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SUMS OF POWERS OF INTEGERS VIA GENERATING FUNCTIONS

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1. INTRODUCTION

Let

$$S_{k,n}(a,d) = a^k + (a+d)^k + (a+2d)^k + \cdots + [a+(n-1)d]^k, \tag{1.1}$$

where k and n are nonnegative integers with $n > 0$, and a and d are complex numbers with $d \neq 0$. We shall use the notation

$$S_{k,n} = S_{k,n}(1,1) = 1^k + 2^k + \cdots + n^k. \tag{1.2}$$

Similarly, let $T_{k,n}(a,d)$ be the alternating sum

$$T_{k,n}(a,d) = a^k - (a+d)^k + (a+2d)^k - \cdots + (-1)^{n-1}[a+(n-1)d]^k, \tag{1.3}$$

and let

$$T_{k,n} = T_{k,n}(1,1) = 1^k - 2^k + 3^k - \cdots + (-1)^{n-1}n^k. \tag{1.4}$$

It is, of course, well known that

$$S_{k,0} = n; \quad S_{k,1} = n(n+1)/2; \quad S_{k,2} = n(n+1)(2n+1)/6;$$

and so on. Many different writers have worked on the problem of finding simple formulas for $S_{k,n}$, and many different methods have been used; see [3], [5], [6], [9], [10] for just a small sampling of recent articles. The formulas for $T_{k,n}$ are certainly less well known.

In the present paper we use generating functions to find new recurrences for $S_{k,n}(a,d)$ and $T_{k,n}(a,d)$. We also show how $S_{k,n}(a,d)$ and $T_{k,n}(a,d)$ can be determined from $S_{k-1,n}(a,d)$ and $T_{k-1,n}(a,d)$, respectively, and we show how $S_{k,n}(a,d)$ and $T_{k,n}(a,d)$ can be expressed in terms of Bernoulli numbers. One of the main results is a new "lacunary" recurrence formula for $S_{k,n}$ with gaps of 6 (Theorem 3.1); that is, we can use the formula to find $S_{m,n}$ for $m = 0, 1, \ldots, 5$; then, using only $S_{m,n}$, we can find $S_{6+m,n}$; then, using only $S_{m,n}$ and $S_{6+m,n}$, we can find $S_{12+m,n}$, and so on. There is a similar recurrence for $T_{k,n}$.

There are several motivations for this paper: (1) A recent article by Wiener [10] dealt with equation (1.1) and generalized some well-known properties of $S_{k,n}$. We show how the formulas of [10] can be derived very quickly and how they can be extended. (2) In a recent article by Howard [4], formulas were found which connected Bernoulli numbers to $T_{k,n}$. Evidently, the properties of $T_{k,n}$ are not well known, so the results of [4] are a stimulus to study $T_{k,n}$ and $T_{k,n}(a,d)$ in some detail. (3) The new lacunary recurrences mentioned above are useful and easy to use, and (in the writer's opinion) they are of considerable interest. In Section 3 we illustrate the formulas by computing $S_{6,n}$, $S_{12,n}$, and $S_{18,n}$; in Section 6 we compute $T_{4,n}$, $T_{10,n}$, and $T_{16,n}$. (4) Perhaps the main purpose of the paper is to show how generating functions provide a simple, unified approach to the study of sums of powers of integers. The many, and often repetitious articles on $S_{k,n}$ that have appeared in the last twenty-five years seem to indicate a need for such a unified approach.

2. RECURRENCES FOR $S_{k,n}(a,d)$

We first note that by (1.1) the exponential generating function for $S_{k,n}(a,d)$ is

$$\sum_{k=0}^{\infty} S_{k,n}(a,d)\frac{x^k}{k!} = e^{ax} + e^{(a+d)x} + \cdots + e^{[a+(n-1)d]x} = \frac{e^{(a+nd)x} - e^{ax}}{e^{dx}-1}. \tag{2.1}$$

We can use (2.1) to prove the next two theorems in a very direct way.

Theorem 2.1: Let $k \geq 0$ and $n > 0$. We have the following recurrences for $S_{k,n}(a,d)$:

$$\sum_{j=0}^{k}\binom{k+1}{j} d^{k+1-j} S_{j,n}(a,d) = (a+nd)^{k+1} - a^{k+1}, \tag{2.2}$$

$$\sum_{j=0}^{k}(-1)^j\binom{k+1}{j} d^{k+1-j} S_{j,n}(a,d) = (-1)^k[(a+nd-d)^{k+1} - (a-d)^{k+1}]. \tag{2.3}$$

Proof: From (2.1) we have

$$(e^{dx}-1)\sum_{k=0}^{\infty} S_{k,n}(a,d)\frac{x^k}{k!} = e^{(a+nd)x} - e^{ax}; \text{ i.e.,}$$

$$\sum_{j=1}^{\infty} d^j \frac{x^j}{j!}\cdot\sum_{k=0}^{\infty} S_{k,n}(a,d)\frac{x^k}{k!} = \sum_{j=0}^{\infty}[(a+nd)^j - a^j]\frac{x^j}{j!}. \tag{2.4}$$

If we examine both sides of (2.4) and equate coefficients of $x^{k+1}/(k+1)!$, we have (2.2). Now if we replace x by $-x$ in (2.1), we have, after simplification,

$$\sum_{k=0}^{\infty}(-1)^k S_{k,n}(a,d)\frac{x^k}{k!} = \frac{e^{(d-a)x} - e^{(d-a-nd)x}}{e^{dx}-1}. \tag{2.5}$$

Multiplying both sides of (2.5) by $e^{dx}-1$ and equating coefficients of $x^{k+1}/(k+1)!$, we have (2.3). This completes the proof.

