Available online at www.sciencedirect.com
SCIENCE DIRECT®

Discrete Mathematics 265 (2003) 323–335

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Some identities related to reciprocal functions

Xiqiang Zhao[a,b,*], Tianming Wang[c]

[a] *Department of Aerodynamics, College of Aerospace Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing Jiangsu 210016, People's Republic of China*

[b] *Department of Mathematics, Shandong Institute of Technology, Zibo Shandong 255012, People's Republic of China*

[c] *Department of Applied Mathematics, Dalian University of Technology, Dalian Liaoning 116024, People's Republic of China*

Received 22 November 1999; received in revised form 11 March 2002; accepted 1 April 2002

Abstract

The concept of Riordan array is used on reciprocal functions, and some identities involving binomial numbers, Stirling numbers and many other special numbers are obtained.

MSC: 05A19; 05A10; 11C20; 05A40

Keywords: Riordan array; Generating function; Combinatorial identity

1. Introduction

In 1991 [4,5,7] Shapiro introduced the concept of the Riordan group, which corresponds to a set of infinite lower-triangular matrices. Riordan groups are particularly important in studying combinatorial identities and combinatorial sums. For example, in 1994 [8], Sprugnoli studied Riordan arrays related to binomial coefficients, coloured walks and Stirling numbers. His work verified that many combinatorial sums can be solved by transforming the generating functions. In 1995 [9], Sprugnoli paid attention to the identities of Abel and Gould, respectively. In this paper, we continue the works of Shapiro and Sprugnoli to discuss some new applications of Riordan arrays. We also obtain many new identities related to special numbers, such as Stirling numbers of both kinds, and Bernoulli numbers.

* Corresponding author.
E-mail address: zhaodss@yahoo.com.cn (X. Zhao).

S0012-365X(02)00584-8

2. Riordan arrays and Lagrange inversion formulas

Notation

$\mathbf{R}$	set of real numbers
$\mathbf{R}[t]$	a ring of formal power series in some indeterminate t
$\mathbf{N}$	$\mathbf{N}=\{0,1,2,\ldots\}$
$f_k=[t^k]f(t)$	$f(t)\in\mathbf{R}[t]$, $f_k=[t^k]f(t)$ denotes the coefficients of t^k in the expansion of $f(t)$ in t
$\bar{f}(t)$	the compositional inverse function of $f(t)$, i.e., $\bar{f}(f(t))=f(\bar{f}(t))=t$
$f(t)=\mathscr{G}\{f_k\}$	$f(t)$ is the ordinary generating function of the sequence $\{f_k\}$.
$f(t)=\mathscr{E}\{\tilde{f}_k\}$	$f(t)$ is the exponential generating function of the sequence $\{\tilde{f}_k\}$.
$\mathrm{ord}(f(t))$	is the smallest integer k for which $f_k\neq 0$, and is called the *order* of $f(t)$

In this paper, we restrict ourselves to the concept of Riordan array as in [7]. This may be described as follows:

Let $g(t),f(t)\in\mathbf{R}[t]$, $g(t)=\sum_k g_k t^k$, $f(t)=\sum_k f_k t^k$ with $f_0=0$ (here we assume $f_1\neq 0$), and $\bar{f}(t)$ as its compositional inverse. The sequence of functions $\{d_k(t)\}_{k\in\mathbf{N}}$ is iteratively defined by

$$d_0(t)=g(t),$$

$$d_k(t)=g(t)(f(t))^k,$$

which also defines an infinite lower-triangular matrix $\{d_{n,k}\mid n,k\in\mathbf{N},\ 0\leqslant k\leqslant n\}$, where $d_{n,k}=[t^n]d_k(t)$. The infinite lower-triangular matrix $\{d_{n,k}\}$ is called a Riordan array in t. And we denote $D=(g(t),f(t))=\{d_{n,k}\}$.

In [9], Sprugnoli proved an important formula [Theorem 3.1, p. 218], which can be used to obtain many identities. Similarly, we give Theorems 1 and 2.

