

# GENERALIZATIONS OF SOME IDENTITIES INVOLVING GENERALIZED SECOND-ORDER INTEGER SEQUENCES

**Zhizheng Zhang and Maixue Liu**

Department of Mathematics, Luoyang Teachers' College, Luoyang, Henan 471022, P. R. China

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In [4], using the method of Carlitz and Ferns [1], some identities involving generalized second-order integer sequences were given. The purpose of this paper is to obtain the more general results.

In the notation of Horadam [2], write  $W_n = W_n(\alpha, \beta, p, q)$  so that

$$W_n = pW_{n-1} - qW_{n-2}, \quad W_0 = \alpha, W_1 = \beta, \quad n \geq 2. \quad (1)$$

If  $\alpha$  and  $\beta$ , assumed distinct, are the roots of  $\lambda^2 - p\lambda + q = 0$ , we have the Binet form (see [2])

$$W_n = A\alpha^n + B\beta^n, \quad (2)$$

where  $A = \frac{\beta - q\alpha}{\alpha - \beta}$  and  $B = \frac{\alpha - q\beta}{\alpha - \beta}$ .

Using this notation, define  $U_n = W_n(0, 1; p, q)$  and  $V_n = W_n(2, p; p, q)$ . The Binet forms for  $U_n$  and  $V_n$  are given by  $U_n = (\alpha^n - \beta^n) / (\alpha - \beta)$  and  $V_n = \alpha^n + \beta^n$ , where  $\{U_n\}$  and  $\{V_n\}$  are the fundamental and primordial sequences, respectively. They have been studied extensively, particularly by Lucas [3].

Throughout this paper, the symbol  $\binom{n}{i, j}$  is defined by  $\binom{n}{i, j} = \frac{n!}{i!j!(n-i-j)!}$ .

To extend the results of [4], we need the following lemma.

**Lemma:** Let  $u = \alpha$  or  $\beta$ , then

$$-q^{m+1} + pq^m u + u^{2(m+1)} = V_m u^{m+2}. \quad (3)$$

**Proof:** Since  $\alpha$  and  $\beta$  are roots of  $\lambda^2 - p\lambda + q = 0$ , we have  $\alpha^2 = p\alpha - q$  and  $\beta^2 = p\beta - q$ . Hence,

$$\begin{aligned} -q^{m+1} + pq^m u + u^{2(m+1)} &= q^m(pu - q) + u^{2(m+1)} = q^m u^2 + u^{2(m+1)} \\ &= u^{m+2}(q^m u^{-m} + u^m) = (\alpha^m + \beta^m)u^{m+2} = V_m u^{m+2}. \end{aligned}$$

This completes the proof of the Lemma.

**Theorem 1:**

$$-q^{m+1}W_k + pq^m W_{k+1} + W_{k+2(m+1)} = V_m W_{k+m+2}. \quad (4)$$

**Proof:** By the Lemma, we have

$$-q^{m+1} + pq^m \alpha + \alpha^{2(m+1)} = V_m \alpha^{m+2} \quad \text{and} \quad -q^{m+1} + pq^m \beta + \beta^{2(m+1)} = V_m \beta^{m+2}.$$

Theorem 1 follows if we multiply both sides of the previous two identities by  $\alpha^k$  and  $\beta^k$ , respectively, and use the Binet form (2).

**Theorem 2:**

$$W_{n+k} = (pq^m)^{-n} \sum_{i+j+s=n} \binom{n}{i, j} (-1)^j q^{(m+1)s} V_m^i W_{(m+2)i+2(m+1)j+k}. \quad (5)$$

$$W_{(m+2)n+k} = V_m^{-n} \sum_{i+j+s=n} \binom{n}{i, j} (-1)^s p^j q^{mj+(m+1)s} W_{2(m+1)i+j+k}. \tag{6}$$

$$W_{2(m+1)n+k} = \sum_{i+j+s=n} \binom{n}{i, j} (-1)^j p^j q^{(m+1)s+mj} V_m^i W_{(m+2)i+j+k}. \tag{7}$$

**Proof:** By using the Lemma and the multinomial theorem, we have

$$(pq^m)^n u^n = \sum_{i+j+s=n} \binom{n}{i, j} (-1)^j q^{(m+1)s} V_m^i u^{(m+2)i+2(m+1)j},$$

$$V_m^n u^{(m+2)n} = \sum_{i+j+s=n} \binom{n}{i, j} (-1)^s p^j q^{mj+(m+1)s} u^{2(m+1)i+j},$$

$$u^{2(m+1)n} = \sum_{i+j+s=n} \binom{n}{i, j} (-1)^j p^j q^{(m+1)s+mj} V_m^i u^{(m+2)i+j}.$$

If we multiply both sides in the preceding identities by  $u^k$  and use the Binet form (2), we obtain (5), (6), and (7), respectively. This completes the proof of Theorem 2.

**Theorem 3:**

$$p^n q^{mn} W_{n+k} - \sum_{j=0}^n \binom{n}{j} (-1)^j q^{(m+1)(n-j)} W_{2(m+1)j+k} \equiv 0 \pmod{V_m}. \tag{8}$$

$$W_{2(m+1)n+k} - (-1)^n q^{mn} W_{2n+k} \equiv 0 \pmod{V_m}. \tag{9}$$

**Proof:** From (5) and (7), by using the decomposition  $\sum_{i+j+s=n} = \sum_{i+j+s=n, i=0} + \sum_{i+j+s=n, i \neq 0}$  and Theorem 2.1 of [4], we get Theorem 3.

**Remark:** When we take  $m = 2, 4,$  and  $8,$  the results of this paper become those of [4].

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### REFERENCES

1. L. Carlitz & H. H. Ferns. "Some Fibonacci and Lucas Identities." *The Fibonacci Quarterly* **8.1** (1970):61-73.
2. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* **3.2** (1965):161-76.
3. E. Lucas. *Theorie des Nombres*. Paris: Blanchard, 1961.
4. Zhizheng Zhang. "Some Identities Involving Generalized Second-Order Integer Sequences." *The Fibonacci Quarterly* **35.3** (1997):265-68.

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