Formulas (2.2) and (2.3) generalize known formulas for $a = d = 1$ [8, p. 159]. We note that (2.2) was found by Wiener [10]; see also [2, p. 169]. Bachmann [1, p. 28] found a recurrence for $S_{k,n}(a,d)$ involving only $S_{j,n-1}$ for $j = 1, \ldots, k$.

We now add (2.2) and (2.3) to obtain the next theorem.

Theorem 2.2: For $k = 1, 2, 3, \ldots$, we have

$$2\sum_{j=0}^{k-1}\binom{2k}{2j} d^{2k-2j} S_{2j,n}(a,d) = (a+nd)^{2k} - (a+nd-d)^{2k} + (a-d)^{2k} - a^{2k},$$

and for $k = 2, 3, \ldots$, we have

$$2\sum_{j=1}^{k-1}\binom{2k-1}{2j-1}d^{2k-2j}S_{2j-1,n}(a,d)=(a+nd)^{2k-1}-(a+nd-d)^{2k-1}+(a-d)^{2k-1}-a^{2k-1}.$$

Theorem 2.2 can be compared to results of Wiener [10] and Riordan [8, p. 160].

3. RECURRENCES WITH INDICES 6*K* + *M*

We now show how to find formulas of the type (2.2) where the index j varies only over integers of the form $6k+m$, with $0\le m\le 5$. To the writer's knowledge, these formulas are new. After stating Theorem 3.1 and its corollary, we give some applications; the proof of Theorem 3.1 is given at the end of this section.

Let θ be the complex number $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$, so

$$\theta^3=1 \quad\text{and}\quad \theta^2+\theta+1=0, \tag{3.1}$$

and define the sequence $\{w_j\}$ in the following way:

$$\begin{aligned} w_j &= 1+(-1)^j(\theta^j+\theta^{6-j}) && \text{for } j=0,1,\ldots,5,\\ w_j &= w_{6+j} && \text{for } j=0,\pm1,\pm2,\ldots. \end{aligned} \tag{3.2}$$

For example, $w_{-1}=w_5=2$. The values of w_j for $j=0,1,\ldots,5$ are given in the following table:

j	0	1	2	3	4	5
w_j	3	2	0	−1	0	2

(3.3)

Theorem 3.1 Let w_j be defined by (3.2) and (3.3). Then, for $m=0,1,\ldots,5$ and $n>0$, $k\ge 0$, we have:

$$6\sum_{j=0}^{k}\binom{6k+m+3}{6j+m}d^{6k-6j+3}S_{6j+m,n}(a,d)$$

$$=\sum_{j=1}^{6k+m+1}\binom{6k+m+3}{j}d^{6k+m+3-j}w_{m-j}[(a-d+nd)^j-(a-d)^j].$$

Corollary: For $m=0,1,\ldots,5$ and $n>0$, $k\ge0$, we have:

$$\binom{6k+m+3}{3}S_{6k+m,n}=-\sum_{j=0}^{k-1}\binom{6k+m+3}{6j+m}S_{6j+m,n}+\frac{1}{6}\sum_{j=1}^{6k+m+1}\binom{6k+m+3}{j}w_{m-j}n^j.$$

The corollary gives us an easy way to write $S_{k,n}$ as a polynomial in n of degree $k+1$. In particular, we have for $m=0,1,\ldots,5$:

$$\binom{m+3}{3}S_{m,n}=\frac{1}{6}\sum_{j=1}^{m+1}\binom{m+3}{j}w_{m-j}n^j.$$

Using the corollary, we easily compute $S_{0,n}=n$ and

$$\binom{9}{3}S_{6,n} = -n + \frac{1}{6}\sum_{j=1}^{7}\binom{9}{j}w_{-j}n^n,$$

By (3.2) and (3.3), the numbers w_{-j} are easy to find. We have $w_{-j} = 2$ for $j = 1, 5, 7$; $w_{-j} = 0$ for $j = 2$ and 4; $w_{-j} = -1$ for $j = 3$; $w_{-j} = 3$ for $j = 6$. Thus, we have

$$S_{6,n} = \frac{1}{42}n - \frac{1}{6}n^3 + \frac{1}{2}n^5 + \frac{1}{2}n^6 + \frac{1}{7}n^7.$$

Continuing in the same way, we have

$$\binom{15}{3}S_{12,n} = -n - \binom{15}{6}S_{6,n} + \frac{1}{6}\sum_{j=1}^{13}\binom{15}{j}w_{-j}n^j,$$

so

$$S_{12,n} = -\frac{691}{2730}n + \frac{5}{3}n^3 - \frac{33}{10}n^5 + \frac{22}{7}n^7 - \frac{11}{6}n^9 + n^{11} + \frac{1}{2}n^{12} + \frac{1}{13}n^{13}.$$

It is easy to keep going:

$$\binom{21}{3}S_{18,n} = -n - \binom{21}{6}S_{6,n} - \binom{21}{12}S_{12,n} + \frac{1}{6}\sum_{j=1}^{19}\binom{21}{j}w_{-j}n^j,$$

which gives

$$S_{18,n} = \frac{43867}{798}n - \frac{3617}{10}n^3 + 714n^5 - \frac{23494}{35}n^7 + \frac{1105}{3}n^9 - \frac{663}{5}n^{11}$$
$$+ 34n^{13} - \frac{34}{5}n^{15} + \frac{3}{2}n^{17} + \frac{1}{2}n^{18} + \frac{1}{19}n^{19}.$$

Another application of Theorem 3.1, involving Bernoulli numbers, is given in Section 4.

