SOME IDENTITIES INVOLVING GENERALIZED SECOND-ORDER INTEGER SEQUENCES

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1. INTRODUCTION

In the notation of Horadam [2], write

$$W_n = W_n(a,b;\, p,q),$$

so that

$$W_n = pW_{n-1} - qW_{n-2}, \quad W_0 = a,\ W_1 = b, \quad n \geq 2. \tag{1.1}$$

If α and β, assumed distinct, are the roots of $\lambda^2 - p\lambda + q = 0$, we have the Binet form [2]:

$$W_n = A\alpha^n + B\beta^n, \tag{1.2}$$

where $A = \frac{b-a\beta}{\alpha-\beta}$ and $B = \frac{a\alpha-b}{\alpha-\beta}$.

The sequence $\{W_n\}$ has been studied in the recent papers of Melham and Shannon [4], [5]. The purpose of this article is to establish some new identities involving W_n by using the method of Carlitz and Ferns [1].

Throughout this paper, the symbol $\binom{n}{i,\,j}$ is defined by $\binom{n}{i,\,j} = \frac{n!}{i!\,j!\,(n-i-j)!}$.

2. THE MAIN RESULTS

Carlitz and Ferns [1] have given a large number of interesting Fibonacci and Lucas identities. By adapting their method to the sequence $\{W_n\}$, we have obtained the following results.

Theorem 2.1:

$$W_{2n+k} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} p^j q^{n-j} W_{j+k}. \tag{2.1}$$

Lemma: Let $u = \alpha$ or β, then

(i) $-pq + (p^2 - q)u = u^3,$ (2.2)

(ii) $-q^3 + pq^2u + u^6 = (p^2 - 2q)u^4,$ (2.3)

(iii) $-q^5 + pq^4u + u^{10} = (p^4 - 4p^2q + 2q^2)u^6,$ (2.4)

(iv) $-q^9 + pq^8u + u^{18} = \Delta u^{10},$ (2.5)

where $\Delta = p^8 - 8p^6q + 20p^4q^2 - 16p^2q^3 + 2q^4$.

Theorem 2.2:

$$(p^2 - q)W_{k+1} - pqW_k = W_{k+3}, \tag{2.6}$$

$$-q^3W_k + pq^2W_{k+1} + W_{k+6} = (p^2 - 2q)W_{k+4}, \tag{2.7}$$

$$-q^5W_k + pq^4W_{k+1} + W_{k+10} = (p^4 - 4p^2q + 2q^2)W_{k+6}, \tag{2.8}$$

$$-q^9W_k + pq^8W_{k+1} + W_{k+18} = \Delta W_{k+10}. \tag{2.9}$$

Theorem 2.3:

$$W_{3n+k} = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^{i+s} p^{2j+s} q^{i+s} W_{i+j+k}, \tag{2.10}$$

$$W_{n+k} = (-q)^{-n} \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^{j} p^{2j+s} q^{s} W_{3i+j+k}. \tag{2.11}$$

Theorem 2.4:

$$W_{n+k} = (pq^2)^{-n} \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^{j} q^{3s} (p^2-2q)^{i} W_{4i+6j+k}, \tag{2.12}$$

$$W_{4n+k} = (p^2-2q)^{-n} \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^{s} p^{j} q^{2j+3s} W_{6i+j+k}, \tag{2.13}$$

$$W_{6n+k} = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^{j} p^{j} q^{3s+2j} (p^2-2q)^{i} W_{4i+j+k}. \tag{2.14}$$

Theorem 2.5:

$$W_{n+k} = (pq^4)^{-n} \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^{j} q^{5s} (p^4-4p^2q+2q^2)^{i} W_{6i+10j+k}, \tag{2.15}$$

$$W_{6n+k} = (p^4-4p^2q+2q^2)^{-n} \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^{s} p^{j} q^{4j+5s} W_{10i+j+k}, \tag{2.16}$$

$$W_{10n+k} = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^{j} p^{j} q^{5s+4j} (p^4-4p^2q+2q^2)^{i} W_{6i+j+k}. \tag{2.17}$$

Theorem 2.6:

$$W_{n+k} = (pq^8)^{-n} \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^{j} q^{9s} \Delta^{i} W_{10i+18j+k}, \tag{2.18}$$

$$W_{10n+k} = \Delta^{-n} \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^{s} p^{j} q^{8j+9s} W_{18i+j+k}, \tag{2.19}$$

$$W_{18n+k} = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^{j} p^{j} q^{9s+8j} \Delta^{i} W_{10i+j+k}. \tag{2.20}$$

