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Incomplete Tribonacci Numbers and Polynomials

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Abstract

We define the incomplete tribonacci sequence of numbers and polynomials. We study recurrence relations, some properties of these numbers and polynomials, and the generating function of the incomplete tribonacci numbers.

1 Introduction

Fibonacci numbers and their generalizations have many interesting properties and applications in many fields of science and art (cf. [10]). The Fibonacci numbers F_n are defined by

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the recurrence relation

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}, \ n \geqslant 1.$$

The first few terms are 0,1,1,2,3,5,8,13,... (sequence A000045)[3]. Another important sequence is the Lucas sequence. These numbers are defined by the recurrence relation

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+1} = L_n + L_{n-1}, \ n \geqslant 1.$$

The first few terms are 2,1,3,7,11,18,29,37,... (sequence A000032). The Fibonacci and Lucas numbers have been studied extensively. In particular, there is a beautiful combinatorial identity for Fibonacci numbers and Lucas numbers (cf. [10]):

$$F_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i}, \qquad L_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j}. \tag{1}$$

In analogy with (1), Filipponi [6] introduced the incomplete Fibonacci numbers $F_n(s)$ and the incomplete Lucas numbers $L_n(s)$. They are defined by

$$F_n(s) = \sum_{j=0}^{s} \binom{n-1-j}{j}, \quad \left(n = 1, 2, 3, \ldots; 0 \leq s \leq \left\lfloor \frac{n-1}{2} \right\rfloor \right);$$

and

$$L_n(s) = \sum_{j=0}^{s} \frac{n}{n-j} \binom{n-j}{j}, \quad \left(n = 1, 2, 3, \ldots; 0 \leq s \leq \left\lfloor \frac{n}{2} \right\rfloor \right).$$

Generating functions of the incomplete Fibonacci and Lucas numbers were determined by Pintér and Srivastava [15]. Djordjević [3] defined and studied incomplete generalized Fibonacci and Lucas numbers. Djordjević and Srivastava [4] defined incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers. Tasci and Cetin Firengiz [21] defined the incomplete Fibonacci and Lucas p-numbers. Tasci et al. [22] defined the incomplete bivariate Fibonacci and Lucas p-polynomials. Ramírez [18] introduced the incomplete k-Fibonacci and k-Lucas numbers, the incomplete $h(x)$-Fibonacci and $h(x)$-Lucas polynomials [17], and the bi-periodic incomplete Fibonacci sequences [16].

The tribonacci numbers are defined by the recurrence relation:

$$t_0 = 0, \quad t_1 = t_2 = 1, \quad t_{n+2} = t_{n+1} + t_n + t_{n-1} \quad \text{for} \ \ n \geqslant 1. \tag{2}$$

The first few terms of the tribonacci numbers are 0, 1, 1, 2, 4, 7, 13, 24, 44, 81,..., (sequence A000073). The tribonacci numbers have been studied in different contexts; see [1, 5, 8, 9, 11, 12, 13, 14, 19]. Alladi and Hoggatt [1] defined the tribonacci triangle; see

	0	1	2	3	4	5	6	7	$\cdots$
0	1								
1	1	1							
2	1	3	1						
3	1	5	5	1					
4	1	7	13	7	1				
5	1	9	25	25	9	1			
6	1	11	41	63	41	11	1		
7	1	13	61	129	129	61	13	1	
$\vdots$				$\vdots$					

Table 1: Tribonacci triangle

Table 1. It was used to derive the expansion of the tribonacci numbers and each element is defined in similar way as in the Pascal Triangle (cf. [10]).