Theorem 1. *Let $D=(g(t),f(t))$ be a Riordan array and $\bar{f}(t)=\sum_k \bar{f}_k t^k$. Then we have*

$$\sum_{k=0}^{\infty} d_{n,k}\bar{f}_k=[t^{n-1}]g(t)=g_{n-1},\quad n>0. \tag{1}$$

Proof. $\sum_{k=0}^{\infty} d_{n,k}\bar{f}_k=\sum_{k=0}^{\infty}[t^n]g(t)(f(t))^k[y^k]\bar{f}(y)=[t^n]g(t)\bar{f}(f(t))=[t^{n-1}]g(t)=g_{n-1}$. □

Example 1. *Let $D=(1/(1-t),t/(1-t))$.* Then $d_{n,k}=\binom{n}{k}$ and $\bar{f}(t)=t/(1+t)$. So we have

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}=1,\quad n>0.$$

Example 2. Let $D_1 = (1, -\ln(1-t))$. Then we have

$$d^{(1)}_{n,k} = [t^n](-\ln(1-t))^k = \frac{k!}{n!}\begin{bmatrix} n \\ k \end{bmatrix},$$

where $\left[{n \atop k}\right]$ denotes the (unsigned) Stirling numbers of the first kind, $f(t) = -\ln(1-t)$, $\bar{f}(t) = 1 - e^{-t}$, and

$$\bar{f}_k = [t^k](1-e^{-t}) = \begin{cases} \frac{(-1)^{k-1}}{k!}, & k>0, \\ 0, & k=0. \end{cases}$$

Therefore, Theorem 1 gives

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{k-1} = n!\delta_{n,1}, \quad n>0,$$

where δ is the Kronecker delta.

Let $D_2 = ((-\ln(1-t))^m, -\ln(1-t))$ and $n>0$. Then

$$d^{(2)}_{n,k} = [t^n](-\ln(1-t))^m(-\ln(1-t))^k = \frac{(m+k)!}{n!}\begin{bmatrix} n \\ m+k \end{bmatrix}$$

and by (1), we have

$$\sum_{k=0}^{n} \binom{m+k}{k} \begin{bmatrix} n \\ m+k \end{bmatrix} (-1)^{k-1} = n \begin{bmatrix} n-1 \\ m \end{bmatrix}.$$

If $\left\{{n \atop k}\right\}$ denotes the Stirling numbers of the second kind, then

$$(e^t - 1)^p = \sum_{k \geqslant p} \frac{p!}{k!} \begin{Bmatrix} k \\ p \end{Bmatrix} t^k.$$

If we consider the Riordan arrays $D_3 = (1, e^t - 1)$ and $D_4 = ((e^t-1)^p, e^t - 1)$, then, by (1), we have the following identities:

$$\sum_{k=1}^{n} (-1)^{k-1}(k-1)! \begin{Bmatrix} n \\ k \end{Bmatrix} = n!\delta_{n,1}, \quad n>0$$

and

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}(k+p)!}{k} \begin{Bmatrix} n \\ k+p \end{Bmatrix} = np! \begin{Bmatrix} n-1 \\ p \end{Bmatrix}, \quad n>0.$$

Furthermore, all of the above identities can be proved by a direct application of the Riordan array concept, for example, if $n>0$, the first identity may be obtained as follows:

$$\begin{aligned}
&\frac{1}{n!}\sum_{k=0}^{n}\left[{n \atop k}\right]\sum_{i=0}^{k-1}\binom{k}{i+1}(-1)^{k-1-i}\\
&\quad=\sum_{k=0}^{n}\frac{k!}{n!}\left[{n \atop k}\right][t^k](\mathrm{e}^t-1)\mathrm{e}^{-t}\\
&\quad=[t^n]\left[\frac{\mathrm{e}^y-1}{\mathrm{e}^y}\,\middle|\, y=\ln\frac{1}{1-t}\right]\\
&\quad=[t^n]\frac{(1-t)^{-1}-1}{(1-t)^{-1}}\\
&\quad=[t^n]t\\
&\quad=\delta_{n,1}.
\end{aligned}$$

Theorem 1′. *The hypotheses are the same as those in Theorem* 1. *Then*, *by using the Lagrange inversion formula* (*see* [1]), *we have*

$$\sum_k d_{n,k}\,\frac{1}{k}[t^{k-1}]\left(\frac{f(t)}{t}\right)^{-k}=[t^{n-1}]g(t),\quad n>0. \tag{2}$$