3. THE PROOFS OF THE MAIN RESULTS

Since α and β are roots of $\lambda^2 - p\lambda + q = 0$, then

$$\alpha^2 = p\alpha - q, \tag{3.1}$$

$$\beta^2 = p\beta - q. \tag{3.2}$$

Now, by the binomial theorem, we have

$$\alpha^{2n} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} p^{j} q^{n-j} \alpha^{j}, \tag{3.3}$$

$$\beta^{2n} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} p^{j} q^{n-j} \beta^{j}. \tag{3.4}$$

Theorem 2.1 follows if we multiply both sides of (3.3) and (3.4) by α^k and β^k, respectively, and use the Binet form (1.2).

The Lemma can be proved by using (3.1) and (3.2). We prove only (2.3) since the proofs of (2.2), (2.4), and (2.5) are similar.

Proof of (2.3): Using (3.1) and (3.2), we have

$$\begin{aligned}-q^3+pq^2u+u^6 &= q^2(pu-q)+u^4(pu-q)\\ &= q^2u^2+pu^5-qu^4 = q^2u^2+pu^3(pu-q)-qu^4\\ &= (p^2-q)u^4+q^2u^2-pqu^3 = (p^2-q)u^4-qu^2(pu-q) = (p^2-2q)u^4.\end{aligned}$$

This completes the proof of (2.3).

Theorem 2.2 can be proved by using the results of the Lemma and proceeding in the same manner as the proof of Theorem 2.1.

The proofs of Theorems 2.3-2.6 are similar. Therefore, we prove only Theorem 2.4.

Proof of Theorem 2.4: By using (2.3) and the multinomial theorem, we have

$$(pq^2)^n u^n = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^j q^{3s}(p^2-2q)^i u^{4i+6j},$$

$$(p^2-2q)^n u^{4n} = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^s p^j q^{2j+3s} u^{6i+j},$$

$$u^{6n} = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^j p^j q^{3s+2j}(p^2-2q)^i u^{4i+j}.$$

If we multiply both sides in the preceding identities by u^k and use the Binet form (1.2), we obtain (2.12), (2.13), and (2.14), respectively. This completes the proof of Theorem 2.4.

4. SOME CONGRUENCE PROPERTIES

From (2.12), (2.15), and (2.18), by using the decomposition

$$\sum_{i+j+s=n} = \sum_{\substack{i+j+s=n\\ i=0}} + \sum_{\substack{i+j+s=n\\ i\neq 0}},$$

we obtain

Theorem 4.1:

$$p^n q^{2n} W_{n+k} - \sum_{j=0}^{n} \binom{n}{j} (-1)^j q^{3n-3j} W_{6j+k} \equiv 0 \pmod{p^2-2q}, \tag{4.1}$$

$$p^n q^{4n} W_{n+k} - \sum_{j=0}^{n} \binom{n}{j} (-1)^j q^{5n-5j} W_{10j+k} \equiv 0 \pmod{p^4-4p^2q+2q^2}, \tag{4.2}$$

$$p^n q^{8n} W_{n+k} - \sum_{j=0}^{n} \binom{n}{j} (-1)^j q^{9n-9j} W_{18j+k} \equiv 0 \pmod{\Delta}. \tag{4.3}$$

From (2.14), (2.17), and (2.20), by also using the above decomposition and Theorem 2.1, we get the following result:

Theorem 4.2:

$$W_{6n+k} - (-1)^n q^{2n} W_{2n+k} \equiv 0 \pmod{p^2 - 2q}, \tag{4.4}$$

$$W_{10n+k} - (-1)^n q^{4n} W_{2n+k} \equiv 0 \pmod{p^4 - 4p^2q + 2q^2}, \tag{4.5}$$

$$W_{18n+k} - (-1)^n q^{8n} W_{2n+k} \equiv 0 \pmod{\Delta}. \tag{4.6}$$

5. A REMARK

Some of the results in this paper are not as "practical" as others. For example, if we put $n = 10$ and $k = 0$ in (2.13), then we seek to find W_{40}. However, on the right-hand side, we need to know $W_6, W_{12}, W_{18}, \ldots, W_{60}$ (and many other terms) in order to find W_{40}. In contrast, (2.14) is more practical since, in order to find W_{60}, we need to know the value of terms whose subscripts are much less than 60.

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