Let $B(n,i)$ be the element in the n-th row and i-th column of the tribonacci triangle. By the definition of the triangle, we have

$$B(n+1,i) = B(n,i) + B(n,i-1) + B(n-1,i-1), \tag{3}$$

where $B(n,0) = B(n,n) = 1$. The sum of elements on the rising diagonal lines in the tribonacci triangle is the tribonacci number t_n (cf. [1]), i.e.,

$$t_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} B(n-1-i,i).$$

Moreover, Barry [2] showed that these coefficients satisfy the relation

$$B(n,i) = \sum_{j=0}^{i} \binom{i}{j} \binom{n-j}{i}.$$

Therefore,

$$t_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} B(n-1-i,i) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{n-1-i-j}{i}. \tag{4}$$

A large class of polynomials can also be defined by Fibonacci-like recurrences and tribonacci-like recurrences, such that yield Fibonacci numbers and tribonacci numbers. Such polynomials are called Fibonacci polynomials [10] and tribonacci polynomials [7], respectively.

[3]Many integer sequences and their properties are given in the *On-Line Encyclopedia of Integer Sequences* [20].

In 1883, Catalan and Jacobsthal introduced Fibonacci polynomials (cf. [10]). The polynomials $F_n(x)$, studied by Catalan, are defined by the recurrence relation

$$F_0(x) = 0, \quad F_1(x) = 1, \quad F_{n+1}(x) = xF_n(x) + F_{n-1}(x), \ n \geqslant 1.$$

The Fibonacci polynomials, studied by Jacobsthal, are defined by

$$J_0(x) = 1, \quad J_1(x) = 1, \quad J_{n+1}(x) = J_n(x) + xJ_{n-1}(x), \ n \geqslant 1.$$

The Lucas polynomials $L_n(x)$, originally studied in 1970 by Bicknell, are defined by

$$L_0(x) = 2, \quad L_1(x) = x, \quad L_{n+1}(x) = xL_n(x) + L_{n-1}(x), \ n \geqslant 1.$$

Hoggatt and Bicknell [7] introduced tribonacci polynomials. The tribonacci polynomials $T_n(x)$ are defined by the recurrence relation

$$T_0(x) = 0, \quad T_1(x) = 1, \quad T_2(x) = x^2, \quad T_{n+2}(x) = x^2T_{n+1}(x) + xT_n(x) + T_{n-1}(x), \ n \geqslant 2.$$

Note that $T_n(1) = t_n$ for all integer positive n. The first few tribonacci polynomials are

$$\begin{array}{ll} T_1(x) = 1, & T_5(x) = x^8 + 3x^5 + 3x^2, \\ T_2(x) = x^2, & T_6(x) = x^{10} + 4x^7 + 6x^4 + 2x, \\ T_3(x) = x^4 + x, & T_7(x) = x^{12} + 5x^9 + 10x^6 + 7x^3 + 1, \\ T_4(x) = x^6 + 2x^3 + 1, & T_8(x) = x^{14} + 6x^{11} + 15x^8 + 16x^5 + 6x^2. \end{array}$$

In analogy with the tribonacci triangle, we define the tribonacci polynomial triangle; see Table 2.

	0	1	2	3	4	5 $\cdots$
0	1					
1	x^2	x				
2	x^4	$2x^3+1$	x^2			
3	x^6	$3x^5+2x^2$	$3x^4+2x$	x^3		
4	x^8	$4x^7+3x^4$	$6x^6+6x^3+1$	$4x^5+3x^2$	x^4	
5	x^{10}	$5x^9+4x^6$	$10x^8+12x^5+3x^2$	$10x^7+12x^4+3x$	$5x^6+4x^3$	x^5
$\vdots$				$\vdots$		

Table 2: Tribonacci polynomial triangle

Let $B(n, i)(x)$ be the element in the n-th row and i-th column of the tribonacci polynomial triangle. Then

$$B(n+1, i)(x) = x^2B(n, i)(x) + xB(n, i-1)(x) + B(n-1, i-1)(x),$$

where $B(n, 0)(x) = x^{2n}$ and $B(n, n) = x^n$.