Example 3. Let

$$D=\left(\frac{t^m}{(1-t)^{m+1}},\frac{t}{(1-t)^{a+1}}\right),$$

then

$$d_{n,k}=\binom{n+ak}{m+(a+1)k}\ \text{(see [8, p. 272])},$$

$$f(t)=\frac{t}{(1-t)^{a+1}},$$

$$\begin{aligned}
\bar{f}_k&=\frac{1}{k}[t^{k-1}]\left(\frac{1}{(1-t)^{a+1}}\right)^{-k}=\frac{1}{k}[t^{k-1}](1-t)^{k(a+1)}\\
&=\frac{1}{k}\binom{k(a+1)}{k-1}(-1)^{k-1}
\end{aligned}$$

and

$$[t^{n-1}]g(t)=[t^{n-1}]\frac{t^m}{(1-t)^{m+1}}=\binom{n-1}{m}.$$

So we have

$$\sum_{k=0}^{n-m} \frac{(-1)^{k-1}}{k} \binom{n+ak}{m+(a+1)k} \binom{k(a+1)}{k-1} = \binom{n-1}{m}, \quad n>0. \qquad \square$$

Theorem 2. *Let $D=(g(t), f(t))$ be a Riordan array, $h(t)$ be the generating function of the sequence $\{h_k\}_{k\in\mathbf{N}}$ and $h(\bar{f}(t)) = \sum_k (h(\bar{f}))_k t^k$. Then we have*

$$\sum_k d_{n,k}(h(\bar{f}))_k = [t^n]g(t)h(t). \tag{3}$$

Proof. $\sum d_{n,k}(h(\bar{f}))_k = [t^n]g(t)h(\bar{f}(f(t))) = [t^n]g(t)h(t). \quad \square$

Example 4. Now we consider the Riordan array $D=(1, \mathrm{e}^t-1)$ and $h(t)=\exp(\mathrm{e}^t-1)$. Then, by (3), we have $\sum (k!/n!) \left\{ {n \atop k} \right\} 1/k! = [t^n]\exp(\mathrm{e}^t-1)$ or $\sum \left\{ {n \atop k} \right\} = \mathscr{B}_n$, where $\mathscr{B}_n$ are the Bell numbers, whose exponential generating function is $\exp(\mathrm{e}^t-1) = \sum_{n=0}^{\infty} (1/n!)\mathscr{B}_n t^n$.

Example 5. Let $B_n^{(k)}$ denote the Bernoulli numbers of high order, whose exponential generating function is

$$\left(\frac{t}{\mathrm{e}^t-1}\right)^k = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}.$$

Let

$$D=(1, \ln(1+t)), \quad h(t) = \left(\frac{1}{t} \ln \frac{1}{1+t}\right)^p.$$

Then

$$d_{n,k} = [t^n](\ln(1+t))^k = \frac{k}{n}[t^{n-k}]\left(\frac{\mathrm{e}^t-1}{t}\right)^{-n} = \frac{k}{n} B_{n-k}^{(n)} \frac{1}{(n-k)!},$$

$\bar{f}(t) = \mathrm{e}^t-1$, $(h(\bar{f}(t)) = (-t/(\mathrm{e}^t-1))^p$, and $(h(\bar{f}))_k = ((-1)^p/k!)B_k^{(p)}$. Therefore, we have

$$\sum_k \frac{k}{n(n-k)!} B_{n-k}^{(n)} \frac{(-1)^p}{k!} B_k^{(p)} = [t^n]\left(\frac{1}{t} \ln \frac{1}{1+t}\right)^p,$$

or

$$\sum_k k(n+p)! \binom{n}{k} B_{n-k}^{(n)} B_k^{(p)} = (-1)^{n+p} p!nn! \left[{n+p \atop p} \right].$$

By the Lagrange inversion formulas of all kinds in [1], we can easily obtain many formulas. These formulas can be used in different cases.

Theorem 3. *Let $D=(g(t),f(t))$ be a Riordan array. Then we have*

$$\sum_{k=1}^{\infty}\frac{q}{k}\,d_{n,k}[t^{k-q}]\left(\frac{f(t)}{t}\right)^{-n}=[t^{n-q}]g(t). \tag{4}$$

Proof. See [1]. □

Theorem 4. *Let $D=(g(t),f(t))$ be a Riordan array and $h(t)=\sum_{k=0}^{\infty}h_kt^k$. Then we have*

$$d_{n,0}h_0+\sum_{k=1}^{n}d_{n,k}\,\frac{1}{k}[t^{k-1}]h'(t)\left(\frac{f(t)}{t}\right)^{-k}=[t^n]g(t)h(t). \tag{5}$$

Proof. See [1]. □

When only knowing the coefficients of powers of $f(t)$ with positive integral exponents, we have the following theorem.

