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A GENERAL SUMMATION FORMULA

BY

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1. In a recent paper[1] I have introduced certain polynomials $\overset{\nu}{x}_{\omega n}$, defined by having to satisfy the equations

$$\Delta \overset{\nu}{x}_{\omega n} = \nu \overset{\nu-1}{x}_{\omega, n-1} \tag{1}$$

$$\underset{\omega}{\Delta} \overset{\nu}{x}_{\omega n} = \nu \overset{\nu-1}{x}_{\omega n}, \tag{2}$$

besides the initial conditions

$$\overset{0}{x}_{\omega n} = 1, \qquad \overset{\nu}{x}_{\omega 0} = x(x-\omega) \dots (x-\nu\omega+\omega). \tag{3}$$

These polynomials are the natural instrument for dealing with some of the most important problems of the theory of interpolation, such as expressing a difference of a certain order and with a given interval in terms of differences with another given interval, or expressing a sum of a certain order and with a given interval in terms of sums with another given interval. In the present paper we intend to occupy ourselves with the latter problem.

2. Proceeding in a way similar to that followed in deriving the generalized Euler—Maclaurin summation-formula[2], we begin by defining a function $\overset{-\nu}{x}_{\omega r}$, distinguished by a bar above the argument, r being a positive integer, which in the semi-closed interval $0 \leq x < r$ is identical with $\overset{\nu}{x}_{\omega r}$, that is

[1] On a Generalization of Nörlunds Polynomials. Det Kgl. Danske Videnskabernes Selskab, Mathematisk-fysiske Meddelelser, VII, 5 (1926). Quoted below as "G. N. P."

[2] Nörlund: Differenzenrechnung, pp. 154—161; Transactions of the American Mathematical Society, Vol. 25, No. 1, pp. 36—46.

$$\overline{x}^{\nu}_{\omega r} = x^{\nu}_{\omega r} \qquad (0 \leq x < r), \tag{4}$$

while for all x (positive, negative or zero)

$$\Delta^r \overline{x}^{\nu}_{\omega r} = 0. \tag{5}$$

As the latter relation is a linear relation between $\overline{x}^{\nu}_{\omega r}$, $\overline{x+1}^{\nu}_{\omega r}, \ldots \overline{x+r}^{\nu}_{\omega r}$, it is seen that $\overline{x}^{\nu}_{\omega r}$ is completely determined by (4) and (5).

It is easy to form an explicit expression for the calculation of $\overline{x}^{\nu}_{\omega r}$. Let k be an integer (positive, negative or zero), and let $0 \leq \theta < 1$. We may, then, always write $x = k + \theta$, and it may be proved that, putting $\binom{\nu}{s} = 0$ for $s > \nu$,

$$\overline{k+\theta}^{\nu}_{\omega r} = \sum_{s=0}^{r-1} \binom{\nu}{s} k^{(s)} \theta^{\nu-s}_{\omega, r-s} \qquad (0 \leq \theta < 1). \tag{6}$$

We only have to show that this expression satisfies (4) and (5). Now, performing Δ^r on both sides of (6) with respect to k, all the terms on the right vanish, so that (5) is satisfied. As regards (4), we assume for a moment $0 \leq k + \theta < r$, so that $0 \leq k \leq r - 1$. As $k^{(s)}$ then vanishes for $s > k$, (6) becomes

$$\overline{k+\theta}^{\nu}_{\omega r} = \sum_{s=0}^{k} \binom{\nu}{s} k^{(s)} \theta^{\nu-s}_{\omega, r-s}$$

or, as $\binom{\nu}{s}$ vanishes for $s > \nu$, and $k^{(s)}$ for $s > k$,

$$\overline{k+\theta}^{\nu}_{\omega r} = \sum_{s=0}^{\nu} \binom{\nu}{s} k^{(s)} \theta^{\nu-s}_{\omega, r-s},$$

or, by G. N. P. (5),

$$\overline{k+\theta}^{\nu}_{\omega r} = (k+\theta)^{\nu}_{\omega r},$$

so that (4) is also satisfied.

It should be noted that if $\nu < r$, (4) is valid for all x, as in that case $\Delta^r \overset{\nu}{x}_{\omega r} = 0$ for all x, so that (5) is satisfied by $\overset{\nu}{x}_{\omega r}$ itself. The formula (6), though valid in all cases, need therefore only be applied if $\nu \geq r$.

3. We are now able to prove that

$$\Delta \overline{x}^{\nu}_{\omega r} = \nu \overline{x}^{\nu-1}_{\omega, r-1}; \tag{7}$$

as $\overline{x}^{\nu}_{\omega 0}$ has not yet been defined, we may put

$$\overline{x}^{\nu}_{\omega 0} = 0, \tag{8}$$

so that (7), owing to (5), also holds for $r = 1$.

We need only difference (6) with respect to k, the result being

$$\begin{aligned}
\Delta \overline{k + \theta}^{\nu}_{\omega r} &= \sum_{s=0}^{r-1} \binom{\nu}{s} s k^{(s-1)} \theta^{\nu-s}_{\omega, r-s} \\
&= \nu \sum_{s=1}^{r-1} \binom{\nu-1}{s-1} k^{(s-1)} \theta^{\nu-s}_{\omega, r-s} \\
&= \nu \sum_{s=0}^{r-2} \binom{\nu-1}{s} k^{(s)} \theta^{\nu-s-1}_{\omega, r-s-1} \\
&= \nu \overline{k + \theta}^{\nu-1}_{\omega, r-1},
\end{aligned}$$

or (7).

4. It follows evidently from (4) and (6) that the function $\overline{x}^{\nu}_{\omega r}$ is continuous in the interval $0 \leq x < r$ and also in the interval between any two consecutive integers. If $\nu < r$, $\overline{x}^{\nu}_{\omega r}$ is identical with $\overset{\nu}{x}_{\omega r}$ and, therefore, continuous for all x. If $\nu > r$, it can be proved that $\overline{x}^{\nu}_{\omega r}$ is still continuous for all x; but in the case of $\nu = r$ we shall arrive at the result that $\overline{x}^{r}_{\omega r}$ possesses points of discon-

tinuity when x is an integer $\geq r$ or ≤ 0; other points of discontinuity do not exist.

