

# SOME IDENTITIES INVOLVING GENERALIZED GENOCCHI POLYNOMIALS AND GENERALIZED FIBONACCI-LUCAS SEQUENCES

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## 1. INTRODUCTION

In [7], Toscano gave some novel identities between generalized Fibonacci-Lucas sequences and Bernoulli-Euler polynomials. Later, Zhang and Guo [9] and Wang and Zhang [8] discussed the case of Bernoulli-Euler polynomials of higher order and generalized the results of Toscano.

The purpose of this paper is to establish some identities containing generalized Genocchi polynomials that, as one application, yield some results of Toscano [7] and Byrd [1] as special cases, as well as other identities involving Bernoulli-Euler and Fibonacci-Lucas numbers.

## 2. SOME LEMMAS

It is well known that a general linear sequence  $S_n(p, q)$  ( $n = 0, 1, 2, \dots$ ) of order 2 is defined by the law of recurrence,

$$S_n(p, q) = pS_{n-1}(p, q) - qS_{n-2}(p, q),$$

with  $S_0, S_1, p$ , and  $q$  arbitrary, provided that  $\Delta = p^2 - 4q > 0$ .

In particular, if  $S_0 = 0, S_1 = 1$ , or  $S_0 = 2, S_1 = p$ , we have generalized Fibonacci and Lucas sequences, respectively, in symbols  $U_n(p, q), V_n(p, q)$ .

If  $\alpha, \beta$  ( $\alpha > \beta$ ) are the roots of the equation  $x^2 - px + q = 0$ , then we have (see [7])

$$U_n(p, q) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n(p, q) = \alpha^n + \beta^n, \quad (1)$$

$$S_n(p, q) = \left(S_1 - \frac{1}{2}S_0\right)U_n(p, q) + \frac{1}{2}S_0V_n(p, q). \quad (2)$$

We assume

$$S_0 = k, \quad S_1 = \frac{1}{2}pk + \left(x - \frac{1}{2}k\right)\Delta^{1/2}$$

and, using (1) and (2), we deduce that

$$S_n(x; p, q) = \left(x - \frac{1}{2}k\right)\Delta^{1/2}U_n(p, q) + \frac{1}{2}kV_n(p, q), \quad (3)$$

$$S_n(x; p, q) = x\alpha^n + (k - x)\beta^n. \quad (4)$$

From this point on, we shall use the brief notation  $U_n, V_n$ , and  $S_n(x)$  to denote  $U_n(p, q), V_n(p, q)$ , and  $S_n(x; p, q)$ , respectively.

By adapting the method of [7], [8], and [9] to  $S_n(x)$ , we have obtained the following results:

$$\begin{aligned}
 & S_n^m(x) + (-1)^\nu S_n^m(k-x) \\
 &= \sum_{r=0}^m \binom{m}{r} \Delta^{r/2} U_n^r \frac{1}{2^{m-r}} k^{m-r} V_n^{m-r} \left(x - \frac{k}{2}\right)^r (1 + (-1)^{\nu+r}) \\
 &= \begin{cases} \frac{1}{2^{m-1}} \sum_{r=0}^{\lfloor m/2 \rfloor} \binom{m}{2r} \Delta^r U_n^{2r} k^{m-2r} V_n^{m-2r} (2x-k)^{2r} & (\nu \text{ even}), \\ \frac{1}{2^{m-1}} \sum_{r=0}^{\lfloor m/2 \rfloor} \binom{m}{2r+1} \Delta^{r+(1/2)} U_n^{2r+1} k^{m-2r-1} V_n^{m-2r-1} (2x-k)^{2r+1} & (\nu \text{ odd}), \end{cases} \tag{5}
 \end{aligned}$$