Theorem 5. *Let $D=(g(t),f(t))$ be a Riordan array, q be a positive integer and $1\leqslant q\leqslant n$. Then we have*

$$\sum_{k=q}^{n}qd_{n,k}\binom{2k-q}{k}\sum_{j=1}^{k-q}\frac{(-1)^j}{k+j}\,f_1^{-k-j}\binom{k-q}{j}[t^{k-q+j}](f(t))^j$$
$$=[t^{n-q}]g(t). \tag{6}$$

Proof. See [1]. □

Example 6. Let

$$D=\left(\frac{1}{(1-t)^{p+1}}\,\ln\frac{1}{1-t},\frac{t}{(1-t)^q}\right)$$

because

$$\mathscr{G}\left\{(H_{m+n}-H_m)\binom{m+n}{m}\right\}=\frac{1}{(1-t)^{m+1}}\,\ln\frac{1}{1-t},$$

then

$$d_{n,k}=(H_{p+n+(q-1)k}-H_{p+qk})\binom{p+n+(q-1)k}{n-k},$$

where $H_n = \sum_{k=1}^{n} 1/k$. By (6), we have

$$\sum_{k=l}^{n} l \binom{2k-l}{k} (H_{p+n+(q-1)k} - H_{p+qk}) \binom{p+n+(q-1)k}{n-k}$$
$$\times \quad \sum_{j=1}^{k-l} \frac{(-1)^j}{k+j} \binom{k-l}{j} \binom{qj+k-l-1}{k-l}$$
$$= (H_{p+n-l} - H_p) \binom{p+n-l}{p}.$$

Theorem 6. *Let three functions $F(x), G(x)$ and $H(y)$ of a real variable be given, where $F(x)$ and $G(x)$ are of class $\mathbf{C}^\infty$ in $x=a$, and $H(y)$ is of class $\mathbf{C}^\infty$ in $y=b=F(a)$, and let $P(x)=H(F(x))$. If we put*

$$g_l = \frac{1}{l!}\frac{d^l G}{dx^l}\bigg|_{x=a}, \quad f_k = \frac{1}{k!}\frac{d^k F}{dx^k}\bigg|_{x=a}, \quad h_n = \frac{1}{n!}\frac{d^n H}{dy^n}\bigg|_{y=b},$$
$$p_m = \frac{1}{m!}\frac{d^m P}{dx^m}\bigg|_{x=a},$$

$f_0=F(a)$, $f_1 \neq 0$, $g_0 \neq 0$, $h_0=H(b)=p_0=P(a)=H(F(a))$, and define the following formal power series:

$$g(t)=\sum_{l\geqslant 0} g_l t^l, \quad f(t)=\sum_{k\geqslant 1} f_k t^k, \quad h(u)=\sum_{n\geqslant 0} h_n u^n, \quad p(t)=\sum_{m\geqslant 0} p_m t^m.$$

Then $(g(t), f(t))$ is a proper Riordan array and we have

$$\sum_{k=0}^{\infty} d_{n,k} h_k = [t^n] g(t) p(t) = \sum_{j=0}^{n} g_j p_{n-j}. \tag{7}$$

Proof. By the definition of the proper Riordan array and $f_1 \neq 0$, we know that $(g(t), f(t))$ is a proper Riordan array, and we have $\sum d_{n,k} h_k = [t^n] g(t) h(f(t))$. On the other hand, from Theorem B in [1, p. 138], we obtain formally $p(t)=h(f(t))$. So we have

$$\sum_{k=0}^{\infty} d_{n,k} h_k = [t^n] g(t) h(f(t)) = [t^n] g(t) p(t) = \sum_{j=0}^{n} g_j p_{n-j}. \qquad \square$$

Example 7. Let $H(t)=t^{-m}$, $G(t)=1/(1-t)$, and

$$F(t)=\frac{e^t-1}{t} = 1 + \frac{t}{2!} + \frac{t^2}{3!} + \cdots (t \neq 0), \quad F(0)=1.$$

Then $f(t)=t/2!+t^2/3!+\cdots$, $g(t)=G(t)$, and $(g(t),f(t))=\{d_{n,k}\}$ is a proper Riordan array, where

$$\begin{aligned} d_{n,k} &= [t^n]\,\frac{1}{1-t}\left(\frac{t}{2!}+\frac{t^2}{3!}+\cdots+\frac{t^n}{(n+1)!}+\cdots\right)^k \\ &= [t^n]\,\frac{1}{1-t}\left(\frac{t}{2!}+\frac{t^2}{3!}+\cdots+\frac{t^n}{(n+1)!}\right)^k \\ &= \sum_{j=0}^{n}\sum_{\substack{k_1+k_2+\cdots+k_n=k\\ k_1+2k_2+\cdots+nk_n=j}}\binom{k}{k_1,k_2,\ldots,k_n}\prod_{i=1}^{n}\frac{1}{((i+1)!)^{k_i}}. \end{aligned}$$