It can be proved by induction on n, that the sum of elements on the rising diagonal lines in the tribonacci polynomial triangle is the tribonacci polynomial $T_n(x)$, i.e.,

$$T_n(x) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} B(n-1-i,i)(x).$$

Moreover,

$$B(n,i)(x) = \sum_{j=0}^{i} \binom{i}{j} \binom{n-j}{i} x^{2n-i-3j}.$$

Therefore,

$$T_n(x) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} B(n-1-i,i)(x) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{n-1-i-j}{i} x^{2n-3(i+j)-2}. \tag{5}$$

From Equations (4) and (5), we introduce the incomplete tribonacci polynomials and incomplete tribonacci numbers. In this way we obtain new recurrence relations, new identities and the generating function of the incomplete tribonacci numbers.

2 Incomplete tribonacci polynomials and incomplete tribonacci numbers

Definition 1. For $n \geqslant 1$, *incomplete tribonacci polynomials* are defined as

$$T_n^{(s)}(x) = \sum_{i=0}^{s} B(n-1-i,i)(x) \tag{6}$$

$$= \sum_{i=0}^{s} \sum_{j=0}^{i} \binom{i}{j} \binom{n-i-j-1}{i} x^{2n-3(i+j)-2}, \ 0 \leqslant s \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{7}$$

We define the *incomplete tribonacci numbers*, $t_n(s)$ as the value of $T_n^{(s)}(x)$ at $x = 1$, i.e., $t_n(s) = T_n^{(s)}(1)$.

In Tables 3 and 4, some values of incomplete tribonacci polynomials and incomplete tribonacci numbers are provided.

From the definitions follow

$$T_n^{\left(\lfloor \frac{n-1}{2} \rfloor\right)}(x) = T_n(x) \ \text{ and } \ t_n\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) = t_n.$$

n/s	0	1	2	3
1	1			
2	x^2			
3	x^4	x^4+x		
4	x^6	x^6+2x^3+1		
5	x^8	$x^8+3x^5+2x^2$	$x^8+3x^5+3x^2$	
6	x^{10}	$x^{10}+4x^7+3x^4$	$x^{10}+4x^7+6x^4+2x$	
7	x^{12}	$x^{12}+5x^9+4x^6$	$x^{12}+5x^9+10x^6+6x^3+1$	$x^{12}+5x^9+10x^6+7x^3+1$
8	x^{14}	$x^{14}+6x^{11}+5x^8$	$x^{14}+6x^{11}+15x^8+12x^5+3x^2$	$x^{14}+6x^{11}+15x^8+16x^5+6x^2$

Table 3: Polynomials $T_n^{(s)}(x)$, for $1 \leqslant n \leqslant 8$ and $0 \leqslant s \leqslant 3$.

Some special cases of (7) are

$$T_n^{(0)}(x) = x^{2n-2},\ (n \geq 1); \tag{8}$$

$$T_n^{(1)}(x) = x^{2n-2} + (n-2)x^{2n-5} + (n-3)x^{2n-8},\ (n \geq 3); \tag{9}$$

$$T_n^{(\lfloor \frac{n-1}{2} \rfloor)}(x) = T_n(x),\ (n \geq 1); \tag{10}$$

$$T_n^{(\lfloor \frac{n-3}{2} \rfloor)}(x) = \begin{cases} T_n(x) - \left(\frac{n}{2}x^{\frac{n+2}{2}} + \frac{n-2}{2}x^{\frac{n-4}{2}}\right), & \text{if } n \geq 3 \text{ and even;} \\ T_n(x) - x^{\frac{n-1}{2}}, & \text{if } n \geq 3 \text{ and odd.} \end{cases} \tag{11}$$

Throughout the rest of this section, we describe some recurrence properties of the polynomials $T_n^{(s)}(x)$ and numbers $t_n(s)$.