Proof of Theorem 3.1: Let

$$A(x) = \frac{e^{(a+nd)x} - e^{ax}}{e^{dx} - 1} = \sum_{k=0}^{\infty} S_{k,n}(a,d)\frac{x^k}{k!}, \tag{3.4}$$

and define $A_0(x)$, $A_1(x)$, and $A_2(x)$ as follows:

$$A_0(x) = \frac{1}{3}[A(x) + A(\theta x) + A(\theta^2 x)] = \sum_{k=0}^{\infty} S_{3k,n}(a,d)\frac{x^{3k}}{(3k)!}, \tag{3.5}$$

$$A_1(x) = \frac{1}{3}[(A(x) + \theta^2 A(\theta x) + \theta A(\theta^2 x)] = \sum_{k=0}^{\infty} S_{3k+1,n}(a,d)\frac{x^{3k+1}}{(3k+1)!}, \tag{3.6}$$

$$A_2(x) = \frac{1}{3}[A(x) + \theta A(\theta x) + \theta^2 A(\theta^2 x)] = \sum_{k=0}^{\infty} S_{3k+2,n}(a,d)\frac{x^{3k+2}}{(3k+2)!}. \tag{3.7}$$

The equalities on the extreme right of (3.5), (3.6), and (3.7) follow from (3.1) and (3.4). Using (3.4) and the equalities on the extreme left of (3.5), (3.6), and (3.7), we can write

$$A_m(x) = \frac{N_m}{3D_m} \quad (m = 0, 1, 2),$$

with $D_0 = D_1 = D_2 = (e^{dx}-1)(e^{\theta dx}-1)(e^{\theta^2 dx}-1)$. Using (3.1), it is easy to compute

$$D_0 = D_1 = D_2 = 6\sum_{k=0}^{\infty} d^{6k+3}\frac{x^{6k+3}}{(6k+3)!}.$$

The formulas for N_0, N_1, and N_2 are more complicated, but they are easy to work out from (3.4), (3.5), (3.6), and (3.7). We first note that, for $m = 0, 1, 2$, we have

$$A_m(x) = \frac{1}{3}[A(x) + \theta^{3-m}A(\theta x) + \theta^m A(\theta^2 x)] = \sum_{k=0}^{\infty} S_{3k+m,n}(a,d)\frac{x^{3k+m}}{(3k+m)!}.$$

Thus, we have:

$$\begin{aligned} N_m &= (e^{(a+nd)x} - e^{ax})(e^{-dx} - e^{d\theta x} - e^{d\theta^2 x} + 1) \\ &+ \theta^{3-m}(e^{(a+nd)\theta x} - e^{a\theta x})(e^{-d\theta x} - e^{dx} - e^{d\theta^2 x} + 1) \\ &+ \theta^m(e^{(a+nd)\theta^2 x} - e^{a\theta^2 x})(e^{-d\theta^2 x} - e^{d\theta x} - e^{dx} + 1). \end{aligned} \tag{3.8}$$

We now multiply, regroup, and expand the terms in (3.8). For example, we have

$$\begin{aligned} e^{(a+nd-d)x} + \theta^{3-m}e^{(a+nd-d)\theta x} + \theta^m e^{(a+nd-d)\theta^2 x} &= \sum_{j=0}^{\infty}(a+nd-d)^j(1+\theta^{3-m+j}+\theta^{m+2j})\frac{x^j}{j!} \\ &= 3\sum_{j=0}^{\infty}(a+nd-d)^{3j+m}\frac{x^{3j+m}}{(3j+m)!}. \end{aligned}$$

Regrouping and expanding the other terms in (3.8), we have, for $m = 0, 1,$ and 2,

$$\begin{aligned} \frac{1}{3}N_m = \sum_{j=0}^{\infty}&[(a+nd+d)^{3j+m} + (a+nd)^{3j+m} + (a+nd-d)^{3j+m} - (a+d)^{3j+m} - a^{3j+m} \\ &- (a-d)^{3j+m}]\frac{x^{3j+m}}{(3j+m)!} - 3\sum_{j=0}^{\infty} d^{3j}\frac{x^{3j}}{(3j)!}\cdot\sum_{j=0}^{\infty}[(a+nd)^{3j+m} - a^{3j+m}]\frac{x^{3j+m}}{(3j+m)!}. \end{aligned}$$