$$\begin{aligned}
 & S_n^m(x) + (-1)^\nu S_n^m(k-x) \\
 &= \sum_{r=0}^m \binom{m}{r} q^{nr} [\beta^{n(m-2r)} + (-1)^\nu \alpha^{n(m-2r)}] x^r (k-x)^{m-r} \\
 &= \begin{cases} \sum_{r=0}^m \binom{m}{r} q^{nr} V_{n(m-2r)} x^r (k-x)^{m-r} & (\nu \text{ even}), \\ -\sum_{r=0}^m \binom{m}{r} q^{nr} \Delta^{1/2} U_{n(m-2r)} x^r (k-x)^{m-r} & (\nu \text{ odd}), \end{cases} \tag{6}
 \end{aligned}$$

$$\begin{aligned}
 & S_n^m(x) + (-1)^\nu S_n^m(k-x) \\
 &= \begin{cases} 2 \binom{2m}{n} q^{nm} x^m (k-x)^m + \sum_{r=0}^{m-1} \binom{2m}{r} q^{nr} V_{2n(m-r)} [x^r (k-x)^{2m-r} + x^{2m-r} (k-x)^r] & (\nu \text{ even}), \\ -\sum_{r=0}^{m-1} \binom{2m}{r} q^{nr} \Delta^{1/2} U_{2n(m-r)} [x^r (k-x)^{2m-r} + x^{2m-r} (k-x)^r] & (\nu \text{ odd}), \end{cases} \tag{7}
 \end{aligned}$$

$$\begin{aligned}
 & S_n^{2m+1}(x) + (-1)^\nu S_n^{2m+1}(k-x) \\
 &= \begin{cases} \sum_{r=0}^m \binom{2m+1}{r} q^{nr} V_{n(2m-2r+1)} [x^r (k-x)^{2m-r+1} + x^{2m-r+1} (k-x)^r] & (\nu \text{ even}), \\ -\sum_{r=0}^m \binom{2m+1}{r} q^{nr} \Delta^{1/2} U_{n(2m-2r+1)} [x^r (k-x)^{2m-r+1} + x^{2m-r+1} (k-x)^r] & (\nu \text{ odd}), \end{cases} \tag{8}
 \end{aligned}$$

and the generating functions

$$\sum_{r=0}^{\infty} \frac{t^r}{r!} U_{nr} = \frac{1}{\Delta^{1/2}} [\exp(t\alpha^n) - \exp(t\beta^n)] \tag{9}$$

and

$$\sum_{r=0}^{\infty} \frac{t^r}{r!} V_{nr} = \exp(t\alpha^n) + \exp(t\beta^n). \tag{10}$$

3. THE MAIN RESULTS

The generalized Genocchi polynomial is defined as (see [4])

$$\sum_{r=0}^{\infty} \frac{t^r}{r!} G_r^{(k)}(x) = \left( \frac{2t}{\exp t + 1} \right)^k \exp(tx). \tag{11}$$

From this definition, it is easy to deduce the following properties (see [4]):

$$G_n^{(k)}(k-x) = (-1)^{n+k} G_n^{(k)}(x), \tag{12}$$

$$G_n^{(k+1)}(x) = \frac{2n(k-x)}{k} G_{n-1}^{(k)}(x) + \frac{2}{k} (n-k) G_n^{(k)}(x). \tag{13}$$

In particular,  $G_n^{(1)}(0) = G_n$  (Genocchi number, see [3]).

From (11), replacing  $t$  by  $\Delta^{1/2} U_n t$ , we have

$$\begin{aligned} \sum_{r=0}^{\infty} G_r^{(k)}(x) \frac{(\Delta^{1/2} U_n t)^r}{r!} &= \frac{2^k \Delta^{k/2} U_n^k t^k}{[\exp(\Delta^{1/2} U_n t) + 1]^k} \exp(x \Delta^{1/2} U_n t) \\ &= \frac{2^k \Delta^{k/2} U_n^k t^k}{[\exp(\alpha^n t) + \exp(\beta^n t)]^k} \exp[t(x\alpha^n + (k-x)\beta^n)]. \end{aligned}$$