Proposition 2. *The non-linear recurrence relation of the incomplete tribonacci polynomials $T_n^{(s)}(x)$ is*

$$T_{n+3}^{(s+1)}(x) = x^2 T_{n+2}^{(s+1)}(x) + x T_{n+1}^{(s)}(x) + T_n^{(s)}(x),\ 0 \leqslant s \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{12}$$

The relation (12) can be transformed into the non-homogeneous recurrence relation

$$T_{n+3}^{(s)}(x) = x^2 T_{n+2}^{(s)}(x) + x T_{n+1}^{(s)}(x) + T_n^{(s)}(x) - \left(xB(n-s,s)(x) + B(n-1-s,s)(x)\right). \tag{13}$$

Proof. We use the Definition 1 to rewrite the right-hand side of (12) as

$$x^2 \sum_{i=0}^{s+1} B(n+1-i,i)(x) + x \sum_{i=0}^{s} B(n-i,i)(x) + \sum_{i=0}^{s} B(n-1-i,i)(x).$$

n/s	0	1	2	3	4	5	6	7
1	1							
2	1							
3	1	2						
4	1	4						
5	1	6	7					
6	1	8	13					
7	1	10	23	24				
8	1	12	37	44				
9	1	14	55	80	81			
10	1	16	77	140	149			
11	1	18	103	232	273	274		
12	1	20	133	364	493	504		
13	1	22	167	544	865	926	927	
14	1	24	205	780	1461	1692	1705	
15	1	26	247	1080	2369	3050	3135	3136
16	1	28	293	1452	3693	5376	5753	5768

Table 4: Numbers $t_n(s)$, for $1 \leqslant n \leqslant 16$ and $0 \leqslant s \leqslant 7$.

Therefore,

$$
\begin{aligned}
&x^2 T_{n+2}^{(s+1)}(x) + xT_{n+1}^{(s)}(x) + T_n^{(s)}(x) = \\
&\qquad = x^2 \sum_{i=0}^{s+1} B(n+1-i,i)(x) + x \sum_{i=1}^{s+1} B(n-i+1,i-1)(x) + \sum_{i=1}^{s+1} B(n-i,i-1)(x) \\
&\qquad = \sum_{i=0}^{s+1} \left(x^2 B(n+1-i,i)(x) + xB(n-i+1,i-1)(x) + B(n-i,i-1)(x) \right) \\
&\qquad - xB(n+1,-1)(x) - B(n,-1)(x) \\
&\qquad = \sum_{i=0}^{s+1} B(n+2-i,i)(x) = T_{n+3}^{(s+1)}(x).
\end{aligned}
$$

□

Corollary 3. *The non-linear recurrence relation of the incomplete tribonacci numbers $t_n(s)$ is*

$$t_{n+3}(s+1) = t_{n+2}(s+1) + t_{n+1}(s) + t_n(s), \ \ 0 \leqslant s \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor . \tag{14}$$

The relation (14) can be transformed into the non-homogeneous recurrence relation

$$t_{n+3}(s) = t_{n+2}(s) + t_{n+1}(s) + t_n(s) - \left(B(n-s,s) + B(n-1-s,s) \right). \tag{15}$$

Corollary 4. *For $n \geq 2s+2$,*

$$\sum_{i=0}^{h-1} t_{n+i}(s) = \frac{1}{2}\left(t_{n+h+2}(s+1) - t_{n+2}(s+1) + t_n(s) - t_{n+h}(s)\right). \tag{16}$$

Proof. We proceed by induction on h. The sum (16) clearly holds for $h = 1$; see (14). Now suppose that the result is true for all $i < h$. We prove it for h:

$$\begin{aligned}\sum_{i=0}^{h} t_{n+i}(s) &= \sum_{i=0}^{h-1} t_{n+i}(s) + t_{n+h}(s)\\ &= \frac{1}{2}\left(t_{n+h+2}(s+1) - t_{n+2}(s+1) + t_n(s) - t_{n+h}(s)\right) + t_{n+h}(s)\\ &= \frac{1}{2}\left(t_{n+h+2}(s+1) - t_{n+2}(s+1) + t_n(s) + t_{n+h}(s)\right)\\ &= \frac{1}{2}\left(t_{n+h+3}(s+1) - t_{n+2}(s+1) + t_n(s) - t_{n+h+1}(s)\right).\end{aligned}$$

□

The following proposition shows the sum of the n-th row of the Table 3.