Since

$$D_m\sum_{k=0}^{\infty} S_{3k+m,n}(a,d)\frac{x^{3k+m}}{(3k+m)!} = \frac{1}{3}N_m, \tag{3.9}$$

we can equate coefficients of $x^{3k+m}/(3k+m)!$ in (3.9) and state the following: For $m = 0, 1, 2$,

$$\begin{aligned} &3\sum_{j=0}^{k}[1+(-1)^{k-j+1}]\binom{3k+m}{3j+m}d^{3k-3j}S_{3j+m,n}(a,d) \\ &= (a+nd+d)^{3k+m} + (a+nd)^{3k+m} + (a+nd-d)^{3k+m} - (a+d)^{3k+m} \\ &\quad - a^{3k+m} - (a-d)^{3k+m} - 3\sum_{j=0}^{k}\binom{3k+m}{3j+m}[(a+nd)^{3j+m} - a^{3j+m}]d^{3k-3j}. \end{aligned} \tag{3.10}$$

At this point we observe how the sums in (3.10) can be simplified. By using properties of θ and the binomial theorem, we see that

$$
\begin{aligned}
&3\sum_{j=0}^{k}\binom{3k+m}{3j+m}(a+nd)^{3j+m}d^{3k-3j} \\
&=(d+a+nd)^{3k+m}+\theta^{2m}(d+a\theta\ +nd\theta)^{3k+m}+\theta^{m}(d+a\theta^2+nd\theta^2)^{3k+m} \\
&=(d+a+nd)^{3k+m}+\theta^{2m}[-d\theta^2+\{(a-d)+nd\}\theta]^{3k+m}+\theta^{m}[-d\theta+\{(a-d)+nd\}\theta^2]^{3k+m} \\
&=(d+a+nd)^{3k+m}+\sum_{j=0}^{3k+m}\binom{3k+m}{j}(-1)^{k+m-j}d^{3k+m-j}(\theta^{m-j}+\theta^{2m+j})(a-d+nd)^j \\
&=(d+a+nd)^{3k+m}+\sum_{j=0}^{3k+m}\binom{3k+m}{j}(-1)^{k}d^{3k+m-j}(-1+w_{m-j})(a-d+nd)^j. \qquad (3.11)
\end{aligned}
$$

We substitute (3.11) [and also (3.11) with $n=0$] into (3.10), and we consider the two cases of k even and k odd. Then using the binomial theorem and the fact that $w_{m-j}=0$ when $j=6k+m+2$, we can easily simplify (3.10) to get Theorem 3.1. This completes the proof.

4. $S_{k,n}(a,d)$ IN TERMS OF BERNOULLI NUMBERS

The Bernoulli polynomial $B_k(x)$ may be defined by means of the generating function

$$\frac{xe^{xz}}{e^x-1}=\sum_{k=0}^{\infty}B_k(z)\frac{x^k}{k!}. \qquad (4.1)$$

When $z=0$, we have the ordinary Bernoulli number B_k, i.e., $B_k(0)=B_k$. It is well known [2, pp. 48-49] that $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, and $B_{2m+1}=0$ for $m>0$. It follows from (4.1) that

$$B_k(z)=\sum_{j=0}^{k}\binom{k}{j}B_{k-j}z^j. \qquad (4.2)$$

Comparing (4.1) and (1.2), we see

$$S_{k,n}(a,d)=\frac{d^k}{k+1}\left[B_{k+1}\left(\frac{a}{d}+n\right)-B_{k+1}\left(\frac{a}{d}\right)\right]. \qquad (4.3)$$

Now, for fixed a and d, suppose we write $S_{k,n}(a,d)$ as a polynomial in $(a-d+nd)$; i.e.,

$$
\begin{aligned}
S_{k,n}(a,d)&=S_{k,n-1}(a,d)+[a+(n-1)d]^k \\
&=v_{k,0}+v_{k,1}(a-d+nd)+v_{k,2}(a-d+nd)^2+\cdots+v_{k,k+1}(a-d+nd)^{k+1}.
\end{aligned} \qquad (4.4)
$$

By (4.2) and (4.3), we have the following result.

Theorem 4.1: If $S_{k,n}(a,d)$ is written as a polynomial in $(a-d+nd)$ and $v_{k,j}$ is defined by (4.4) for $j=1,2,\ldots,k+1$, then

$$
\begin{aligned}
&v_{k,j}=\frac{1}{k+1}\binom{k+1}{j}d^{k-j}B_{k+1-j}\quad(1\le j\le k-1), \\
&v_{k,k}=\frac{1}{2},\ v_{k,k+1}=\frac{1}{d(k+1)},\ v_{k,0}=\frac{d^k}{k+1}\left[B_{k+1}-B_{k+1}\left(\frac{a}{d}\right)\right].
\end{aligned}
$$

When $a = d = 1$, Theorem 4.1 gives the well-known result [2, pp. 154-55]:

$$S_{k,n} = S_{k,n}(1, 1) = n^k + \frac{1}{k+1}\sum_{j=1}^{k+1}\binom{k+1}{j}B_{k+1-j}n^j.$$

Here, we can give another application of Theorem 3.1. Since $v_{k,1} = B_k$ for $k > 1$, and $a = d = 1$, we see from the corollary to Theorem 3.1 that, for $m = 0, \ldots, 5$ and $6k + m > 1$,

$$\binom{6k+m+3}{3}B_{6k+m} = -\sum_{j=1}^{k-1}\binom{6k+m+3}{6j+m}B_{6j+m} + \frac{1}{6}(6k+m+3)w_{m-1}. \tag{4.5}$$

Formula (4.5) is a lacunary recurrence for the Bernoulli numbers that is equivalent to a formula of Ramanujan [7, pp. 3-4]. See also [8, pp. 136-37].