Therefore,

$$[\exp(\alpha^n t) + \exp(\beta^n t)]^k \sum_{r=0}^{\infty} G_r^{(k)}(x) \frac{(\Delta^{1/2} U_n t)^r}{r!} = 2^k \Delta^{k/2} U_n^k t^k \exp(t S_n(x)).$$

Hence,

$$\left( \sum_{r=0}^{\infty} \frac{t^r}{r!} V_{nr} \right)^k \left( \sum_{r=0}^{\infty} G_r^{(k)}(x) \frac{(\Delta^{1/2} U_n t)^r}{r!} \right) = 2^k \Delta^{k/2} U_n^k t^k \exp(t S_n(x)).$$

We now expand the product figuring in the left member into a power series of  $t$ , compare with the expansion of the right member, and obtain

$$\begin{aligned} \sum_{r=0}^m \binom{m}{r} \Delta^{r/2} U_n^r G_r^{(k)}(x) (m-r)! \sum_{r_1+\dots+r_k=m-r} \frac{V_{nr_1}}{r_1!} \dots \frac{V_{nr_k}}{r_k!} \\ = (m)_k 2^k \Delta^{k/2} U_n^k S_n^{m-k}(x). \end{aligned} \tag{14}$$

If we replace  $x$  by  $k-x$  in (14) and use (12), we find

$$\begin{aligned} \sum_{r=0}^m \binom{m}{r} \Delta^{r/2} U_n^r (-1)^r G_r^{(k)}(x) (m-r)! \sum_{r_1+\dots+r_k=m-r} \frac{V_{nr_1}}{r_1!} \dots \frac{V_{nr_k}}{r_k!} \\ = (m)_k (-1)^k 2^k \Delta^{k/2} U_n^k S_n^{m-k}(x). \end{aligned} \tag{15}$$

From (14), (15), (5), (7), and (8), we have

$$\begin{aligned} \sum_{r=0}^{[m/2]} \binom{m}{2r} \Delta^r U_n^{2r} G_{2r}^{(k)}(x) (m-2r)! \sum_{r_1+\dots+r_k=m-2r} \frac{V_{nr_1}}{r_1!} \dots \frac{V_{nr_k}}{r_k!} \\ = (m)_k 2^{k-1} \Delta^{k/2} U_n^k [S_n^{m-k}(x) + (-1)^k S_n^{m-k}(x)] \end{aligned}$$

$$= \begin{cases} \frac{\binom{m}{k}}{2^{m-2k}} \Delta^{k/2} U_n^k \sum_{r=0}^{[(m-k)/2]} \binom{m-k}{2r} \Delta^r U_n^{2r} k^{m-k-2r} V_n^{m-k-2r} (2x-k)^{2r} & (k \text{ even}), \\ \frac{\binom{m}{k}}{2^{m-2k}} \Delta^{(k+1)/2} U_n^k \sum_{r=0}^{[(m-k)/2]} \binom{m-k}{2r+1} \Delta^r U_n^{2r+1} k^{m-k-2r-1} V_n^{m-k-2r-1} (2x-k)^{2r+1} & (k \text{ odd}), \end{cases} \quad (16)$$

$$= \begin{cases} (m)_k 2^{k-1} U_n^k (1 + (-1)^{m-k}) q^{n(m-k)/2} \binom{m-k}{(m-k)/2} x^{\frac{m-k}{2}} (k-x)^{\frac{m-k}{2}} + (m)_k 2^{k-1} \Delta^{k/2} U_n^k \\ \times \sum_{r=0}^{m-k-1} \binom{m-k}{r} q^{nr} V_{n(m-k-2r)} [x^r (k-x)^{m-k-r} + x^{m-k-r} (k-x)^r] & (k \text{ even}), \\ -(m)_k 2^{k-1} \Delta^{(k+1)/2} U_n^k \sum_{r=0}^{[\frac{m-k-2}{2}]} \binom{m-k}{r} q^{nr} U_{n(m-k-2r)} [x^r (k-x)^{m-k-r} + x^{m-k-r} (k-x)^r] & (k \text{ odd}). \end{cases} \quad (17)$$