And $h_k=(-1)^k\langle m\rangle_k$, where $\langle m\rangle_k$ is the rising factorial of m of order k. So from (7), we have

$$\sum_{j=0}^{n}\sum_{\sigma(j)}(-1)^k\langle m\rangle_k\binom{k}{k_1,k_2,\ldots,k_n}\prod_{i=1}^{n}\frac{1}{((i+1)!)^{k_i}}=\sum_{j=0}^{n}\frac{B_j^{(m)}}{j!},$$

where $\sigma(j)$ denotes set of partitions of $j\,(j\in\mathbf{N})$, represented by $1^{k_1}2^{k_2}\cdots j^{k_j}$ with $k_1+2k_2+\cdots+jk_j=j$, $k_i\in\mathbf{N}$, $(i=1,2,\ldots,j)$.

3. The exponential Riordan array

For the exponential generating function of a sequence, we have

Definition. Let $g(t)=\mathscr{E}\{\tilde{g}_k\}$, $f(t)=\mathscr{E}\{\tilde{f}_k\}\in\mathbf{R}[t]$, $\operatorname{ord}(g(t))=0$, $\operatorname{ord}(f(t))=1$. For an infinite lower triangular array $D=\{d_{n,k}\mid n,k\in\mathbf{N},0\leqslant k\leqslant n\}$, if for fixed k, $\mathscr{E}\{d_{n,k}\}=g(t)((f(t))^k/k!)(k\geqslant 0)$, then we write $D=\langle g(t),f(t)\rangle$ and say that $\langle g(t),f(t)\rangle$ is an exponential Riordan array.

Let $\langle g(t),f(t)\rangle$ and $\langle d(t),h(t)\rangle$ be two exponential Riordan arrays. Let $\langle d(t),h(t)\rangle*\langle g(t),f(t)\rangle=\langle d(t)g(h(t)),f(h(t))\rangle$. Then the set of all exponential Riordan arrays form an infinite group and $\langle 1,t\rangle$ is its unit element. Just as in [8,9,12], we can obtain many identities related to the exponential Riordan arrays. Besides, the exponential Riordan arrays are directly related to the classical Umbral Calculus. The interested readers can see the relative papers and the works of Rota, Roman and Knuth [3,6].

Let $f(t)=\sum_{k=1}^{\infty}\tilde{f}_k t^k/k!$, $g(t)=\sum_{k=1}^{\infty}\tilde{g}_k t^k/k!\in\mathbf{R}[t]$, $f(g(t))=g(f(t))$, and $f(0)=g(0)=0$. We define number pair $\{A_1(n,k),A_2(n,k)\}$ as follows:

$$d(t)\,\frac{(f(t))^k}{k!}=\sum_{n\geqslant k}A_1(n,k)\frac{t^n}{n!},\quad d(t)\,\frac{(g(t))^k}{k!}=\sum_{n\geqslant k}A_2(n,k)\,\frac{t^n}{n!}.$$

If $f(g(t))=g(f(t))=t$ and $d(t)=1$, then $\{A_1(n,k),A_2(n,k)\}$ is the *generalized Stirling number pair* in [2].

Theorem 7. *For number pair* $\{A_1(n,k),A_2(n,k)\}$ *and* $n>0$, *we have*

$$\sum_{k=1}^{n} A_1(n,k)\tilde{g}_k = \sum_{k=1}^{n} A_2(n,k)\tilde{f}_k = \delta_{n,1}. \tag{8}$$

If, in particular, $\{A_1(n,k),A_2(n,k)\}$ *is the Stirling number pair, we have*

$$\tilde{g}_n = \sum_{1\leqslant j\leqslant k\leqslant n} A_2(n,k)A_2(k,j)\tilde{f}_j, \tag{9}$$

$$\tilde{f}_n = \sum_{1\leqslant j\leqslant k\leqslant n} A_1(n,k)A_1(k,j)\tilde{g}_j. \tag{10}$$

Proof. We have

$$\begin{aligned} d(t)\,g(f(t)) &= \sum_{k=1}^{\infty} d(t)\,\frac{(f(t))^k}{k!}\,\tilde{g}_k = \sum_{k=1}^{\infty} \tilde{g}_k \sum_{n\geqslant k} A_1(n,k)\,\frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} A_1(n,k)\tilde{g}_k\right)\frac{t^n}{n!}. \end{aligned}$$

On the other hand,

$$\begin{aligned} d(t)\,f(g(t)) &= \sum_{k=1}^{\infty} d(t)\,\frac{(g(t))^k}{k!}\,\tilde{f}_k = \sum_{k=1}^{\infty} \tilde{f}_k \sum_{n\geqslant k} A_2(n,k)\,\frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} A_2(n,k)\tilde{f}_k\right)\frac{t^n}{n!}. \end{aligned}$$

Then we obtain (8) by identifying the coefficients of $t^n/n!$ in $d(t)\,f(g(t))$ and $d(t)\,g(f(t))$.