5. FINDING $S_{k,n}(a, d)$ FROM $S_{k-1,n}(a, d)$

Several writers, like Khan [6], have pointed out that when $a = d = 1$, if we know just $S_{k-1,n}$, we can evaluate $S_{k,n}$. Using (2.1), it is easy to prove this and to generalize it. First of all, we can use mathematical induction on (2.2) to prove that $S_{k,n}(a, d)$ is a polynomial in n of degree $k + 1$, with constant term equal to 0. (That also follows from Section 4.) Thus, for fixed a and d, we can write

$$S_{k,n}(a, d) = c_{k,1}n + c_{k,2}n^2 + \cdots + c_{k,k+1}n^{k+1}. \tag{5.1}$$

Theorem 5.1: For fixed a and d, let $c_{k,j}$ be defined by (5.1) for $j = 1, \ldots, k+1$. Then, for $k \geq 1$, we have

$$c_{k,j} = \frac{kd}{j}c_{k-1,j-1} \quad (j = 2, \ldots, k+1), \tag{5.2}$$

$$c_{k,1} = a^k - c_{k,2} - c_{k,3} - \cdots - c_{k,k+1}. \tag{5.3}$$

Proof: Define the polynomial $P_k(z)$ by means of the generating function

$$\sum_{k=0}^{\infty} P_k(z)\frac{x^k}{k!} = \frac{e^{(a+zd)x} - e^{ax}}{e^{dx} - 1}, \tag{5.4}$$

so $P_k(n) = S_{k,n}(a, d)$. This implies that

$$P_k(z) = c_{k,1}z + c_{k,2}z^2 + \cdots + c_{k,k+1}z^{k+1} \tag{5.5}$$

for all positive integers z, and hence for all complex numbers z. Now we differentiate both sides of (5.4) with respect to z to obtain

$$\sum_{k=0}^{\infty} P_k'(z)\frac{x^k}{k!} = \frac{dxe^{(a+zd)x}}{e^{dx} - 1} = \frac{dx(e^{(a+zd)x} - e^{ax})}{e^{dx} - 1} + \frac{dxe^{ax}}{e^{dx} - 1}. \tag{5.6}$$

We recall the definition of the Bernoulli polynomials, formula (4.1), and we see that

$$\frac{dxe^{ax}}{e^{dx} - 1} = \sum_{k=0}^{\infty} d^k B_k\left(\frac{a}{d}\right)\frac{x^k}{k!}, \tag{5.7}$$

and we note that $B_k(\frac{a}{d})$ is independent of z. From (5.4), (5.6), and (5.7), we have

$$\sum_{k=0}^{\infty} P_k'(z)\frac{x^k}{k!} = d\sum_{k=0}^{\infty} P_k(z)\frac{x^{k+1}}{k!} + \sum_{k=0}^{\infty} d^k B_k\left(\frac{a}{d}\right)\frac{x^k}{k!}. \tag{5.8}$$

Equating coefficients of $x^k / k!$ in (5.8), we have

$$P_k'(z) = dkP_{k-1}(z) + d^k B_k\left(\frac{a}{d}\right). \tag{5.9}$$

Thus, by using (5.5) and equating coefficients of z^{j-1} in (5.9), we have

$$c_{k,j} = \frac{kd}{j} c_{k-1,j-1} \quad (j = 2, \ldots, k+1).$$

Also, by (5.1) and the fact that $S_{k,1}(a, d) = a^k$, we have

$$d^k B_k\left(\frac{a}{d}\right) = c_{k,1} = a^k - c_{k,2} - c_{k,3} - \cdots - c_{k,k+1},$$

and the proof is complete.

Thus, if we know the coefficients of $S_{k-1,n}(a, d)$, we can determine the coefficients $c_{k,j}$ of $S_{k,n}(a, d)$ for $j = 2, \ldots, k+1$ from (5.2) and then compute $c_{k,1}$ from (5.3). For example,

$$S_{0,n}(a, d) = c_{0,1}n = n,$$

so, by (5.2) and (5.3), we have $c_{1,2} = \frac{d}{2}c_{0,1} = \frac{d}{2}$, and $c_{1,1} = a - \frac{d}{2}$; that is,

$$S_{1,n}(a, d) = c_{1,1}n + c_{1,2}n^2 = \left(a - \frac{d}{2}\right)n + \frac{d}{2}n^2. \tag{5.10}$$

Equivalently, by (5.9), we can integrate to find $P_k(z)$:

$$P_k(z) = dk\int P_{k-1}(z)\,dz + d^k B_k\left(\frac{a}{d}\right)z, \tag{5.11}$$