Let us first assume $\nu > r$. The relation (5), written in full, is

$$\overline{x+r}^{\,\nu}_{\omega r} - \binom{r}{1}\overline{x+r-1}^{\,\nu}_{\omega r} + \binom{r}{2}\overline{x+r-2}^{\,\nu}_{\omega r} - \ldots + (-1)^r \overline{x}^{\,\nu}_{\omega r} = 0. \quad (9)$$

This is valid for all x. Putting $x = 0$, and making use of (4), we have

$$\overline{r}^{\,\nu}_{\omega r} - \binom{r}{1}(r-1)^{\nu}_{\omega r} + \binom{r}{2}(r-2)^{\nu}_{\omega r} - \ldots + (-1)^r 0^{\nu}_{\omega r} = 0$$

or

$$\Delta^r 0^{\nu}_{\omega r} + \overline{r}^{\,\nu}_{\omega r} - r^{\nu}_{\omega r} = 0.$$

But if $\nu > r$, $\Delta^r 0^{\nu}_{\omega r} = \nu^{(r)} 0^{\nu - r}_{\omega 0}$ vanishes, and we have, therefore,

$$\overline{r}^{\,\nu}_{\omega r} = r^{\nu}_{\omega r} \qquad (\nu > r).$$

It follows that $\overline{x}^{\,\nu}_{\omega r}$ is, for $\nu > r$, continuous in the closed interval $0 \leq x \leq r$, and (9) shows clearly that $\overline{x}^{\,\nu}_{\omega r}$ must then be continuous for all x, as was to be proved.

It remains to investigate the case $\nu = r$. We obtain by (6) for $\nu = r$

$$\overline{k+\theta}^{\,r}_{\omega r} = \sum_{s=0}^{r-1}{}' \binom{r}{s} k^{(s)} \theta^{r-s}_{\omega, r-s}$$

$$= \sum_{s=0}^{r}{}' \binom{r}{s} k^{(s)} \theta^{r-s}_{\omega, r-s} - k^{(r)}$$

or, by G. N. P. (5),

$$\overline{k+\theta}^{\,r}_{\omega r} = (k+\theta)^{r}_{\omega r} - k^{(r)} \qquad (0 \leq \theta < 1). \qquad (10)$$

Hence we have, for $\theta = 0$,

$$\overline{k}^r_{\omega r} = k^r_{\omega r} - k^{(r)} \tag{11}$$

and for $\theta \to 1$

$$\overline{k+1-0}^r_{\omega r} = (k+1)^r_{\omega r} - k^{(r)}$$

or, replacing k by $k-1$,

$$\overline{k-0}^r_{\omega r} = k^r_{\omega r} - (k-1)^{(r)}. \tag{12}$$

Subtracting (12) from (11), we have

$$\begin{aligned}\overline{k}^r_{\omega r} - \overline{k-0}^r_{\omega r} &= (k-1)^{(r)} - k^{(r)} \\ &= -\Delta (k-1)^{(r)}\end{aligned}$$

or finally

$$\overline{k}^r_{\omega r} - \overline{k-0}^r_{\omega r} = -r(k-1)^{(r-1)}. \tag{13}$$

This expression shows that $\overline{x}^r_{\omega r}$ has discontinuities at all the points $x = r,\ r+1,\ r+2, \ldots$ and $x = 0,\ -1,\ -2, \ldots$; other discontinuities do not exist, as the expression on the right of (13) vanishes, if k has one of the values $1, 2, \ldots, r-1$ $(r > 1)$.

It is worth noting that the amount of the discontinuity, or the height of the "jump", is independent of ω, as appears from (13).

5. The relation G. N. P. (34), or

$$x^\nu_{\omega n} = (-1)^\nu\, (n-x)^\nu_{-\omega, n} \tag{14}$$

also holds, with an obvious reservation, for the functions $\overline{x}^\nu_{\omega r}$. We begin by noting that instead of (6) we may use the following relation for the calculation of $\overline{x}^\nu_{\omega r}$

$$\overline{r-k-1+\theta}^\nu_{\omega r} = (-1)^\nu \sum_{s=0}^{r-1} \binom{\nu}{s} k^{(s)} (1-\theta)^{\nu-s}_{-\omega, r-s} \tag{15}$$

where k and θ have the same meanings as before. For, putting $x = r - k - 1 + \theta$, it may be proved that the expression (15) satisfies (4) and (5), as we proceed to show.

As regards (5), it is seen at once that, differencing r times on both sides with respect to $-k$, all the terms on the right vanish, so that (5) is satisfied.

Next, we assume $0 \leq x < r$, that is $0 \leq r - k - 1 + \theta < r$, so that $0 \leq r - k - 1 \leq r - 1$, or $0 \leq k \leq r - 1$. If k is comprised between these limits we may, as above, replace the upper limit of summation in (15) by ν, so that

$$\overline{r-k-1+\theta}^{\nu}_{\omega r} = (-1)^{\nu} \sum_{s=0}^{\nu}{}' \binom{\nu}{s} k^{(s)} (1-\theta)^{\nu-s}_{-\omega, r-s}$$

or, by G. N. P. (5),

$$\overline{r-k-1+\theta}^{\nu}_{\omega r} = (-1)^{\nu} (k+1-\theta)^{\nu}_{-\omega, r}$$

or finally, by (14),

$$\overline{r-k-1+\theta}^{\nu}_{\omega r} = (r-k-1+\theta)^{\nu}_{\omega r},$$

so that also (4) is satisfied.