If $d(t)=1$ and $f(g(t))=g(f(t))=t$, then $\{A_1(n,k)\}*\{A_2(n,k)\}=\langle 1,f(t)\rangle*\langle 1,g(t)\rangle=\langle 1,t\rangle$, thus $A_1(n,k)A_2(n,k)=I$ (infinite unit matrix). And we obtain the inverse relation:

$$a_n = \sum_{k=1}^{n} A_1(n,k)b_k,$$

$$b_n = \sum_{k=1}^{n} A_2(n,k)a_k.$$

Let $a_n=\sum_{k=1}^{n} A_1(n,k)\tilde{g}_k$ in (8). By the above inverse relation, we have $\tilde{g}_n=\sum_{k=1}^{n} A_2(n,k)a_k$. Then, by (8), we obtain

$$\tilde{g}_n = \sum_{k=1}^{n} A_2(n,k)\sum_{j=1}^{k} A_2(k,j)\tilde{f}_j = \sum_{1\leqslant j\leqslant k\leqslant n} A_2(n,k)A_2(k,j)\tilde{f}_j.$$

The proof of (10) is similar to that of (9). □

Example 8. Let $\{A_1(n,k),A_2(n,k)\}$ be the generalized Stirling number pair. If $\{A_1(n,k)\}=\langle g_1(t),f_1(t)\rangle$, $\{A_2(n,k)\}=\langle g_2(t),f_2(t)\rangle$, then we denote simply $\{A_1(n,k),A_2(n,k)\}$ as $\{\langle g_1(t),f_1(t)\rangle,\ \langle g_2(t),f_2(t)\rangle\}$.

For the generalized Stirling number pair

$$\left\{(-1)^n L_{n,k}, \frac{1}{k!}\binom{k}{n-k}\right\} = \left\{\left\langle 1, \frac{t}{1-t}\right\rangle, \left\langle 1, \frac{t}{1+t}\right\rangle\right\},$$

where $L_{n,k}$ is the Lah number, which has the expression

$$L_{n,k}=(-1)^n\,\frac{n!}{k!}\binom{n-1}{k-1}$$

by Theorem 7, we can obtain the following identities:

$$\sum_{k=1}^{n}(-1)^{n+k-1}k!L_{n,k}=\sum_{k=1}^{n}\binom{k}{n-k}=\delta_{n,1},\quad n>0,$$

$$n!=\sum_{1\leqslant j\leqslant k\leqslant n}(-1)^{j-1}\frac{1}{k!}\binom{k}{n-k}\binom{j}{k-j},\quad n>0,$$

$$(-1)^n n!=\sum_{1\leqslant j\leqslant k\leqslant n}(-1)^{n+k}j!L_{n,k}L_{k,j},\quad n>0.$$

Let $f(t)\in\mathbf{R}[t]$ and $g(t)=f(f(t))$. Then $f(g(t))=g(f(t))$. So we have the following example.

Example 9. Let $f(t)=\mathrm{e}^t-1$ and $g(t)=f(f(t))=\mathrm{e}^{\mathrm{e}^t-1}-1=\sum_{m\geqslant 1}\mathscr{B}_m\frac{t^m}{m!}$, where $\mathscr{B}_m$ is the Bell number. Then $\tilde{f}_k=1\ (k\geqslant 1)$, $\tilde{g}_k=\mathscr{B}_k\ (k\geqslant 1)$.

If $d(t)=1$, then we have

$$d(t)\,\frac{(f(t))^k}{k!}=\frac{1}{k!}(\mathrm{e}^t-1)^k=\sum_{n\geqslant k}\left\{ {n \atop k} \right\}\frac{t^n}{n!}$$

and

$$d(t)\,\frac{(g(t))^k}{k!}=\frac{1}{k!}\left(\sum_{m\geqslant 1}\mathscr{B}_m\frac{t^m}{m!}\right)^k=\sum_{n\geqslant k}B_{n,k}(\mathscr{B}_1,\ldots,\mathscr{B}_{n-k+1})\frac{t^n}{n!}$$

where $B_{n,k}(x_1,\ldots,x_{n-k+1})$ is the partial Bell polynomial.

