GENERATING FUNCTIONS OF THE INCOMPLETE GENERALIZED FIBONACCI AND GENERALIZED LUCAS NUMBERS

Gospava B. Djordjević

University of Niš, Faculty of Technology
16 000 Leskovac, Serbia and Montenegro

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1. INTRODUCTION

The polynomials $\Phi_n(p,q;x)$ are studied in [1]. In this note we consider these polynomials for $p=0$ and $q=-1$. Namely, we introduce the polynomials $U_n(x)=\Phi_n(0,-1;x)$ with $U_0(x)=0, U_1(x)=1, U_2(x)=x$, and $V_n(x)=\Phi_n(0,-1;x)$ with $V_0(x)=2, V_1(x)=x$ and $V_2(x)=x^2$. So we have the following recurrence relations:

$$U_n(x)=xU_{n-1}(x)+U_{n-3}(x), n\geq 3, U_0(x)=0, U_1(x)=1, U_2(x)=x, \tag{1.1}$$

and

$$V_n(x)=xV_{n-1}(x)+V_{n-3}(x), n\geq 3, V_0(x)=2, V_1(x)=x, V_2(x)=x^2. \tag{1.2}$$

Let's note that $U_n(x)$ are the generalized Fibonacci polynomials, and $V_n(x)$ are the generalized Lucas polynomials.

Using the standard method, we can prove that the polynomials $U_n(x)$ and $V_n(x)$ possess generating functions as follows

$$U(t)=(1-xt-t^3)^{-1}=\sum_{n=0}^{\infty}U_{n+1}(x)t^n, \tag{1.3}$$

and

$$V(t)=(2-xt)/(1-xt-t^3)=\sum_{n=0}^{\infty}V_n(x)t^n. \tag{1.4}$$

Using (1.3) and (1.4), respectively, we find that

$$U_{n+1}(x)=\sum_{j=0}^{[n/3]}\binom{n-2j}{j}x^{n-2j}$$

is an explicit representation of the polynomials $U_{n+1}(x)$, and

$$V_n(x)=\sum_{j=0}^{[n/3]}\frac{n-j}{n-2j}\binom{n-2j}{j}x^{n-2j}$$

is an explicit representation of $V_n(x)$.

If $x = 1$ in (1.1) and (1.2), we get two sequences of numbers: $\{U_n(1)\}$ and $\{V_n(1)\}$. These sequences we denote by $\{U_n\}$ and $\{V_n\}$, respectively. Obviously, these sequences satisfy the recurrence relations

$$U_n = U_{n-1} + U_{n-3}, n > 3, U_1 = U_2 = U_3 = 1,$$

and

$$V_n = V_{n-1} + V_{n-3}, n > 3, V_1 = V_2 = 1, V_3 = 3.$$

It is easy to prove the relation

$$V_n = U_{n+1} + U_{n-2}.$$

The incomplete Fibonacci and Lucas numbers are studied in [3]. Namely, the corresponding generating functions of these numbers are found. Similarly, the incomplete generalized Fibonacci numbers and generalized Lucas numbers are discussed in this note and the corresponding generating functions are found.

2. INCOMPLETE NUMBERS

The incomplete generalized Fibonacci numbers $\{U_n(k)\}$ are defined by

$$U_n(k) = \sum_{j=0}^{k} \binom{n-1-2j}{j}, \ n = 1, 2, \ldots, 0 \le k \le [(n-1)/3], \tag{2.1}$$

and the incomplete generalized Lucas numbers $\{V_n(k)\}$ are defined by

$$V_n(k) = \sum_{j=0}^{k} \frac{n-j}{n-2j} \binom{n-2j}{j}, \ n = 1, 2, \ldots, 0 \le k \le [n/3]. \tag{2.2}$$

¿From (2.1) and (2.2), we see that

$$U_n([(n-1)/3]) = U_n \quad \text{and} \quad V_n([n/3]) = V_n,$$

where U_n and V_n are the generalized Fibonacci and generalized Lucas numbers, respectively.

The purpose of this paper is to derive the generating functions for these classes of numbers. First, we are going to prove the following statement.

Lemma 1: *Let $\{s_n\}_{n \in N}$ be a complex sequence satisfying the recurrence relation*

$$s_n = s_{n-1} + s_{n-3} + r_n, \quad n > 2,$$

where $r_n : N \to C$ is a given sequence. Then the generating function $F(t)$ of s_n is

$$F(t) = \frac{G(t) + s_0 - r_0 + t(s_1 - s_0 - r_1) + t^2(s_2 - s_1 - r_2)}{1 - t - t^3},$$

where $G(t)$ denotes the generating function of r_n.

Proof: This statement is a special case of a known result [3, p. 592, Lemma with $a = b = 1$], so the proof will be omitted.

Now we are going to prove the following theorem.