Having thus established (15), we may, if we exclude the value $\theta = 0$, replace θ by $1-\theta$ in (15). Changing the sign of ω, we thus obtain for $0 < \theta < 1$

$$\overline{r-k-\theta}^{\nu}_{-\omega, r} = (-1)^{\nu} \sum_{s=0}^{r-1}{}' \binom{\nu}{s} k^{(s)} \theta^{\nu-s}_{\omega, r-s}. \tag{16}$$

But comparison of this relation and (6) shows that

$$\overline{x}^{\nu}_{\omega r} = (-1)^{\nu}\, \overline{r-x}^{\nu}_{-\omega, r}, \tag{17}$$

provided that x is not an integer. If x is an integer, (17) is still valid for $\nu \neq r$, as in that case $\overline{x}^{\nu}_{\omega r}$ is continuous for all x. But the case $\nu = r$ must be treated by putting

$\theta = 0$ in (6) and letting $\theta \to 0$ in (16), the result being the relation

$$\bar{k}^{r}_{\omega r} = (-1)^r \, \overline{r - k - 0}^{\,r}_{-\omega, r}. \tag{18}$$

6. We shall now assume that

$$0 < \omega < 1, \qquad 0 \leq \theta < 1 - \omega, \tag{19}$$

so that $0 < \theta + \omega < 1$. It follows that (6) remains valid, if θ is replaced by $\theta + \omega$, and we therefore obtain

$$\begin{aligned} \underset{\omega}{\Delta} \overline{k + \theta}^{\,\nu}_{\omega r} &= \sum_{s=0}^{r-1}{}' \binom{\nu}{s} k^{(s)} (\nu - s)\, \theta^{\nu - s - 1}_{\omega, r - s} \\ &= \nu \sum_{s=0}^{r-1}{}' \binom{\nu - 1}{s} k^{(s)}\, \theta^{\nu - s - 1}_{\omega, r - s} \\ &= \nu\, \overline{k + \theta}^{\,\nu - 1}_{\omega r} \end{aligned}$$

or

$$\underset{\omega}{\Delta} \bar{x}^{\nu}_{\omega r} = \nu\, \bar{x}^{\nu - 1}_{\omega r} \qquad \begin{pmatrix} 0 < \omega < 1 \\ k - 1 \leq x < k - \omega \end{pmatrix} \tag{20}$$

where k has one of the values $0, \pm 1, \pm 2, \ldots$

If, in this formula, we let $\omega \to 0$, the symbol $\underset{0}{\Delta}$ may be replaced by the symbol of Differentiation D at every point where the derivative exists; at points where it does not exists $\underset{0}{\Delta}$ means the differential coefficient to the right.

For $x \to k - \omega$ we find from (20)

$$\frac{1}{\omega}\left(\overline{k - 0}^{\,\nu}_{\omega r} - \overline{k - \omega}^{\,\nu}_{\omega r}\right) = \nu\, \overline{k - \omega}^{\,\nu - 1}_{\omega r}$$

whence

$$\underset{\omega}{\Delta} \overline{k - \omega}^{\,\nu}_{\omega r} = \nu\, \overline{k - \omega}^{\,\nu - 1}_{\omega r} + \frac{1}{\omega}\left(\bar{k}^{\nu}_{\omega r} - \overline{k - 0}^{\,\nu}_{\omega r}\right). \tag{21}$$

It follows that, if $\nu \neq r$, (20) is still valid for $x = k-\omega$, while in the case $\nu = r$ we obtain, by (13),

$$\underset{\omega}{\Delta}\,\overline{k-\omega}_{\omega r}^{\,r} = r\,\overline{k-\omega}_{\omega r}^{\,r-1} - \frac{r}{\omega}(k-1)^{(r-1)}. \tag{22}$$

The supplementary term in (22), representing the discontinuity, vanishes for $k = 1, 2, \ldots, r-1$, $(r > 1)$, so that (20) is still valid for $\nu = r$, $x = k-\omega$, if k has one of the values $1, 2, \ldots, r-1$, $(r > 1)$.

7. After these preliminaries, the problem of summation may be attacked. We assume henceforth that $\frac{1}{\omega}$ is a positive integer > 1 whence follows, in particular, that the condition $0 < \omega < 1$, implied in (20), is satisfied.

Let h be a parameter, positive, negative or zero, of which we may dispose afterwards, and let us consider the expression

$$V_s^{(\nu)} = -\omega \sum_{\mu=0}^{\frac{1}{\omega}-1} \frac{\overline{\mu\omega+h}_{\omega s}^{\,\nu+s}}{(\nu+s)!}\,\underset{\omega}{\Delta}^{\nu+1} f(x+1-\omega-\mu\omega). \tag{23}$$

This expression may be transformed in the following way which is equivalent with partial summation. As $\underset{\omega}{\Delta}^{\nu+1} = \underset{\omega}{\Delta}^{\nu}\underset{\omega}{\Delta}$, we have

$$V_s^{(\nu)} = -\sum_{\mu=0}^{\frac{1}{\omega}-1} \frac{\overline{\mu\omega+h}_{\omega s}^{\,\nu+s}}{(\nu+s)!}\,\underset{\omega}{\Delta}^{\nu} f(x+1-\mu\omega)$$
$$+\sum_{\mu=0}^{\frac{1}{\omega}-1} \frac{\overline{\mu\omega+h}_{\omega s}^{\,\nu+s}}{(\nu+s)!}\,\underset{\omega}{\Delta}^{\nu} f(x+1-\omega-\mu\omega)$$

or, if in the first sum we replace μ by $\mu+1$,

$$V_s^{(\nu)} = -\sum_{\mu=-1}^{\frac{1}{\omega}-2} \frac{\overline{\mu\omega+\omega+h_{\omega s}}^{\,\nu+s}}{(\nu+s)!} \mathop{\Delta}_{\omega}^{\nu} f(x+1-\omega-\mu\omega)$$
$$+\sum_{\mu=0}^{\frac{1}{\omega}-1} \frac{\overline{\mu\omega+h_{\omega s}}^{\,\nu+s}}{(\nu+s)!} \mathop{\Delta}_{\omega}^{\nu} f(x+1-\omega-\mu\omega)$$

which may be reduced to

$$V_s^{(\nu)} = -\omega \sum_{\mu=0}^{\frac{1}{\omega}-1} \frac{\mathop{\Delta}_{\omega} \overline{\mu\omega+h_{\omega s}}^{\,\nu+s}}{(\nu+s)!} \mathop{\Delta}_{\omega}^{\nu} f(x+1-\omega-\mu\omega)$$
$$+\frac{\overline{1+h_{\omega s}}^{\,\nu+s}}{(\nu+s)!} \mathop{\Delta}_{\omega}^{\nu} f(x) - \frac{\bar{h}_{\omega s}^{\,\nu+s}}{(\nu+s)!} \mathop{\Delta}_{\omega}^{\nu} f(x+1).$$