[The dz in (5.11) should not be confused with the complex variable d.] The constant of integration is 0, and $c_{k,1} = d^k B_k(\frac{a}{d})$ can be found by means of (5.3). When $a = d = 1$ and $k > 1$, $B_k(\frac{a}{d})$ is the k^{th} Bernoulli number. We illustrate (5.11) by finding $S_{2,n}(a, d)$. From (5.10) and (5.11) we have, after integrating $S_{1,n}(a, d)$ with respect to n and multiplying by $3d$,

$$S_{2,n}(a, d) = d^2 B_2\left(\frac{a}{d}\right)n + \left(ad - \frac{d^2}{2}\right)n^2 + \frac{d^2}{3}n^3,$$

so by (5.3) we have

$$S_{2,n}(a, d) = \left(a^2 - ad + \frac{d^2}{6}\right)n + \left(ad - \frac{d^2}{2}\right)n^2 + \frac{d^2}{3}n^3.$$

6. RECURRENCES FOR $T_{k,n}(a, d)$

Let $T_{k,n}(a,d)$ be defined by (1.3), with $k \geq 0$, $n > 0$, and $d \neq 0$. This type of sum is discussed briefly by Bachmann [1, pp. 27-29] and Turner [9]. We note that $T_{k,n}(a,d)$ can be expressed in terms of $S_{k,n}(a,d)$ in the following ways:

$$T_{k,2n+1}(a,d) = S_{k,n+1}(a,2d) - S_{k,n}(a+d,2d),$$
$$T_{k,2n}(a,d) = S_{k,n}(a,2d) - S_{k,n}(a+d,2d).$$

Also, if $a = d = 1$, then

$$T_{k,2n} = S_{k,2n} - 2^{k-1}S_{k,n}, \tag{6.1}$$

which makes some of the formulas for $T_{k,n}$ trivial in light of the results of Sections 1-5.

In the remainder of the paper, we find formulas for $T_{k,n}(a,d)$ that correspond to the ones for $S_{k,n}(a,d)$. The essential tool is the generating function

$$\sum_{k=0}^{\infty} T_{k,n}(a,d)\frac{x^k}{k!} = e^{ax} - e^{(a+d)x} + \cdots + (-1)^{n-1}e^{(a+nd-d)x} = \frac{(-1)^{n-1}e^{(a+nd)x} + e^{ax}}{e^{dx}+1}. \tag{6.2}$$

The following three theorems are analogs of Theorems 2.1, 2.2, and 3.1, and they are proved in exactly the same way as those earlier theorems. The proofs, which use (6.2) instead of (2.1), are omitted.

Theorem 6.1: We have the following two recurrences for $T_{k,n}(a,d)$: For $k \geq 1, n > 0$,

$$2T_{k+1,n}(a,d) + \sum_{j=0}^{k}\binom{k+1}{j}d^{k+1-j}T_{j,n}(a,d) = (-1)^{n-1}(a+nd)^{k+1} + a^{k+1}, \tag{6.3}$$

$$2T_{k+1,n}(a,d) + \sum_{j=0}^{k}\binom{k+1}{j}(-d)^{k+1-j}T_{j,n}(a,d) = (-1)^{n+1}(a+nd-d)^{k+1} + (a-d)^{k+1}. \tag{6.4}$$

Formula (6.3) generalizes a formula of Turner [9].

Theorem 6.2: For $k = 1, 2, 3, \ldots$, we have

$$2\sum_{j=0}^{k-1}\binom{2k-1}{2j}d^{2k-2j-1}T_{2j,n}(a,d)$$
$$= (-1)^{n-1}(a+nd)^{2k-1} + (-1)^n(a+nd-d)^{2k-1} - (a-d)^{2k-1} + a^{2k-1},$$

and

$$2\sum_{j=1}^{k}\binom{2k}{2j-1}d^{2k-2j+1}T_{2j-1,n}(a,d)$$
$$= (-1)^{n-1}(a+nd)^{2k} + (-1)^n(a+nd-d)^{2k} - (a-d)^{2k} + a^{2k}.$$

Theorem 6.3: Let w_j be defined by (3.2) and (3.3). Then, for $m = 0, 1, \ldots, 5$, and $n > 0, k \geq 0$, we have

$$8T_{6k+m,n}(a,d) = -6\sum_{j=0}^{k-1}\binom{6k+m}{6j+m}d^{6k-6j}T_{6j+m,n}(a,d)$$
$$+\sum_{j=0}^{6k+m-1}\binom{6k+m}{j}d^{6k+m-j}w_{m-j}[(-1)^{n-1}(a-d+nd)^j+(a-d)^j]$$
$$+4[(-1)^{n-1}(a-d+nd)^{6k+m}+(a-d)^{6k+m}].$$

Corollary: Let $n>0$, $k\geq 0$. For $m=0,1,\ldots,5$, and m and k not both 0, we have

$$8T_{6k+m,n} = -6\sum_{j=0}^{k-1}\binom{6k+m}{6j+m}T_{6j+m,n}+(-1)^{n-1}\sum_{j=0}^{6k+m-1}\binom{6k+m}{j}w_{m-j}n^j$$
$$+4(-1)^{n-1}n^{6k+m}+[1+(-1)^{n-1}]w_m.$$

Note that, for $m=1,2,\ldots,5$, we have

$$8T_{m,n} = (-1)^{n-1}\sum_{j=1}^{m-1}\binom{m}{j}w_{m-j}n^j+4(-1)^{n-1}n^m+[1+(-1)^{n-1}]w_m.$$