Therefore, we obtain number pair

$$\{A_1(n,k),A_2(n,k)\}=\left\{\left\{ {n \atop k} \right\},B_{n,k}(\mathscr{B}_1,\ldots,\mathscr{B}_{n-k+1})\right\}$$

$$=\{\langle 1,\mathrm{e}^t-1)\rangle,\langle 1,\mathrm{e}^{\mathrm{e}^t-1}-1\rangle\}.$$

From (8), we have the following identity:

$$\sum_{k=1}^{n} \left\{ {n \atop k} \right\} \mathscr{B}_k = \sum_{k=1}^{n} B_{n,k}(\mathscr{B}_1, \ldots, \mathscr{B}_{n-k+1}).$$

If $d(t) = \mathrm{e}^t - 1$, then we have

$$d(t)\frac{(f(t))^k}{k!} = \frac{1}{k!}(\mathrm{e}^t - 1)^{k+1} = (k+1) \sum_{n \geqslant k+1} \left\{ {n \atop k+1} \right\} \frac{t^n}{n!}$$

and

$$\begin{aligned} d(t)\frac{(g(t))^k}{k!} &= \frac{1}{k!}(\mathrm{e}^t - 1)(\mathrm{e}^{\mathrm{e}^t - 1} - 1)^k \\ &= \sum_{n \geqslant 1} \frac{t^n}{n!} \sum_{n \geqslant k} B_{n,k}(\mathscr{B}_1, \ldots, \mathscr{B}_{n-k+1}) \frac{t^n}{n!} \\ &= \sum_{n \geqslant k} \left[\sum_{i=k}^{n} \binom{n}{i} B_{i,k}(\mathscr{B}_1, \ldots, \mathscr{B}_{i-k+1}) \right] \frac{t^n}{n!}. \end{aligned}$$

Therefore, we have

$$A_1(n,k) = \begin{cases} 0, & n = k, \\ (k+1) \left\{ {n \atop k+1} \right\}, & n \geqslant k+1 \end{cases}$$

and

$$A_2(n,k) = \sum_{i=k}^{n} \binom{n}{i} B_{i,k}(\mathscr{B}_1, \ldots, \mathscr{B}_{i-k+1}).$$

From (8), we have

$$\sum_{k=1}^{n-1} (k+1) \left\{ {n \atop k+1} \right\} \mathscr{B}_k = \sum_{k=1}^{n} \sum_{i=k}^{n} \binom{n}{i} B_{i,k}(\mathscr{B}_1, \ldots, \mathscr{B}_{i-k+1}).$$

Example 10. In [11], Xu Lizhi (L.C. Hsu) and Yu Hongquan have defined the generalized Stirling number pair $\{S(n,k,\alpha,\beta), S(n,k,\beta,\alpha)\}$ as follows:

$$\frac{1}{k!}\left(\frac{(1+\alpha t)^{\beta/\alpha} - 1}{\beta} \right)^k = \sum_{n \geqslant k} S(n,k,\alpha,\beta)\frac{t^n}{n!},$$

$$\frac{1}{k!}\left(\frac{(1+\beta t)^{\alpha/\beta} - 1}{\alpha} \right)^k = \sum_{n \geqslant k} S(n,k,\beta,\alpha)\frac{t^n}{n!},$$

i.e.

$$\{S(n,k,\alpha,\beta), S(n,k,\beta,\alpha)\} = \left\{\left\langle 1, \frac{(1+\alpha t)^{\beta/\alpha}-1}{\beta}\right\rangle, \left\langle 1, \frac{(1+\beta t)^{\alpha/\beta}-1}{\alpha}\right\rangle\right\}.$$

Let $f(t) = 1/\beta((1+\alpha t)^{\beta/\alpha}-1)$ and $g(t) = 1/\alpha((1+\beta t)^{\alpha/\beta}-1)$. Then $\tilde{f}_k = (\beta-\alpha)\cdots(\beta-(k-1)\alpha)$, $\tilde{g}_k = (\alpha-\beta)\cdots(\alpha-(k-1)\beta)$, and we have, by (8) and (9), respectively, that

$$\sum_{k=1}^{n} S(n,k,\alpha,\beta)(\alpha-\beta)\cdots(\alpha-(k-1)\beta) = \delta_{n,1}, \quad n>0, \tag{11}$$

$$(\alpha-\beta)\cdots(\alpha-(n-1)\beta) = \sum_{1\leqslant j\leqslant k\leqslant n} S(n,k,\beta,\alpha)S(k,j,\beta,\alpha) \times(\beta-\alpha)\cdots(\beta-(j-1)\alpha), \quad n>0, \tag{12}$$

while the other two identities involving $S(n,k,\beta,\alpha)$ may be obtained in a like way since α and β are symmetric.