Assuming $s \geq 1$, we have, by (7),

$$\overline{1+h_{\omega s}}^{\,\nu+s} = \bar{h}_{\omega s}^{\,\nu+s} + (\nu+s)\, \bar{h}_{\omega, s-1}^{\,\nu+s-1},$$

so that $V_s^{(\nu)}$ may be written

$$\left.\begin{aligned} V_s^{(\nu)} &= -\omega \sum_{\mu=0}^{\frac{1}{\omega}-1} \frac{\mathop{\Delta}_{\omega} \overline{\mu\omega+h_{\omega s}}^{\,\nu+s}}{(\nu+s)!} \mathop{\Delta}_{\omega}^{\nu} f(x+1-\omega-\mu\omega) \\ &\quad + \frac{\bar{h}_{\omega, s-1}^{\,\nu+s-1}}{(\nu+s-1)!} \mathop{\Delta}_{\omega}^{\nu} f(x) - \frac{\bar{h}_{\omega s}^{\,\nu+s}}{(\nu+s)!} \Delta \mathop{\Delta}_{\omega}^{\nu} f(x). \end{aligned}\right\} \quad (24)$$

We now assume that $\nu \geq 1$ and that h is a multiple of ω, that is, $h = p\omega$ where p denotes an integer (positive, negative or zero). In that case we have, according to No. 6, as $\nu + s \neq s$,

$$\mathop{\Delta}_{\omega} \overline{\mu\omega+h_{\omega s}}^{\,\nu+s} = (\nu+s)\, \overline{\mu\omega+h_{\omega s}}^{\,\nu+s-1},$$

so that we obtain from (24), by (23),

$$V_s^{(\nu)} = V_s^{(\nu-1)} + \frac{\overline{h}_{\omega,\,s-1}^{\nu+s-1}}{(\nu+s-1)!} \mathop{\Delta}_{\omega}^{\nu} f(x) - \frac{\overline{h}_{\omega s}^{\nu+s}}{(\nu+s)!} \Delta \mathop{\Delta}_{\omega}^{\nu} f(x). \quad (25)$$

Performing the operation Δ^{s-1} on both sides of (25) and summing from $s = 1$ to $s = r$, we find, putting

$$R_\nu = \sum_{s=1}^{r} \Delta^{s-1} V_s^{(\nu)} \quad (26)$$

and taking account of (8),

$$R_\nu = R_{\nu-1} - \frac{\overline{h}_{\omega r}^{\nu+r}}{(\nu+r)!} \Delta^r \mathop{\Delta}_{\omega}^{\nu} f(x). \quad (27)$$

Summing on both sides of this equation from $\nu = 1$ to $\nu = m$, we obtain

$$R_m = R_0 - \sum_{\nu=1}^{m} \frac{\overline{h}_{\omega r}^{\nu+r}}{(\nu+r)!} \Delta^r \mathop{\Delta}_{\omega}^{\nu} f(x). \quad (28)$$

It remains to investigate R_0. According to (24)

$$\left.\begin{aligned} V_s^{(0)} &= -\omega \sum_{\mu=0}^{\frac{1}{\omega}-1} \frac{\mathop{\Delta}_{\omega} \overline{\mu\omega + h_{\omega s}}^{\,s}}{s!} f(x+1-\omega-\mu\omega) \\ &\quad + \frac{\overline{h}_{\omega,\,s-1}^{s-1}}{(s-1)!} f(x) - \frac{\overline{h}_{\omega s}^{s}}{s!} \Delta f(x). \end{aligned}\right\} \quad (29)$$

Now it follows from No. 6 that we have generally

$$\mathop{\Delta}_{\omega} \overline{\mu\omega + h_{\omega s}}^{\,s} = s\, \overline{\mu\omega + h_{\omega s}}^{\,s-1},$$

exception being made at the point

$$\mu\omega + h = k - \omega \tag{30}$$

where the term

$$-\frac{s}{\omega}(k-1)^{(s-1)}$$

must be added to the right-hand side, producing a term

$$+\binom{k-1}{s-1} f(x+1-k+h)$$

in $V_s^{(0)}$. We therefore oblain from (29), performing the operation Δ^{s-1} on both sides and summing from $s = 1$ to $s = r$

$$\left.\begin{aligned} R_0 = -\omega \sum_{s=1}^{r}{}' \sum_{\mu=0}^{\frac{1}{\omega}-1}{}' \frac{\overline{\mu\omega+h}_{\omega s}^{\,s-1}}{(s-1)!} \Delta^{s-1} f(x+1-\omega-\mu\omega) \\ + \sum_{s=1}^{r}{}' \binom{k-1}{s-1} \Delta^{s-1} f(x+1-k+h) - \frac{\bar{h}_{\omega r}^{r}}{r!} \Delta^{r} f(x). \end{aligned}\right\} \tag{31}$$

This expression may be simplified, if we assume $0 \leq h < r$. In that case $\bar{h}_{\omega r}^{r}$ may be replaced by $h_{\omega r}^{r}$, and it may be concluded from (30) that $r \geq k \geq 1$, so that

$$\sum_{s=1}^{r}{}' \binom{k-1}{s-1} \Delta^{s-1} f(x+1-k+h)$$

$$= \sum_{s=1}^{k}{}' \binom{k-1}{s-1} \Delta^{s-1} f(x+1-k+h) = f(x+h),$$

as follows from the identity

$$\begin{aligned} f(x+h) &= (1+\Delta)^{k-1} E^{-k+1} f(x+h) \\ &= (1+\Delta)^{k-1} f(x+1-k+h). \end{aligned}$$

