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SOME IDENTITIES FOR THE GENERALIZED FIBONACCI AND LUCAS FUNCTIONS

Fengzhen Zhao and Tianming Wang
Dalian University of Technology, 116024 Dalian, China
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1. INTRODUCTION

In this paper, we consider the generalized Fibonacci and Lucas functions, which may be defined by

$$U(x)=\frac{\alpha^x-e^{i\pi x}\beta^x}{\alpha-\beta} \tag{1}$$

and

$$V(x)=\alpha^x+e^{i\pi x}\beta^x, \tag{2}$$

where $\alpha=(p+\sqrt{\Delta})/2$, $\beta=(p-\sqrt{\Delta})/2>0$, $\Delta=p^2-4q$, p and q are integers with $q>0$, and x is an arbitrary real number. It is clear that $U(x)=F(2x)$ and $V(x)=L(2x)$ when $p=3$ and $q=1$, where $F(x)$ and $L(x)$ are the Fibonacci and Lucas functions, respectively (see [2]).

In [2], R. André-Jeannin proved that well-known identities for Fibonacci and Lucas numbers are again true for $F(x)$ and $L(x)$. Basic results regarding these topics can be found in [1]. Some special cases of the functions $U(x)$ and $V(x)$ are treated in [4] and referred to in the Remark in [8]. The aim of this paper is to establish some identities for $U(x)$ and $V(x)$. We are interested in calculating the summation of reciprocals of products of $U(x)$ and $V(x)$.

2. MAIN RESULTS

From the definitions of the generalized Fibonacci and Lucas functions, we can obtain the main results of this paper.

Theorem: Assume that *n*, *r*, and *s* are positive integers, and *x* is an arbitrary real number. Then

$$\sum_{k=1}^{n}\frac{e^{i\pi(k-1)rx}q^{(k-1)rx}}{U(krx-rx+sx)U(krx+sx)}=\frac{U(nrx)}{U(rx)U(sx)U(sx+nrx)} \tag{3}$$

and

$$\sum_{k=1}^{n}\frac{e^{i\pi(k-1)rx}q^{(k-1)rx}}{V(sx+krx-rx)V(sx+krx)}=\frac{U(nrx)}{U(rx)V(sx)V(sx+nrx)}. \tag{4}$$

Proof: From (1) and (2), it is easy to verify that $U(x)$ and $V(x)$ satisfy

$$\frac{V(sx)}{U(sx)}-\frac{V(sx+rx)}{U(sx+rx)}=\frac{2e^{i\pi sx}q^{sx}U(rx)}{U(sx)U(sx+rx)} \tag{5}$$

and

$$\frac{U(sx)}{V(sx)}-\frac{U(sx+rx)}{V(sx+rx)}=\frac{-2e^{i\pi sx}q^{sx}U(rx)}{V(sx)V(sx+rx)}. \tag{6}$$

In (5) and (6), replacing *s* by *s*, $s+r$, $s+2r$, ..., $s+(n-1)r$, and adding the results we can obtain

$$\sum_{k=1}^{n}\frac{e^{i\pi(k-1)rx}q^{(k-1)rx}}{U(sx+krx-rx)U(sx+krx)}=\frac{e^{-i\pi sx}q^{-sx}}{2U(rx)}\left(\frac{V(sx)}{U(sx)}-\frac{V(sx+nrx)}{U(sx+nrx)}\right)$$

and

$$\sum_{k=1}^{n}\frac{e^{i\pi(k-1)rx}q^{(k-1)rx}}{V(sx+krx-rx)V(sx+krx)}=\frac{e^{-i\pi sx}q^{-sx}}{2U(rx)}\left(\frac{U(sx)}{V(sx)}-\frac{U(sx+nrx)}{V(sx+nrx)}\right).$$

From (5) and (6), we can prove that the equalities (3) and (4) hold. □

Remark: From (1) and (2), we can show that the following relations are valid:

$$V(2rx)-e^{i\pi(r-s)x}q^{(r-s)x}V(2sx)=\Delta U(rx-sx)U(rx+sx); \tag{7}$$

$$V(2rx)+e^{i\pi(r-s)x}q^{(r-s)x}V(2sx)=V(rx-sx)V(rx+sx); \tag{8}$$

$$U^2(rx)-e^{i\pi(r-s)x}q^{(r-s)x}U^2(sx)=U(rx-sx)U(rx+sx); \tag{9}$$

$$V^2(rx)-e^{i\pi(r-s)x}q^{(r-s)x}V^2(sx)=\Delta U(rx-sx)U(rx+sx). \tag{10}$$

From (7) and (8), we have

$$\sum_{k=0}^{n}\frac{e^{i\pi krx}q^{krx}}{V(2krx+rx+2sx)-e^{i\pi(sx+krx)}q^{sx+krx}V(rx)}=\sum_{k=0}^{n}\frac{e^{i\pi krx}q^{krx}}{\Delta U(krx+sx)U(krx+rx+sx)}$$

and

$$\sum_{k=0}^{n}\frac{e^{i\pi krx}q^{krx}}{V(2krx+rx+2sx)+e^{i\pi(sx+krx)}q^{sx+krx}V(rx)}=\sum_{k=0}^{n}\frac{e^{i\pi krx}q^{krx}}{V(krx+sx)V(krx+rx+sx)}.$$