Proposition 5. *The following equality holds:*

$$\sum_{s=0}^{l} T_n^{(s)}(x) = (l+1)T_n(x) - \sum_{i=0}^{l}\sum_{j=0}^{i} i\binom{i}{j}\binom{n-i-j-1}{i}x^{2n-3(i+j)-2}, \tag{17}$$

where $l = \left\lfloor \frac{n-1}{2} \right\rfloor$.

Proof. We know

$$T_n^{(s)}(x) = \sum_{i=0}^{s} B(n-1-i, i)(x).$$

Then, we have

$$\begin{aligned}\sum_{s=0}^{l} T_n^{(s)}(x) &= B(n-1-0,0)(x) + (B(n-1-0,0)(x) + B(n-1-1,1)(x)) + \cdots\\ &+ (B(n-1-0,0)(x) + B(n-1-1,1)(x) + \cdots + B(n-1-l,l)(x))\\ &= (l+1)B(n-1-0,0)(x) + lB(n-1-1,1)(x) + \cdots + B(n-1-l,l)(x).\end{aligned}$$

Hence

$$\begin{aligned}\sum_{s=0}^{l} T_n^{(s)}(x) &= \sum_{i=0}^{l}(l+1-i)B(n-i-1,i)(x)\\ &= \sum_{i=0}^{l}(l+1)B(n-i-1,i)(x) - \sum_{i=0}^{l} iB(n-i-1,i)(x)\\ &= (l+1)T_n(x) - \sum_{i=0}^{l} iB(n-i-1,i)(x).\end{aligned}$$

□

The following corollary shows the sum of the n-th row of the Table 4. It is obtained from (17) with $x = 1$.

Corollary 6. *The following equality holds:*

$$\sum_{s=0}^{l} t_n(s) = (l+1)t_n - \sum_{i=0}^{l}\sum_{j=0}^{i} i\binom{i}{j}\binom{n-i-j-1}{i},$$

where $l = \left\lfloor \frac{n-1}{2} \right\rfloor$.

Let

$$a_n = \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor}\sum_{j=0}^{i} i\binom{i}{j}\binom{n-i-j-1}{i}.$$

We conjecture that the generating function of $(a_n)_{n=0}^{\infty}$ is

$$\frac{z^3+z^4}{(1-z-z^2-z^3)^2} = (z^3+z^4)T^2(z),$$

where $T(z)$ is the generating function of the tribonacci numbers and $T^2(z)$ is the generating function of the convolution of tribonacci sequence; see sequence A073778. If this conjecture is true, then

$$\sum_{s=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} t_n(s) = (l+1)t_n - \sum_{j=0}^{n}(t_{j-2}+t_{j-3})t_{n-j+1}.$$

3 Generating function of the incomplete tribonacci numbers

In this section, we give the generating functions of the incomplete tribonacci numbers.

Lemma 7. *Let $(s_n)_{n=0}^{\infty}$ be a complex sequence satisfying the following non-homogeneous and non-linear recurrence relation*

$$s_n = s_{n-1} + s_{n-2} + s_{n-3} + r_n, \ (n > 2), \tag{18}$$

where $(r_n)_{n=0}^{\infty}$ is a given complex sequence. Then the generating function $U(t)$ of the sequence $(s_n)_{n=0}^{\infty}$ is

$$U(t) = \frac{G(t) + s_0 - r_0 + (s_1 - s_0 - r_1)t + (s_2 - s_1 - s_0 - r_2)t^2}{1 - t - t^2 - t^3} \tag{19}$$

where $G(t)$ denotes the generating function of $(r_n)_{n=0}^{\infty}$.