We thus obtain from (31)

$$\left.\begin{aligned} R_0 = -\omega \sum_{s=1}^{r} \sum_{\mu=0}^{\frac{1}{\omega}-1} \frac{\overline{\mu\omega + h}_{\omega s}^{\,s-1}}{(s-1)!} \Delta^{s-1} f(x+1-\omega-\mu\omega) \\ + f(x+h) - \frac{h_{\omega r}^{r}}{r!} \Delta^{r} f(x). \end{aligned}\right\} \quad (32)$$

Finally, we insert this expression in (28) where $\bar{h}_{\omega r}^{\nu+r}$ may now be replaced by $h_{\omega r}^{\nu+r}$. Noting that $\overline{\mu\omega + h}_{\omega s}^{\,s-1} = (\mu\omega + h)_{\omega s}^{s-1}$ according to No. 2, and writing $\mu\omega = 1 - \omega - \nu\omega$, we find

$$\left.\begin{aligned} f(x+h) = \omega \sum_{s=1}^{r} \sum_{\nu=0}^{\frac{1}{\omega}-1} \frac{(h+1-\omega-\nu\omega)_{\omega s}^{s-1}}{(s-1)!} \Delta^{s-1} f(x+\nu\omega) \\ + \sum_{\nu=0}^{m} \frac{h_{\omega r}^{\nu+r}}{(\nu+r)!} \Delta^{r} \mathop{\Delta}_{\omega}^{\nu} f(x) + R_m \end{aligned}\right\} \quad (33)$$

where

$$0 \leq h = p\omega < r \quad (34)$$

and, by (26) and (23),

$$R_m = -\omega \sum_{s=1}^{r} \sum_{\nu=0}^{\frac{1}{\omega}-1} \frac{\overline{h+1-\omega-\nu\omega}_{\omega s}^{\,m+s}}{(m+s)!} \Delta^{s-1} \mathop{\Delta}_{\omega}^{m+1} f(x+\nu\omega). \quad (35)$$

8. The formula (33) is a generalization of Euler-Maclaurin's formula which is obtained for $r = 1$, $\omega \to 0$.

Our formula may be transformed in several ways. Thus, by G. N. P. (38), it may be written

$$\left.\begin{aligned} f(x+h) &= \omega \sum_{s=0}^{r-1} \sum_{\nu=0}^{\frac{1}{\omega}-1} \binom{h-\nu\omega}{s} \Delta^s f(x+\nu\omega) \\ &+ \sum_{\nu=0}^{m} \frac{h_{\omega r}^{\nu+r}}{(\nu+r)!} \Delta^r \underset{\omega}{\Delta}^{\nu} f(x) + R_m . \end{aligned}\right\} \quad (36)$$

The formula is in reality an identity between the $m+\frac{r}{\omega}+1$ equidistant values of $f(t)$

$$f(x),\ f(x+\omega),\ f(x+2\omega),\ \ldots\ f(x+m\omega+r). \quad (37)$$

If, for $f(t)$, we take a polynomial of degree not exceeding m, we have $R_m = 0$, and comparison with G. N. P. (59) shows that we have

$$\omega \sum_{s=0}^{r-1} \sum_{\nu=0}^{\frac{1}{\omega}-1} \binom{h-\nu\omega}{s} \Delta^s f(x+\nu\omega) = \sum_{\nu=0}^{r-1} \frac{h_{\omega r}^{\nu}}{\nu!} \Delta^r \underset{\omega}{\Delta}^{\nu-r} f(x). \quad (38)$$

This relation has thus been proved for a polynomial. In order to extend this formula to other functions than polynomials, we note that $\Delta^r \underset{\omega}{\Delta}^{-r} f(x)$ has a definite meaning whether $f(x)$ is a polynomial or not, as the various meanings of $\underset{\omega}{\Delta}^{-r} f(x)$ only differ by a function which is cancelled by the subsequent application of Δ^r, $\frac{1}{\omega}$ being an integer. We have, in fact

$$\Delta^r \underset{\omega}{\Delta}^{-r} = \left[\left(1+\omega \underset{\omega}{\Delta}\right)^{\frac{1}{\omega}} - 1\right]^r \underset{\omega}{\Delta}^{-r} \quad (39)$$

or, on comparison with G. N. P. (17),

$$\Delta^r \underset{\omega}{\Delta}^{-r} = \sum_{s=0}^{\left(\frac{1}{\omega}-1\right)r} \frac{0^s_{\omega,-r}}{s!} \underset{\omega}{\Delta}^{s} \quad (40)$$

which may also, by G. N. P. (39), be written

$$\Delta^r \underset{\omega}{\Delta}^{-r} = \sum_{s=0}^{\left(\frac{1}{\omega}-1\right)r} \frac{\Delta^r 0^{s+r}_{\omega 0}}{(s+r)!} \underset{\omega}{\Delta}^{s}, \tag{41}$$

so that the operation $\Delta^r \underset{\omega}{\Delta}^{-r}$ has a well defined meaning whether it is applied to a polynomial or to any other function.

It is now seen that (38) is an identity between the $\frac{r}{\omega}$ values of $f(t)$

$$f(x),\ f(x+\omega),\ f(x+2\omega),\ \ldots\ f(x+r-\omega),$$

both sides being linear functions of these values with coefficients that are independent of $f(t)$. It follows that although (38) was only proved for polynomials, it is valid for any function $f(t)$.

We may therefore write (33) or (36) in the form

$$f(x+h) = \sum_{\nu=0}^{r+m} \frac{h^{\nu}_{\omega r}}{\nu!} \Delta^r \underset{\omega}{\Delta}^{\nu-r} f(x) + R_m. \tag{42}$$

For $\omega \to 0$ we obtain from this a formula due to Nörlund (Differenzenrechnung, p. 160).