To illustrate Theorem 6.3, we first calculate $T_{4,n}$:

$$8T_{4,n} = (-1)^{n-1}\left[4n^4+\binom{4}{3}2n^3+\binom{4}{2}0n^2+\binom{4}{1}(-1)n+0\right];$$

so

$$T_{4,n} = (-1)^{n-1}\left(-\frac{1}{2}n+n^3+\frac{1}{2}n^4\right).$$

Then

$$8T_{10,n} = -6\binom{10}{4}T_{4,n}+(-1)^{n-1}\sum_{j=1}^{9}\binom{10}{j}w_{4-j}n^j+(-1)^{n-1}4n^{10},$$

which gives us

$$T_{10,n} = (-1)^{n-1}\left(\frac{155}{2}n-\frac{255}{2}n^3+63n^5-15n^7+\frac{5}{2}n^9+\frac{1}{2}n^{10}\right).$$

Continuing in the same way, we have

$$8T_{16,n} = -6\binom{16}{4}T_{4,n}-6\binom{16}{10}T_{10,n}+(-1)^{n-1}\sum_{j=1}^{15}\binom{16}{j}w_{4-j}n^j+(-1)^{n-1}4n^{16},$$

which gives us

$$T_{16,n} = (-1)^{n-1}\left(-\frac{929569}{2}n+764540n^3-377286n^5+88660n^7\right.$$
$$\left.-12155n^9+1092n^{11}-70n^{13}+4n^{15}+\frac{1}{2}n^{16}\right).$$

$T_{k,n}(a,d)$ IN TERMS OF GENOCCHI NUMBERS

The Euler polynomial $E_k(z)$ may be defined by the generating function [2, pp. 48-49]

$$\frac{2e^{xz}}{e^x+1}=\sum_{k=0}^{\infty}E_k(z)\frac{x^k}{k!}. \tag{7.1}$$

For $z=1$, we have

$$E_k(1)=\frac{2(2^{k+1}-1)}{k+1}B_{k+1}=-\frac{1}{k+1}G_{k+1}, \tag{7.2}$$

where B_{k+1} is a Bernoulli number and G_{k+1} is called a Genocchi number [2, p. 49]. The Genocchi numbers are integers such that $G_{2m+1}=0$ for $m>0$; the first few are $G_0=0$, $G_1=1$, $G_2=-1$, $G_4=1$, $G_6=-3$. It follows from (7.1) that

$$E_k(z)=\sum_{j=0}^{k}\binom{k}{j}\frac{G_{k-j+1}}{k-j+1}z^j. \tag{7.3}$$

Comparing (7.1) and (6.2), we see that

$$T_{k,n}(a,d)=\frac{d^k}{2}\left[(-1)^{n-1}E_k\left(\frac{a}{d}+n\right)+E_k\left(\frac{a}{d}\right)\right]. \tag{7.4}$$

By (7.3) and (7.4), we have the following result. For fixed a and d, if we write $T_{k,n}(a,d)$ as a polynomial in $(a-d+nd)$, i.e.,

$$\begin{aligned}T_{k,n}(a,d)&=T_{k,n-1}(a,d)+(a-d+nd)^k\\&=u_{k,0}+u_{k,1}(a-d+nd)+u_{k,2}(a-d+nd)^2+\cdots+u_{k,k}(a-d+nd)^k,\end{aligned} \tag{7.5}$$

then we have explicit formulas for the coefficients $u_{k,j}$ in terms of Genocchi numbers.

Theorem 7.1: If $T_{k,n}(a,d)$ is written as a polynomial in $(a-d+nd)$ and $u_{k,j}$ is defined by (7.5) for $j=0,1,\ldots,k$, then

$$u_{k,j}=\frac{(-1)^n d^{k-j}}{2(k-j+1)}\binom{k}{j}G_{k-j+1}\quad(1\le j\le k-1),$$

$$u_{k,0}=\frac{(-1)^n d^k}{2(k+1)}G_{k+1}+\frac{d^k}{2}E_k\left(\frac{a}{d}\right),\quad u_{k,k}=\frac{(-1)^{n-1}}{2}.$$

When $a=d=1$, we have

$$T_{k,n}=T_{k,n}(1,1)=\frac{(-1)^{n-1}}{2}n^k+\frac{(-1)^n}{2}\sum_{j=0}^{k-1}\binom{k}{j}\frac{G_{k-j+1}}{k-j+1}n^j-\frac{G_{k+1}}{2(k+1)}. \tag{7.6}$$

When n is even, (7.6) follows from (6.1) and the formulas in Section 6 [1, p. 27].

For example, $T_{3,n}=1^3-2^3+\cdots+(-1)^{n-1}n^3$

$$=\frac{(-1)^{n-1}}{2}n^3+\frac{3(-1)^n}{4}G_2n^2+\frac{(-1)^n}{2}G_3n+\frac{[(-1)^n-1]}{8}G_4=\frac{(-1)^{n-1}}{8}[4n^3+6n^2-1+(-1)^n].$$

An application of Theorem 6.3 that is analogous to (4.5) is the following. If n is odd and $a = d = 1$, by Theorem 7.1 we have $u_{k,0} = -G_{k+1}/(k+1)$. Thus, by Theorems 7.1 and 6.3 we have, for $m = 1, 2, \ldots, 5$,

$$8G_{6k+m+1} = -6\sum_{j=0}^{k-1}\binom{6k+m+1}{6j+m+1}G_{6j+m+1} - 2(6k+m+1)w_m,$$

which is equivalent to a formula of Ramanujan [7, p. 12].