Proof. Since $U(t)$ and $G(t)$ are the generating functions of $(s_n)_{n=0}^{\infty}$ and $(r_n)_{n=0}^{\infty}$, respectively. Their power series representations are

$$U(t) = s_0 + s_1 t + s_2 t^2 + \cdots + s_k t^k + \cdots ,$$
$$G(t) = r_0 + r_1 t + r_2 t^2 + \cdots + r_k t^k + \cdots .$$

Note that,

$$tU(t) = s_0 t + s_1 t^2 + s_2 t^3 + \cdots + s_k t^{k+1} + \cdots ,$$
$$t^2 U(t) = s_0 t^2 + s_1 t^3 + s_2 t^4 + \cdots + s_k t^{k+2} + \cdots ,$$
$$t^3 U(t) = s_0 t^3 + s_1 t^4 + s_2 t^5 + \cdots + s_k t^{k+3} + \cdots .$$

Therefore

$$(1 - t - t^2 - t^3)U(t) - G(t) = (s_0 - r_0) + (s_1 - s_0 - r_1)t + (s_2 - s_1 - s_0 - r_2)t^2.$$

Then the Equation (19) follows. □

Theorem 8. *Let $Q_s(z)$ be the generating function of the incomplete tribonacci numbers $t_n(s)$. Then*

$$Q_s(z) = \frac{t_{2s+1} + (t_{2s+2} - t_{2s+1})z + (t_{2s+3} - t_{2s+2} - t_{2s+1} - 2)z^2 - (z^2 + z^3)\dfrac{(1+z)^s}{(1-z)^{s+1}}}{1 - z - z^2 - z^3}.$$

Proof. Let s be a fixed positive integer and $t_n(s)$ the n-th incomplete tribonacci number. Since $Q_s(z)$ is the generating function of the $t_n(s)$, we have $Q_s(z) = \sum_{i=0}^{\infty} t_i(s)z^i$.

From (7) with $x = 1$ and (15), we get $t_n(s) = 0$ for $0 \leq n < 2s+1$, $t_{2s+1}(s) = t_{2s+1}, t_{2s+2}(s) = t_{2s+2}$ and $t_{2s+3}(s) = t_{2s+3} - 1$, and that

$$t_n(s) = t_{n-1}(s) + t_{n-2}(s) + t_{n-3}(s) - \left(B(n-3-s,s) + B(n-4-s,s)\right). \tag{20}$$

Now let

$$s_0 = t_{2s+1}(s), \quad s_1 = t_{2s+2}(s), \quad s_2 = t_{2s+3}(s) \quad \text{and}$$

$$s_n = t_{n+2s+1}(s).$$

Also let $r_0 = r_1 = 0, r_1 = 1$ and

$$\begin{aligned} r_n &= B(n+s-2,s) + B(n+s-3,s) \\ &= \sum_{j=0}^{s} \binom{s}{j}\binom{n+s-2-j}{s} + \sum_{j=0}^{s} \binom{s}{j}\binom{n+s-3-j}{s}. \end{aligned}$$

The generating function of the sequence $(r_n)_{n\geq 0}$ is computed using the methods expounded in [23, page 127]. Hence the generating function is equal to

$$(z^2+z^3)\frac{(1+z)^s}{(1-z)^{s+1}}.$$

Thus, from Lemma 7, we get the generating function $Q_s(z)$ of sequence $(t_n(s))_{n=0}^{\infty}$. □

Example 9. The generating functions of the incomplete tribonacci numbers for $s = 1, 2, 3, 4$ are

$$\begin{aligned} Q_1(z) &= \frac{2}{(z-1)^2} = 2 + 4z + 6z^2 + 8z^3 + 10z^4 + 12z^5 + 14z^6 + 16z^7 + 18z^8 + \cdots \\ Q_2(z) &= \frac{-5z^2+8z-7}{(z-1)^3} = 7 + 13z + 23z^2 + 37z^3 + 55z^4 + 77z^5 + 103z^6 + 133z^7 + \cdots \\ Q_3(z) &= \frac{-12z^3+48z^2-52z+24}{(z-1)^4} = 24 + 44z + 80z^2 + 140z^3 + 232z^4 + 364z^5 + \cdots \\ Q_4(z) &= \frac{-45z^4+192z^3-338z^2+256z-81}{(z-1)^5} = 81 + 149z + 273z^2 + 493z^3 + 865z^4 + \cdots \end{aligned}$$

An open problem is to find the generating function of the incomplete tribonacci polynomials.

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