We also obtain

$$\begin{aligned} & \sum_{r=0}^{[m/2]} \binom{m}{2r+1} \Delta^{r+(1/2)} U_n^{2r+1} G_{2r+1}^{(k)}(x) (m-2r-1)! \sum_{r_1+\dots+r_k=m-2r-1} \frac{V_{nr_1}}{r_1!} \dots \frac{V_{nr_k}}{r_k!} \\ &= (m)_k 2^{k-1} \Delta^{k/2} U_n^k [S_n^{m-k}(x) - (-1)^k S_n^{m-k}(x)] \\ &= \begin{cases} \frac{\binom{m}{k}}{2^{m-2k}} \Delta^{k/2} U_n^k \sum_{r=0}^{[(m-k)/2]} \binom{m-k}{2r+1} \Delta^{r+(1/2)} U_n^{2r+1} k^{m-k-2r-1} V_n^{m-k-2r-1} (2x-k)^{2r+1} & (k \text{ even}), \\ \frac{\binom{m}{k}}{2^{m-2k}} \Delta^{k/2} U_n^k \sum_{r=0}^{[(m-k)/2]} \binom{m-k}{2r} \Delta^r U_n^{2r} k^{m-k-2r} V_n^{m-k-2r-1} (2x-k)^{2r} & (k \text{ odd}), \end{cases} \quad (18) \end{aligned}$$

$$= \begin{cases} (m)_k 2^{k-1} \Delta^{k/2} U_n^k (1 + (-1)^{m-k}) q^{n(m-k)/2} \binom{m-k}{(m-k)/2} x^{(m-k)/2} (k-x)^{(m-k)/2} \\ + (m)_k 2^{k-1} \Delta^{k/2} U_n^k \sum_{r=0}^{[(m-k-2)/2]} \binom{m-k}{r} q^{nr} V_{n(m-k-2r)} [x^r (k-x)^{m-k-r} + x^{m-k-r} (k-x)^r] & (k \text{ odd}), \\ -(m)_k 2^{k-1} \Delta^{(k+1)/2} U_n^k \sum_{r=0}^{[(m-k-2)/2]} \binom{m-k}{r} q^{nr} U_{n(m-k-2r)} [x^r (k-x)^{m-k-r} \\ + x^{m-k-r} (k-x)^r] & (k \text{ even}). \end{cases} \quad (19)$$

#### 4. SOME CONSEQUENCES

If we take  $x = k/2$  in (16) and (18), then

$$\begin{aligned} & \sum_{r=0}^{[m/2]} \binom{m}{2r} \Delta^r U_n^{2r} G_{2r}^{(k)}(k/2) (m-2r)! \sum_{r_1+\dots+r_k=m-2r} \frac{V_{nr_1}}{r_1!} \dots \frac{V_{nr_k}}{r_k!} \\ &= \begin{cases} \frac{\binom{m}{k}}{2^{(m-2k)}} \Delta^{k/2} U_n^k k^{m-k} V_n^{m-k} & (k \text{ even}), \\ 0 & (k \text{ odd}), \end{cases} \quad (20) \end{aligned}$$

$$\sum_{r=0}^{[m/2]} \binom{m}{2r+1} \Delta^r U_n^{2r} G_{2r+1}^{(k)} (k/2)(m-2r-1)! \sum_{r_1+\dots+r_k=m-2r-1} \frac{V_{nr_1}}{r_1!} \dots \frac{V_{nr_k}}{r_k!}$$

$$= \begin{cases} 0 & (k \text{ even}), \\ \frac{\binom{m}{k}}{2^{(m-2k)}} \Delta^{(k-1)/2} U_n^{k-1} k^{m-k} V_n^{m-k} & (k \text{ odd}). \end{cases} \quad (21)$$