9. If we impose certain restrictions on $f(t)$, the remainder R_m may be put into the convenient form

$$R_m = \omega \sum_{\nu=0}^{\infty} \frac{\overline{h-\omega-\nu\omega}^{\,m+r}_{\omega r}}{(m+r)!} \Delta^r \underset{\omega}{\Delta}^{m+1} f(x+\nu\omega). \tag{43}$$

The assumption we make about $f(t)$ is that the expression

$$\sum_{\nu=0}^{\infty} \overline{-\nu\omega}_{\omega r}^{\,m+r} f(z+\nu\omega) \tag{44}$$

must be convergent for $z \geq x$. This condition is, for instance, satisfied, if

$$\lim_{t \to \infty} t^{r+\varepsilon} f(t) = 0 \qquad (\varepsilon > 0); \tag{45}$$

for it follows from (6) that $\overline{x}_{\omega r}^{\nu}$ does not increase more rapidly than $|x|^{r-1}$. The condition (44) does not imply any analytical property of $f(t)$ but only concerns the rapidity with which the function must decrease for $t \to \infty$. It is clear that if (44) is satisfied for a given value of r, it is also satisfied for any smaller value of r.

We may now prove (43) by induction, writing $R_m^{(r)}$ instead of R_m in order to indicate that R_m depends on r. Let us first prove the formula for $r = 1$. We have by (35)

$$R_m^{(1)} = -\omega \sum_{\nu=0}^{\frac{1}{\omega}-1} \frac{\overline{h+1-\omega-\nu\omega}_{\omega 1}^{\,m+1}}{(m+1)!} \mathop{\Delta}_{\omega}^{m+1} f(x+\nu\omega).$$

If to the right-hand side we add the expression

$$\omega \sum_{\nu=0}^{\infty} \frac{\overline{h+1-\omega-\nu\omega}_{\omega 1}^{\,m+1} - \overline{h-\omega-\nu\omega}_{\omega 1}^{\,m+1}}{(m+1)!} \mathop{\Delta}_{\omega}^{m+1} f(x+\nu\omega)$$

which is convergent according to hypothesis, and vanishes identically, as $\overline{x}_{\omega 1}^{\,m+1}$ is periodical with the period 1, we obtain after an obvious reduction

$$R_m^{(1)} = \omega \sum_{\nu=\frac{1}{\omega}}^{\infty} \frac{\overline{h+1-\omega-\nu\omega}_{\omega 1}^{m+1}}{(m+1)!} \underset{\omega}{\Delta}^{m+1} f(x+\nu\omega)$$

$$-\omega \sum_{\nu=0}^{\infty} \frac{\overline{h-\omega-\nu\omega}_{\omega 1}^{m+1}}{(m+1)!} \underset{\omega}{\Delta}^{m+1} f(x+\nu\omega)$$

or, writing $\nu + \frac{1}{\omega}$ instead of ν in the first sum and reducing,

$$R_m^{(1)} = \omega \sum_{\nu=0}^{\infty} \frac{\overline{h-\omega-\nu\omega}_{\omega 1}^{m+1}}{(m+1)!} \Delta \underset{\omega}{\Delta}^{m+1} f(x+\nu\omega),$$

so that (43) is valid for $r = 1$.

It remains to show that if (43) is valid for $r = s-1$, it is also valid for $r = s$. Now, by (35),

$$R_m^{(s)} = R_m^{(s-1)} - \omega \sum_{\nu=0}^{\frac{1}{\omega}-1} \frac{\overline{h+1-\omega-\nu\omega}_{\omega s}^{m+s}}{(m+s)!} \Delta^{s-1} \underset{\omega}{\Delta}^{m+1} f(x+\nu\omega);$$

hence, if (43) is valid for $r = s-1$,

$$R_m^{(s)} = \omega \sum_{\nu=0}^{\infty} \frac{\overline{h-\omega-\nu\omega}_{\omega, s-1}^{m+s-1}}{(m+s-1)!} \Delta^{s-1} \underset{\omega}{\Delta}^{m+1} f(x+\nu\omega)$$

$$-\omega \sum_{\nu=0}^{\frac{1}{\omega}-1} \frac{\overline{h+1-\omega-\nu\omega}_{\omega s}^{m+s}}{(m+s)!} \Delta^{s-1} \underset{\omega}{\Delta}^{m+1} f(x+\nu\omega).$$

But, as

$$\overline{h-\omega-\nu\omega}_{\omega, s-1}^{m+s-1} = \frac{\overline{h+1-\omega-\nu\omega}_{\omega s}^{m+s} - \overline{h-\omega-\nu\omega}_{\omega s}^{m+s}}{m+s},$$

we find immediately, on reduction,

$$R_m^{(s)} = \omega \sum_{\nu=0}^{\infty} \frac{\overline{h - \omega - \nu\omega}_{\omega s}^{m+s}}{(m+s)!} \Delta^s \Delta_{\omega}^{m+1} f(x+\nu\omega),$$

and the proof is completed.

10. Before proceeding to establish the desired general summation-formula, we shall make a few remarks about repeated summation. The symbol Δ_{ω}^{-1} is generally defined in such a way that

$$\Delta_{\omega}^{-1} f(x) = \varphi(x) + \psi_{\omega}(x),$$

$\varphi(x)$ being any particular solution of the difference equation $\Delta_{\omega} \varphi(x) = f(x)$, and $\psi_{\omega}(x)$ being an arbitrary periodic function with the period ω. It will now be advantageous to fix the meaning of Δ_{ω}^{-1}. We put[1], assuming the convergence,

$$\Delta_{\omega}^{-1} f(x) = -\omega \sum_{\nu=0}^{\infty} f(x+\nu\omega), \tag{46}$$

and it is obvious that, with this definition, $\Delta_{\omega} \Delta_{\omega}^{-1} f(x) = f(x)$, as it should be. For the applications of the operation Δ_{ω}^{-1}, thus defined, to summation between finite limits, the condition that (46) must be convergent is not a restriction of real importance; for, as we do not assume that $f(x)$ is an analytical function, the summation-process (46) may be applied to any table of finite extent, provided we put $f(t) = 0$ for values of t outside the range of the table.