Theorem 1: *Let k be a positive integer. Then*

$$R_k(t) := \sum_{j=0}^{\infty} U_k(j)t^j = t^{3k+1}\left(\frac{A}{1-t-t^3} - \frac{t^3}{(1-t)^{k+1}(1-t-t^3)}\right), \tag{2.3}$$

where

$$A = U_{3k} + t(U_{3k+1} - U_{3k}) + t^2(U_{3k+2} - U_{3k+1}),$$

and

$$S_k(t) := \sum_{j=0}^{\infty} V_k(j)t^j = t^{3k}\left(\frac{B}{1-t-t^3} - \frac{t^3(2-t)}{(1-t)^{k+1}(1-t-t^3)}\right), \tag{2.4}$$

where

$$B = V_{3k+1} + t(V_{3k} - V_{3k-1}) + t^2(V_{3k+1} - V_{3k}).$$

Proof: In the proof of this theorem we use Lemma 1. Namely, let k be a fixed positive integer. From (2.1) and (2.2) it follows that $U_n(k) = 0$ for $0 \le n \le 3k+1, U_{3k+1}(k) = U_{3k}, U_{3k+2}(k) = U_{3k+1}, U_{3k+3}(k) = U_{3k+2}$, and

$$U_n(k) = U_{n-1}(k) + U_{n-3}(k) - \binom{n-4-2k}{n-4-3k}, \quad \text{if } n \ge 3k+4. \tag{2.5}$$

Let $s_0 = U_{3k+1}(k),\ s_1 = U_{3k+2}(k),\ s_2 = U_{3k+3}(k)$ and $s_n = U_{n+3k+1}(k)(n > 2)$, and $r_0 = r_1 = r_2 = 0$ and $r_n = \binom{n-3+k}{n-3}$.

It is simple to prove that $G(t) = t^3(1-t)^{-(k+1)}$ is the generating function of the sequence $r_n = \binom{n-3+k}{n-3}$.

¿From Lemma 1 and (2.5), it follows that the generating function $\Phi_k(t)$ of the sequence s_n satisfies the equality

$$\Phi_k(t) = \frac{U_{3k} + t(U_{3k+1} - U_{3k}) + t^2(U_{3k+2} - U_{3k+1})}{1-t-t^3} - \frac{t^3}{(1-t)^{k+1}(1-t-t^3)}.$$

Finally, the generating function $R_k(t)$ of the sequence $\{U_n(k)\}$ is $t^{3k+1}\Phi_k(t)$, and it immediately yields (2.3).

For the sequence $\{V_n(k)\}$ we have that: $V_n(k) = 0$ for $n < 3k, V_{3k}(k) = V_{3k-1}, V_{3k+1}(k) = V_{3k}, V_{3k+2}(k) = V_{3k+1}$ and

$$V_n(k) = V_{n-1}(k) + V_{n-3}(k) - \frac{n-3-k}{n-3-2k}\binom{n-3-2k}{n-3-3k}, \text{ for } n \ge 3k+3. \tag{2.6}$$

Let

$$s_0 = V_{3k}(k), s_1 = V_{3k+1}(k), s_2 = V_{3k+2}(k), s_n = V_{n+3k}(k)\ (n > 2). \tag{2.7}$$

Also, let

$$r_0 = r_1 = r_2 = 0, r_n = \frac{n-3+2k}{n-3+k}\binom{n-3+k}{n-3}. \tag{2.8}$$

Thus, we find that the generating function of the sequence r_n is

$$G(t) = t^3(2-t)(1-t)^{-(k+1)},$$

and the generating function $\Psi_k(t)$ of s_n satisfies the equality

$$\Psi_k(t)(1-t-t^3) + \frac{t^3(2-t)}{(1-t)^{k+1}} = s_0 - r_0 + t(s_1 - s_0 - r_1) + t^2(s_2 - s_1 - r_2).$$

So, by (2.7) and (2.8), it follows that

$$\Psi_k(t) = \frac{V_{3k}(k) + t(V_{3k+1}(k) - V_{3k}(k)) + t^2(V_{3k+2}(k) - V_{3k+1}(k))}{1-t-t^3}$$
$$- \frac{t^3(2-t)}{(1-t)^{k+1}(1-t-t^3)}.$$

Finally, the generating function $S_k(t)$ of the sequence $\{V_n(k)\}$ is $t^{3k}\Psi_k(t)$.

3. GENERALIZATION

In this section we introduce the generalized Fibonacci polynomials $f_{n,m}(x)$ and generalized Lucas polynomials $l_{n,m}(x)$ by:

$$f_{n,m}(x) = xf_{n-1,m}(x) + f_{n-m,m}(x), \quad n > m, \tag{3.1}$$

with $f_{n,m}(x) = x^{n-1}$, if $n = 1, 2, \ldots, m$, and

$$l_{n,m}(x) = xl_{n-1,m}(x) + l_{n-m,m}(x), \quad n > m, \tag{3.2}$$

with $l_{n,m}(x) = x^n$, if $n = 1, 2, \ldots, m$.