Taking  $k = 2$  in (20), again using  $G_{2r}(1) = -G_{2r}$  (see [4]) and recurrence relation (13), we get

$$\sum_{r=0}^{[m/2]} \binom{m}{2r} \Delta^{r-1} U_n^{2(r-1)} (1-2r) G_{2r} \sum_j^{m-2r} \binom{m-2r}{j} V_{nj} V_{n(m-2r-j)} = 2V_n^{m-2} m(m-1). \quad (22)$$

Taking  $k = 1$  in (21), using  $G_{2r+1}(1/2) = \frac{(2r+1)E_{2r}}{2^{2r}}$ , where  $E_{2r}$  is the Euler number (see [4]), and  $m \rightarrow m+1$ , we get (22) of [8], namely,

$$\sum_{r=0}^{[m/2]} \binom{m}{2r} \Delta^r U_n^{2r} \frac{E_{2r}}{2^{2r}} V_{n(m-2r)} = \frac{1}{2^{m-1}} V_n^m. \quad (23)$$

In (22), using  $G_{2r} = 2(1-2^{2r})B_{2r}$ , where  $B_{2r}$  is the Bernoulli number (see [2], [4]), we obtain

$$\sum_{r=0}^{[m/2]} \binom{m}{2r} \Delta^{r-1} U_n^{2(r-1)} (1-2r)(1-2^{2r})B_{2r} \sum_j^{m-2r} \binom{m-2r}{j} V_{nj} V_{n(m-2r-j)} = V_n^{m-2} m(m-1). \quad (24)$$

If we take  $p = 1$  and  $q = -1$ , then  $U_n(1, -1) = F_n$  (Fibonacci number),  $V_n(1, -1) = L_n$  (Lucas number), and from (23) and (24) it follows that

$$\sum_{r=0}^{[m/2]} \binom{m}{2r} 5^r F_n^{2r} \frac{E_{2r}}{2^{2r}} L_{n(m-2r)} = \frac{1}{2^{m-1}} L_n^m, \quad (25)$$

$$\sum_{r=0}^{[m/2]} \binom{m}{2r} 5^{r-1} F_n^{2r} (1-2r)(1-2^{2r})B_{2r} \sum_{j=0}^{m-2r} \binom{m-2r}{j} L_{nj} L_{n(m-2r-j)} = m(m-1)L_n^{m-2}, \quad (26)$$

where (25) is the result of Byrd [1].

If we take  $p = 2$  and  $q = -1$ , then  $U_n(2, -1) = P_n$  (Pell number),  $V_n(2, -1) = Q_n$  (Pell-Lucas number, see [5]), and from (23) and (24), it follows that

$$\sum_{r=0}^{[m/2]} \binom{m}{2r} 2^r P_n^{2r} \frac{E_{2r}}{2^{2r}} Q_{n(m-2r)} = \frac{1}{2^{m-1}} Q_n^m, \quad (27)$$

$$\sum_{r=0}^{[m/2]} \binom{m}{2r} 8^{r-1} P_n^{2r} (1-2r)(1-2^{2r})B_{2r} \sum_{j=0}^{m-2r} \binom{m-2r}{j} Q_{nj} Q_{n(m-2r-j)} = m(m-1)Q_n^{m-2}. \quad (28)$$