8. FINDING $T_{k,n}(a,d)$ FROM $T_{k-1,n}(a,d)$

We proceed as we did for $S_{k,n}(a,d)$. By using induction on (6.3), we can prove that $T_{k,n}(a,d)$ is a polynomial in n of degree k, and the constant term is 0 if n is even. That also follows from the results of Section 7. Thus, for fixed a and d, we can write

$$T_{k,n}(a,d) = \begin{cases} t_{k,1}n + t_{k,2}n^2 + \cdots + t_{k,k}n^k & (n \text{ even}), \\ h_{k,0} + h_{k,1}n + \cdots + h_{k,k}n^k & (n \text{ odd}). \end{cases} \tag{8.1}$$

Using the generating function (6.2), we prove the next theorem just as we proved Theorem 5.1.

Theorem 8.1: For fixed a and d, let $t_{k,j}$ and $h_{k,j}$ be defined by (8.1) for $j = 0, \ldots, k$. Then for $k \geq 1$ we have

$$t_{k,j} = \frac{kd}{j}t_{k-1,j-1} \quad (j = 2, \ldots, k); \qquad h_{k,j} = \frac{kd}{j}h_{k-1,j-1} \quad (j = 2, \ldots, k);$$

$$t_{k,1} = -\frac{kd^k}{2}E_{k-1}\left(\frac{a}{d}\right) = \frac{1}{2}[a^k - (a+d)^k] - (2t_{k,2} + 2^2t_{k,3} + \cdots + 2^{k-1}t_{k,k});$$

$$h_{k,0} = a^k - h_{k,1} - \cdots - h_{k,k};$$

$$h_{k,1} = -\frac{kd^k}{2}E_{k-1}\left(\frac{a}{d}\right) + kd\,h_{k-1,0} = t_{k,1} + kd\,h_{k-1,0}.$$

Thus, if we know $T_{k-1,n}(a,d)$, we can use Theorem 8.1 to find $T_{k,n}(a,d)$. For example,

$$T_{0,n}(a,d) = \begin{cases} 0 & (n \text{ even}), \\ 1 & (n \text{ odd}); \end{cases} \qquad T_{1,n}(a,d) = \begin{cases} -(d/2)n & (n \text{ even}), \\ (a - d/2) + (d/2)n & (n \text{ odd}). \end{cases}$$

Then, for n even, we have $T_{2,n}(a,d) = t_{2,1}n + t_{2,2}n^2$, with

$$t_{2,2} = dt_{1,1} = -d^2/2; \qquad t_{2,1} = \frac{1}{2}[a^2 - (a+d)^2] + d^2 = -ad + d^2/2.$$

Thus,

$$T_{2,n}(a,d) = \left(\frac{1}{2}d^2 - ad\right)n - \frac{1}{2}d^2n^2 \quad (n \text{ even}).$$

For n odd, $T_{2,n}(a,d) = h_{2,0} + h_{2,1}n + h_{2,2}n^2$, with

$$h_{2,2} = dh_{1,1} = d^2/2; \quad h_{2,1} = t_{2,1} + 2dh_{1,0} = ad - d^2/2; \quad h_{2,0} = a^2 - h_{2,1} - h_{2,2} = a^2 - ad.$$

Thus,

$$T_{2,n}(a,d) = (a^2 - ad) + \left(ad - \frac{1}{2}d^2\right)n + \frac{1}{2}d^2n^2 \quad (n \text{ odd}).$$

Equivalently, by Theorem 8.1, we can integrate to find $S_{k,n}(a,d)$:

$$S_{k,n}(a,d) = kd\int S_{k-1,n}(a,d)\,dn - \frac{kd^k}{2}E_{k-1}\left(\frac{a}{d}\right)n + h_k(n),$$

where $E_{k-1}(\frac{a}{d})$ can be found by means of Theorem 8.1, and

$$h_k(n) = \begin{cases} 0 & (n \text{ even}), \\ h_{k,0} = a^k - h_{k,1} - h_{k,2} - \cdots - h_{k,k} & (n \text{ odd}). \end{cases}$$

9. FINAL COMMENTS

In summary, we have used generating functions to prove and generalize some of the basic formulas for sums of powers of integers. In particular, we have used the generating function technique to find: recurrence relations for $S_{k,n}(a,d)$ and $T_{k,n}(a,d)$; explicit formulas (involving Bernoulli numbers) for $S_{k,n}(a,d)$ and $T_{k,n}(a,d)$ if they are written as polynomials in n; methods for finding $S_{k,n}(a,d)$ and $T_{k,n}(a,d)$ from $S_{k-1,n}(a,d)$ and $T_{k-1,n}(a,d)$. Some of the results are old (and scattered in the literature), and most of the proofs are straightforward. However, the writer believes that many of the generalizations are new, and he believes that Theorems 3.1 and 6.1 give us new recurrence formulas that are of interest.

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