.1.5.

[12] D. Merlini, R. Sprugnoli, and M. C. Verri, Lagrange inversion: when and how, *Acta Appl. Math.*, **94** (2006), 233–249.

[13] D. Merlini, R. Sprugnoli, and M. C. Verri, The method of coefficients, *Amer. Math. Monthly*, **114** (2007), 40–57.

[14] L. W. Shapiro, S. Getu, W-J. Woan, and L.C. Woodson, The Riordan group, *Discr. Appl. Math.*, **34** (1991), 229–239.

[15] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*. Published electronically at `http://oeis.org`, 2013.

[16] N. J. A. Sloane, The on-line encyclopedia of integer Sequences, *Notices Amer. Math. Soc.*, **50** (2003), 912–915.

[17] R. Sprugnoli, Riordan arrays and combinatorial sums, *Discrete Math.*, **132** (1994), 267–290.

[18] R. P. Stanley, *Enumerative Combinatorics*, Volume 2, Cambridge University Press, 2001.

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