In particular, taking $\alpha = 1$ and letting $\beta \to 0$, we easily find that $S(n,k,1,0) = (-1)^{n+k} \left[{n \atop k}\right]$ and $S(n,k,0,1) = \left\{{n \atop k}\right\}$ just stand for the ordinary Stirling numbers of the first and second kinds, respectively.

If taking $\alpha = 1$ and $\beta = \theta$, then $S(n,k,1,\theta) = (-1)^{n+k}S_1(n,k|\theta)$ and $S(n,k,\theta,1) = S(n,k|\theta)$, where $S_1(n,k|\theta)$ and $S(n,k|\theta)$ are called the degenerate Stirling numbers of the first and second kind by Carlitz [10] and defined by

$$\frac{1}{k!}\left(\frac{1-(1-t)^{\theta}}{\theta}\right)^{k} = \sum_{n\geqslant k} S_1(n,k|\theta)\frac{t^n}{n!},$$

$$\frac{1}{k!}((1+\theta t)^{\mu}-1)^{k} = \sum_{n\geqslant k} S(n,k|\theta)\,\frac{t^n}{n!},$$

where $\theta\mu = 1$. Moreover we have, by (11) and (12), respectively, the following identities,

$$\sum_{k=1}^{n}(-1)^{n+k}S_1(n,k|\theta)(1-\theta)\cdots(1-(k-1)\theta) = \sum_{k=1}^{n} S(n,k|\theta)(\theta-1)\cdots(\theta-k+1)) = \delta_{n,1}, \quad n>0,$$

$$(1-\theta)\cdots(1-(n-1)\theta) = \sum_{1\leqslant j\leqslant k\leqslant n} S(n,k|\theta)S(k,j|\theta) \times(\theta-1)\cdots(\theta-j+1), \quad n>0,$$

$$(\theta-1)\cdots(\theta-n+1)=\sum_{1\leqslant j\leqslant k\leqslant n} S_1(n,k|\theta)S_1(k,j|\theta)\times(1-\theta)\cdots(1-(j-1)\theta), \quad n>0.$$

Acknowledgements

This work was supported by the National Natural Science Foundation of China (19771014). The author wishes to thank the referee for many useful suggestions that led to the improvement and revision of this note.

References

[1] L. Comtet, Advanced Combinatorics, Reidel, Dordrecht, 1974.
[2] L.C. Hsu, Generalized Stirling number pairs associated with inverse relations, The Fibonacci Quart. 25 (1987) 346–351.
[3] D.E. Knuth, The Art of Computer Programming, Vol. I–III, Addison-Wesley, Reading, MA, 1968–1973.
[4] D. Merlini, D.G. Rogers, R. Sprugnoli, M.C. Verri, On some alternative characterizations of Riordan arrays, Canad. J. Math. 49(2) (1997) 301–320.
[5] D.G. Rogers, Pascal triangles, Catalan numbers and renewal arrays, Discrete Math. 22 (1978) 301–310.
[6] S.M. Roman, G.C. Rota, The Umbral calculus, Adv. in Math. 27 (1978) 95–188.
[7] L.W. Shapiro, S. Getu, W.J. Woan, L. Woodson, The Riordan group, Discrete Appl. Math. 34 (1991) 229–239.
[8] R. Sprugnoli, Riordan arrays and combinatorial sums, Discrete Math. 132 (1994) 267–290.
[9] R. Sprugnoli, Riordan arrays and the Abel–Gould identity, Discrete Math. 142 (1995) 213–233.
[10] Wang Tianming, Yu Hongquan, Yao Hong, A matrix representation of combinatorial numbers with applications, J. Dalian Univ. Technol. 36 (1996) 380–385.
[11] Xu Lizhi(L.C. Hsu), Yu Hongquan, A unified approach to a class of Stirling-type pairs, Appl. Math. -JCU 12(B) (1997) 225–232.
[12] Yin Dongsheng, Umbral calculus and Hsu-Riordan array, Ph.D. Thesis, Dalian University of Technology, 1999.