The symbol Δ_{ω}^{-1}, defined in this particular way, is commutative with Δ_{ω}; for we have

[1] Compare NÖRLUND: Differenzenrechnung, p. 41.

$$\underset{\omega}{\Delta}^{-1}\underset{\omega}{\Delta} f(x) = -\omega \sum_{\nu=0}^{\infty} \frac{f(x+\nu\omega+\omega)-f(x+\nu\omega)}{\omega}$$

$$= -\sum_{\nu=0}^{\infty} f(x+\nu\omega+\omega) + \sum_{\nu=0}^{\infty} f(x+\nu\omega)$$

$$= f(x) = \underset{\omega}{\Delta}\underset{\omega}{\Delta}^{-1} f(x).$$

From (46) follows, for $\omega = 1$,

$$\Delta^{-1} f(x) = -\sum_{\nu=0}^{\infty} f(x+\nu), \tag{47}$$

and it is easily proved that any two of the symbols Δ, Δ^{-1}, $\underset{\omega}{\Delta}$, $\underset{\omega}{\Delta}^{-1}$ are commutative if, in exchanging the order of two symbols of summation, we assume the absolute convergence of the double sum.

The operation $\underset{\omega}{\Delta}^{-1}$ may be repeated, always assuming the convergence; and we find in the case of absolute convergence

$$\underset{\omega}{\Delta}^{-r} f(x) = (-\omega)^r \sum_{\nu=0}^{\infty} \binom{\nu+r-1}{r-1} f(x+\nu\omega). \tag{48}$$

For this formula is valid for $r = 1$; but being valid for any particular value of r, it is also valid for the following one, as

$$\underset{\omega}{\Delta}^{-1}\underset{\omega}{\Delta}^{-r} f(x) = (-\omega)^{r+1} \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} \binom{\nu+r-1}{r-1} f(x+\mu\omega+\nu\omega)$$

$$= (-\omega)^{r+1} \sum_{s=0}^{\infty}\sum_{\mu=0}^{s} \binom{s-\mu+r-1}{r-1} f(x+s\omega)$$

$$= (-\omega)^{r+1} \sum_{s=0}^{\infty} \binom{s+r}{r} f(x+s\omega).$$

From (48) we find, putting $\omega = 1$,

$$\Delta^{-r} f(x) = (-1)^r \sum_{\nu=0}^{\infty} \binom{\nu + r - 1}{r - 1} f(x + \nu). \tag{49}$$

11. In order to derive the desired summation-formula, we will for a moment assume that $f(t)$ is a function that vanishes beyond a certain range, say, for $t > N$. We may then, in (42) and (43), perform the operation Δ^{-r} on both sides, and thus obtain

$$\Delta^{-r} f(x+h) = \sum_{\nu=0}^{r+m} \frac{h_{\omega r}^{\nu}}{\nu!} \underset{\omega}{\Delta}^{\nu - r} f(x) + R, \tag{50}$$

$$R = \omega \sum_{\nu=0}^{\infty} \frac{\overline{h - \omega - \nu\omega}_{\omega r}^{\,m+r}}{(m+r)!} \underset{\omega}{\Delta}^{m+1} f(x + \nu\omega). \tag{51}$$

But this formula evidently remains valid for $N \to \infty$, if all the sums are convergent.

A sufficient condition for the validity of (50) and (51) is, therefore, that the condition (45) is satisfied in which case all the sums are absolutely convergent. In particular, (50) and (51) may be applied to summation between finite limits, if we put $f(t) = 0$ for values of t outside the range of the table. The parameter h must satisfy the condition (34), and $\frac{1}{\omega}$ is a positive integer > 1.

By keeping the first term on the right of (50) apart, it is seen that the formula may be used for the approximate calculation of $\underset{\omega}{\Delta}^{-r} f(x)$ or $\Delta^{-r} f(x+h)$ if, besides one of these sums, we know the sums of lower order $\underset{\omega}{\Delta}^{1-r} f(x)$, $\underset{\omega}{\Delta}^{2-r} f(x)$, ...

12. The simplest and most important case of (50) is obtained for $r = 1$. The result may be written

$$\omega \sum_{\nu=0}^{\infty} f(x+\nu\omega) = \sum_{\nu=0}^{\infty} f(x+h+\nu) + \sum_{\nu=1}^{m+1} \frac{\overset{\nu}{h}_{\omega 1}}{\nu!} \underset{\omega}{\Delta}^{\nu-1} f(x) + R \tag{52}$$

where $0 \leq h = p\omega < 1$, and

$$R = \omega \sum_{\nu=0}^{\infty} \frac{\overline{h-\omega-\nu\omega}^{\,m+1}_{\omega 1}}{(m+1)!} \underset{\omega}{\Delta}^{m+1} f(x+\nu\omega). \tag{53}$$

The explicit expression of $\overset{\nu}{x}_{\omega 1}$ is, according to G. N. P. (46),

$$\overset{\nu}{x}_{\omega 1} = \sum_{s=0}^{\nu} \binom{\nu}{s} 0^{\nu-s}_{\omega 1} \overset{s}{x}_{\omega 0}, \tag{54}$$

and $\overline{x}^{\nu}_{\omega 1}$ is a function, periodical with the period 1, which in the interval $0 \leq x < 1$ is identical with $\overset{\nu}{x}_{\omega 1}$.

From (52) and (53), Euler's summation formula is obtained by letting $\omega \to 0$; we need not go into details.

13. It is not always practical to use (50) for summation between finite limits, but another formula may be derived from (42) as follows.

Let x be an integer (this restriction being of no real consequence), and let γ be another integer, supposed to be constant. We put[1], for $x < \gamma$,

$$S' f(x) = \sum_{x}^{\gamma-1} f(x) = \sum_{\nu=0}^{\gamma-1-x} f(x+\nu) \tag{55}$$

while $S' f(x) = 0$ for $x \geq \gamma$. Hence, on repeating the operation S' r times,

[1] Compare Steffensen: Interpolation (Baltimore 1927), art. 111 (where β is written for $\gamma - 1$).

$$S^{(r)} f(x) = \sum_{\nu = x}^{\gamma-1} \binom{\nu - x + r - 1}{r-1} f(\nu) = \sum_{\nu = 0}^{\gamma - 1 - x} \binom{\nu + r - 1}{r - 1} f(x + \nu), \quad (56)$$

as may be proved by induction, or be concluded from (49) (putting $f(t) = 0$ for $t \geq \gamma$).