Remark: *For $m = 2$ and $m = 3$, we have*:

$$f_{n,2}(x) = F_n(x) \text{ (Fibonacci polynomials)},$$
$$l_{n,2}(x) = L_n(x) \text{ (Lucas polynomials)},$$
$$f_{n,3}(x) = U_n(x), l_{n,3}(x) = V_n(x).$$

Using the standard methods, we find that

$$f_{n,m}(x) = \sum_{j=0}^{[(n-1)/m]} \binom{n-1-(m-1)j}{j} x^{n-1-mj}, \tag{3.3}$$

and

$$l_{n,m}(x)=\sum_{j=0}^{[n/m]}\frac{n-(m-2)j}{n-(m-1)j}\binom{n-(m-1)j}{j}x^{n-mj}, \tag{3.4}$$

are explicit representations of the polynomials $f_{n,m}(x)$ and $l_{n,m}(x)$, respectively.

For $x=1$ in (3.3) and (3.4), we obtain two sequences of numbers $\{f_{n,m}\}$ and $\{l_{n,m}\}$. Hence, we get

$$f_{n,m}(k)=\sum_{j=0}^{k}\binom{n-1-(m-1)j}{j}, n=1,3,\ldots,0\le k\le[(n-1)/m], \tag{3.5}$$

which are the incomplete generalized Fibonacci numbers, and

$$l_{n,m}(k)=\sum_{j=0}^{k}\frac{n-(m-2)j}{n-(m-1)j}\binom{n-(m-1)j}{j}, n=1,2,\ldots,0\le k\le[n/m], \tag{3.6}$$

which are the incomplete generalized Lucas numbers.

Therefore, we can observe that $f_{n,m}([(n-1)/m])=f_{n,m}$ and $l_{n,m}([n/m])=l_{n,m}$; where $f_{n,m}$ and $l_{n,m}$ denote generalized Fibonacci and generalized Lucas numbers, respectively.

Our main purpose is to determine generating functions of the sequences $\{f_{n,m}(k)\}$ and $\{l_{n,m}(k)\}$.

First, we can prove the following statement.

Lemma 2: *Let $\{s_n\}$ be a complex sequence satisfying the recurrence relation*

$$s_n=s_{n-1}+s_{n-m}+r_n(n>m),$$

where $r_n:N\to C$ is a given sequence. Then the generating function $\Theta(t)$ of s_n is

$$\Theta(t)=\frac{H(t)+s_0-r_0+\sum_{i=1}^{m-1}t^i(s_i-s_{i-1}-r_i)}{1-t-t^m},$$

where $H(t)$ is generating function of the sequence r_n.

Proof: This proof is similar to the proof of Lemma 1.

The following satement represents the main result of this note.

Theorem 2: *Let k be a positive integer. Then*

$$R_k^m(t):=\sum_{j=0}^{\infty}f_{k,m}(j)t^j=t^{mk+1}\left(\frac{A_m}{1-t-t^m}-\frac{t^m}{(1-t)^{k+1}(1-t-t^m)}\right), \tag{3.7}$$

where

$$A_m=f_{mk,m}+\sum_{i=1}^{m-1}t^i(f_{mk+i,m}-f_{mk+i-1,m})$$

and

$$S_k^m(t) := \sum_{j=0}^{\infty} l_{k,m}(j)t^j = t^{mk}\left(\frac{B_m}{1-t-t^m} - \frac{t^m(2-t)}{(1-t)^{k+1}(1-t-t^m)}\right), \tag{3.8}$$

where

$$B_m = l_{mk-1,m} + \sum_{i=1}^{m-1} t^i (l_{mk+i,m} - l_{mk+i-1,m}).$$

Proof: The proof of Theorem 2 is based on Lemma 2. Namely, if

$$\Theta(t) = s_0 + s_1 t + s_2 t^2 + \cdots + s_n t^n + \cdots = \sum_{n=0}^{\infty} s_n t^n,$$

then

$$t\Theta(t) = \sum_{n=0}^{\infty} s_n t^{n+1}, t^m\Theta(t) = \sum_{n=0}^{\infty} s_n t^{n+m},$$

and

$$H(t) = \sum_{n=0}^{\infty} r_n t^n,$$

so we get

$$\begin{aligned}\Theta(t)(1-t-t^m) - H(t) = s_0 - r_0 + t(s_1 - s_0 - r_1) + t^2(s_2 - s_1 - r_2) + \cdots \\ + t^{m-1}(s_{m-1} - s_{m-2} - r_{m-1}).\end{aligned} \tag{3.9}$$