### 5. RESULTS IN TERMS OF THE POLYNOMIALS $\Xi_n(u)$ , $T_n(u)$ , $\Omega_n(u)$ , AND $\Psi_n(u)$

The Bernoulli and Euler polynomials allow themselves to be expressed as follows:

$$\begin{aligned} B_{2n}(x) &= \Xi_n(u), \quad u = x^2 - x, \quad n = 0, 1, 2, \dots, \\ E_{2n}(x) &= T_n(u), \quad u = x^2 - x, \quad n = 0, 1, 2, \dots, \\ B_{2n-1}(x) &= (2x-1)\Omega_{n-1}(u), \quad u = x^2 - x, \quad n = 1, 2, \dots, \\ E_{2n-1}(x) &= (2x-1)\Psi_{n-1}(u), \quad u = x^2 - x, \quad n = 1, 2, \dots, \end{aligned}$$

where  $\Xi, T, \Omega,$  and  $\Psi$  are all polynomials in  $u$  (see, e.g., Subramanian and Devanathan [6]). Applying (15), (16), (20), and (21) of [7], we get the following:

$$\sum_{r=0}^{[(m-1)/2]} \binom{m}{2r} \Delta^r U_n^{2r} U_{n(m-2r)} \Xi_r(u) = \frac{m}{2^{m-1}} U_n \sum_{r=0}^{[(m-1)/2]} \binom{m-1}{2r} \Delta^r U_n^{2r} V_n^{m-2r-1} (1+4u)^r, \quad (29)$$

$$\sum_{r=0}^{[(m-2)/2]} \binom{m}{2r+1} \Delta^r U_n^{2r+1} U_{n(m-2r-1)} \Omega_r(u) = \frac{m}{2^{m-1}} U_n \sum_{r=0}^{[(m-2)/2]} \binom{m-1}{2r+1} \Delta^r U_n^{2r+1} V_n^{m-2r-2} (1+4u)^r, \quad (30)$$

$$\sum_{r=0}^{[m/2]} \binom{m}{2r} \Delta^r U_n^{2r} V_{n(m-2r)} T_r(u) = \frac{1}{2^{m-1}} U_n \sum_{r=0}^{[m/2]} \binom{m}{2r} \Delta^r U_n^{2r} V_n^{m-2r} (1+4u)^r, \quad (31)$$

$$\sum_{r=0}^{[(m-1)/2]} \binom{m}{2r+1} \Delta^r U_n^{2r+1} V_{n(m-2r-1)} \Psi_r(u) = \frac{1}{2^{m-1}} U_n \sum_{r=0}^{[(m-1)/2]} \binom{m-1}{2r+1} \Delta^r U_n^{2r+1} V_n^{m-2r-1} (1+4u)^r. \quad (32)$$

### 6. A REMARK

From our main results (16), (17), (18), and (19), according to different choices of  $k, x, p,$  and  $q,$  using recurrence relation (13), we can obtain many interesting identities.

Our results have been numerically checked and found to be correct for  $m \leq 5.$

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### REFERENCES

1. P. F. Byrd. "Relations between Euler and Lucas Numbers." *The Fibonacci Quarterly* **13.2** (1975):111-14.
2. L. Carlitz. "Some Unusual Congruences for the Bernoulli and Genocchi Numbers." *Duke Math. J.* **35** (1968):563-66.
3. L. Comtet. *Advanced Combinatorics.* Boston: Reidel, 1974.
4. A. F. Horadam. "Genocchi Polynomials." In *Applications of Fibonacci Numbers 4*:145-66. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1991.
5. A. F. Horadam & Bro. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.1** (1985):7-20.
6. P. R. Subramanian & V. Devanathan. "Generation of Angular Momentum Traces and Euler Polynomials by Recurrence Relations." *J. Phys. A: Math. Gen.* **18** (1985):2909-15.
7. L. Toscano. "Recurring Sequences and Bernoulli-Euler Polynomials." *J. Comb. Inform. & System Sci.* **4** (1979):303-08.
8. Tianming Wang & Zhizheng Zhang. "Recurrence Sequences and Norlund-Euler Polynomials." *The Fibonacci Quarterly* **34.4** (1996):314-19.
9. Zhizheng Zhang & Lizhou Guo. "Recurrence Sequences and Bernoulli Polynomials of Higher Order." *The Fibonacci Quarterly* **33.4** (1995):359-62.

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