It is now easy to prove that for $s \leq r$

$$S^{(s)} \Delta^r f(x) = (-1)^s \left[\Delta^{r-s} f(x) - \sum_{\nu = 0}^{s-1} (-1)^\nu \binom{\gamma - 1 - x + \nu}{\nu} \Delta^{r-s+\nu} f(\gamma) \right]. \quad (57)$$

For this formula is valid for $s = 1$, as, by (55),

$$\begin{aligned} S' \Delta^r f(x) &= S' \Delta^{r-1} \Delta f(x) \\ &= S' \Delta^{r-1} f(x+1) - S' \Delta^{r-1} f(x) \\ &= \Delta^{r-1} f(\gamma) - \Delta^{r-1} f(x); \end{aligned}$$

and being valid for any particular value of s, (57) is proved to be valid also for the following one, on performing the operation S' on both sides and noting that

$$S' \binom{\gamma - 1 - x + \nu}{\nu} = \binom{\gamma - x + \nu}{\nu + 1}.$$

Similarly, we put

$$\underset{\omega}{S'} f(x) = \omega \left[f(x) + f(x + \omega) + \ldots + f(\gamma - \omega) \right] \quad (58)$$

besides $\underset{\omega}{S'} f(x) = 0$ for $x \geq \gamma$; whence, by induction or by (48),

$$\underset{\omega}{S^{(r)}} f(x) = \omega^r \sum_{\nu = 0}^{\frac{\gamma - x}{\omega} - 1} \binom{\nu + r - 1}{r - 1} f(x + \nu\omega). \quad (59)$$

If now, in (42), we interpret $\underset{\omega}{\Delta}^{\nu-r} f(x)$ for $\nu < r$ as $(-1)^{\nu-r} \underset{\omega}{S}^{(r-\nu)} f(x)$, this formula may be written

$$\left.\begin{aligned} f(x+h) &= \sum_{\nu=0}^{r-1} (-1)^{\nu+r} \frac{h_{\omega r}^{\nu}}{\nu!} \Delta^{r} \underset{\omega}{S}^{(r-\nu)} f(x) \\ &\quad + \sum_{\nu=0}^{m} \frac{h_{\omega r}^{\nu+r}}{(\nu+r)!} \Delta^{r} \underset{\omega}{\Delta}^{\nu} f(x) + R_m \end{aligned}\right\} \quad (60)$$

where R_m has the meaning (43).

Finally, performing the operation $S^{(r)}$ on both sides of (60), and taking into account that, according to (57),

$$S^{(r)} \Delta^{r} f(x) = (-1)^{r} \left[f(x) - \sum_{\nu=0}^{r-1} (-1)^{\nu} \binom{\gamma-1-x+\nu}{\nu} \Delta^{\nu} f(\gamma) \right], \quad (61)$$

we find, as $\underset{\omega}{S}^{(r-\nu)} f(\gamma) = 0$ for $\nu < r$,

$$\left.\begin{aligned} S^{(r)} f(x+h) &= \sum_{\nu=0}^{r-1} (-1)^{\nu} \frac{h_{\omega r}^{\nu}}{\nu!} \underset{\omega}{S}^{(r-\nu)} f(x) \\ &\quad + (-1)^{r} \sum_{\nu=0}^{m} \frac{h_{\omega r}^{\nu+r}}{(\nu+r)!} \left[\underset{\omega}{\Delta}^{\nu} f(x) \right. \\ &\quad \left. - \sum_{\mu=0}^{r-1} (-1)^{\mu} \binom{\gamma-1-x+\mu}{\mu} \Delta^{\mu} \underset{\omega}{\Delta}^{\nu} f(\gamma) \right] + R \end{aligned}\right\} \quad (62)$$

where

$$\left.\begin{aligned} R &= (-1)^{r} \omega \sum_{\nu=0}^{\infty} \frac{\overline{h-\omega-\nu\omega}_{\omega r}^{\,m+r}}{(m+r)!} \left[\underset{\omega}{\Delta}^{m+1} f(x+\nu\omega) \right. \\ &\quad \left. - \sum_{\mu=0}^{r-1} (-1)^{\mu} \binom{\gamma-1-x+\mu}{\mu} \Delta^{\mu} \underset{\omega}{\Delta}^{m+1} f(\gamma+\nu\omega) \right]. \end{aligned}\right\} \quad (63)$$

This formula is more cumbrous in appearance than (50), but has the advantage over the latter that the remainder may tend to a limit for $\omega \to 0$ which is not the case if (50) is applied to summation between finite limits by assuming that $f(t)$ vanishes beyond a certain range.

14. In the particular case where $r = 1$ we obtain from (62) and (63)

$$\underset{\omega}{S'} f(x) = S' f(x+h) + \sum_{\nu=0}^{m} \frac{h_{\omega 1}^{\nu+1}}{(\nu+1)!}\left[\underset{\omega}{\Delta}^{\nu} f(x) - \underset{\omega}{\Delta}^{\nu} f(\gamma)\right] + R \qquad (64)$$

where

$$R = \omega \sum_{\nu=0}^{\infty} \frac{\overline{h-\omega-\nu\omega}^{\,m+1}_{\omega 1}}{(m+1)!}\left[\underset{\omega}{\Delta}^{m+1} f(x+\nu\omega) - \underset{\omega}{\Delta}^{m+1} f(\gamma+\nu\omega)\right]. \qquad (65)$$

$S' f(x+h)$ has the value

$$S' f(x+h) = f(x+h) + f(x+h+1) + \ldots + f(\gamma-1+h)$$

while

$$\underset{\omega}{S'} f(x) = \omega\left[f(x) + f(x+\omega) + \ldots + f(\gamma-\omega)\right].$$

The parameter h must satisfy the condition

$$0 \leq h = p\omega < 1.$$

By letting $\omega \to 0$ we may, from (64) and (65), derive Euler's summation-formula.

(64) and (65) may also be obtained directly from (50) and (51) by writing γ for x and deducting.

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