By the method used to obtain (3) and (4), we obtain the equalities

$$\sum_{k=0}^{n}\frac{e^{i\pi krx}q^{krx}}{V(2krx+rx+2sx)-e^{i\pi(sx+krx)}q^{sx+krx}V(rx)}=\frac{U(nrx+rx)}{\Delta U(rx)U(sx)U(sx+nrx+rx)}$$

and

$$\sum_{k=0}^{n}\frac{e^{i\pi krx}q^{krx}}{V(2krx+rx+2sx)+e^{i\pi(sx+krx)}q^{sx+krx}V(rx)}=\frac{U(nrx+rx)}{U(rx)V(sx)V(sx+nrx+rx)}.$$

Using (9) and (10) and applying the method used to obtain (3), we obtain the equalities

$$\sum_{k=0}^{n-1}\frac{e^{2i\pi krx}q^{2krx}}{U^2(2krx+rx+sx)-e^{i\pi(sx+2krx)}q^{sx+2krx}U^2(rx)}=\frac{U(2nrx)}{U(2rx)U(sx)U(sx+2nrx)}$$

and

$$\sum_{k=0}^{n}\frac{e^{2i\pi krx}q^{2krx}}{V^2(2krx+rx+sx)-e^{i\pi(sx+2krx)}q^{sx+2krx}V^2(rx)}=\frac{U(2nrx)}{\Delta U(2rx)U(sx)U(sx+2nrx)}.$$

Letting x be a positive real number and $\left|\frac{\beta}{\alpha}\right|<1$, due to

$$\lim_{n\to+\infty}\frac{U(nx)}{U(nx+rx)}=\frac{1}{\alpha^{rx}}\quad\text{and}\quad\lim_{n\to+\infty}\frac{U(nx)}{V(nx+rx)}=\frac{1}{\sqrt{\Delta}\alpha^{rx}},$$

we immediately have the following corollary.

Corollary: Suppose that r and s are positive integers, and x is a positive real number. If $\left|\frac{\beta}{\alpha}\right|<1$, then we have

$$\sum_{k=1}^{\infty} \frac{e^{i\pi(k-1)rx} q^{(k-1)rx}}{U(sx+krx-rx)U(sx+krx)} = \frac{1}{\alpha^{sx}U(rx)U(sx)}$$

and

$$\sum_{k=1}^{\infty} \frac{e^{i\pi(k-1)rx} q^{(k-1)rx}}{V(sx+krx-rx)V(sx+krx)} = -\frac{1}{\sqrt{\Delta}\,\alpha^{sx}U(rx)V(sx)}.$$

We note that formulas (3.3), (3.4), (3.5), (3.6), (3.9), and (3.10) in [6] are special cases of the Theorem and the Corollary.

Valuable references connected with the main results of this paper are [3], [5], and [7].

Finally, we give some special cases of the Theorem and the Corollary. If $p = 3$ and $q = 1$ in (3) and (4), we obtain

$$\sum_{k=1}^{n} \frac{e^{i\pi(k-1)rx}}{F(2krx-2rx+2sx)F(2krx+2sx)} = \frac{F(2nrx)}{F(2rx)F(2sx+2nrx)}$$

and

$$\sum_{k=1}^{n} \frac{e^{i\pi(k-1)rx}}{L(2krx-2rx+2sx)L(2krx+2sx)} = \frac{F(2nrx)}{F(2rx)L(2sx)L(2sx+2nrx)}.$$

If $p = 3$, $q = 1$, and $x = 1$ in the Corollary, we have

$$\sum_{k=1}^{\infty} \frac{(-1)^{(k-1)r}}{F(2s+2kr-2r)F(2s+2kr)} = \frac{2^s}{(3+\sqrt{5})^s F(2r)F(2s)}$$

and

$$\sum_{k=1}^{\infty} \frac{(-1)^{(k-1)r}}{L(2s+2kr-2r)L(2s+2kr)} = \frac{2^s}{\sqrt{5}(3+\sqrt{5})^s F(2r)L(2s)}.$$

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