Let k be a fixed positive integer. It is known (see [2]) that $f_{n,m}(k) = 0$ if $0 \leq n < mk + 1$, $f_{mk+1,m}(k) = f_{mk,m}$, $f_{mk+2,m}(k) = f_{mk+1,m}, f_{mk+3,m}(k) = f_{mk+2,m}, \ldots, f_{mk+m,m}(k) = f_{mk+m-1,m}$, and that

$$f_{n,m}(k) = f_{n-1,m}(k) + f_{n-m,m}(k) = \binom{n-1-m-(m-1)k}{n-m-1-mk}, \ n \geq mk+m+1.$$

Set

$$\begin{aligned}s_0 = f_{mk+1,m}(k), s_1 = f_{mk+2,m}(k), \ldots, s_{m-1} = f_{mk+m,m}(k), \\ s_n = f_{n+mk+1,m}(k).\end{aligned} \tag{3.10}$$

Also, let

$$r_0 = r_1 = \cdots = r_{m-1} = 0 \quad \text{and} \quad r_n = \binom{n-m+k}{n-m}. \tag{3.11}$$

The generating function of the sequence r_n is $H(t) = t^m(1-t)^{-(k+1)}$. Hence, from (3.9) and using (3.10) and (3.11), we have

$$\Theta(t)(1-t-t^m) + \frac{t^m}{(1-t)^{k+1}} = f_{mk,m} + t(f_{mk+1,m} - f_{mk,m})$$

$$+ t^2(f_{mk+2,m} - f_{mk+1,m}) + \cdots + t^{m-1}(f_{mk+m-1,m} - f_{mk+m-2,m}).$$

Thus, we get

$$\Theta(t) = \frac{A_m}{1-t-t^m} - \frac{t^m}{(1-t)^{k+1}(1-t-t^m)},$$

where

$$A_m = f_{mk,m} + t(f_{mk+1,m} - f_{mk,m}) + t^2(f_{mk+2,m} - f_{mk+1,m}) + \cdots$$

$$+ t^{m-1}(f_{mk+m-1,m} - f_{mk+m-2,m}) = f_{mk,m} + \sum_{i=1}^{m-1} t^i(f_{mk+i,m} - f_{mk+i-1,m}).$$

The generating function $R_k^m(t)$ of $\{f_{n,m}(k)\}$ is $t^{mk+1}\Theta(t)$.

In the proof of (3.8), we use the following facts:

$$\begin{aligned}
&l_{n,m}(k) = 0 \text{ if } 0 \le n < mk,\ l_{mk,m}(k) = l_{mk-1,m},\\
&l_{mk+1,m}(k) = l_{mk,m}, \ldots, l_{mk+m-1,m}(k) = l_{mk+m-2,m} \text{ and}\\
&l_{n,m}(k) = l_{n-1,m}(k) - l_{n-m,m}(k) - \alpha_m, \quad n \ge mk+m,
\end{aligned} \tag{3.12}$$

where

$$\alpha_m = \frac{n-m-(m-2)k}{n-m-(m-1)k}\binom{n-m-(m-1)k}{n-m-mk}.$$

Let

$$s_0 = l_{mk,m}(k), s_1 = l_{mk+1,m}(k), \ldots, s_{m-1} = l_{mk+m-1,m}(k), s_n = l_{n+mk,m}(k).$$

Furthermore, let

$$r_0 = r_1 = \cdots = r_{m-1} = 0 \text{ and } r_n = \frac{n-m+2k}{n-m+k}\binom{n-m+k}{n-m}.$$

So, we find that $K(t) = t^m(2-t)(1-t)^{-(k+1)}$ is the generating function of the sequence $\{r_n\}$, and the generating function of the sequence $\{s_n\}$ satisfies the following equality

$$L(t)(1-t-t^m) + K(t) = s_0 + t(s_1 - s_0) + t^2(s_2 - s_1) + \cdots + t^{m-1}(s_{m-1} - s_{m-2}),$$

i.e.

$$L(t) = \frac{l_{mk,m}(k) + \sum_{i=1}^{m-1} t^i (l_{mk+i,m}(k) - l_{mk+i-1,m}(k))}{1 - t - t^m} - \frac{t^m(2-t)}{(1-t)^{k+1}(1-t-t^m)}.$$

Thus, using (3.12), we get

$$L(t) = \frac{l_{mk-1,m} + \sum_{i=1}^{m-1} t^i (l_{mk+i,m} - l_{mk+i-1,m})}{1 - t - t^m} - \frac{t^m(2-t)}{(1-t)^{k+1}(1-t-t^m)}.$$

The generating function $S_k^m(t)$ of the sequence $\{l_{n,m}(k)\}$ is $t^{mk}L(t)$ (see (3.8)). The proof of Theorem 2 is complete.

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