

SOME IDENTITIES INVOLVING THE FIBONACCI POLYNOMIALS*

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1. INTRODUCTION AND RESULTS

As usual, the Fibonacci polynomials $F(x) = \{F_n(x)\}$, $n = 0, 1, 2, \dots$, are defined by the second-order linear recurrence sequence

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x) \tag{1}$$

for $n \geq 0$ and $F_0(x) = 0$, $F_1(x) = 1$. Let

$$\alpha = \frac{x + \sqrt{x^2 + 4}}{2} \quad \text{and} \quad \beta = \frac{x - \sqrt{x^2 + 4}}{2}$$

denote the roots of the characteristic polynomial $\lambda^2 - x\lambda - 1$ of the sequence $F(x)$, then the terms of the sequence $F(x)$ (see [2]) can be expressed as

$$F_n(x) = \frac{1}{\alpha - \beta} \{\alpha^n - \beta^n\}$$

for $n = 0, 1, 2, \dots$

If $x = 1$, then the sequence $F(1)$ is called the Fibonacci sequence, and we shall denote it by $F = \{F_n\}$.

The various properties of $\{F_n\}$ were investigated by many authors. For example, Duncan [1] and Kuipers [3] proved that $(\log F_n)$ is uniformly distributed mod 1. Robbins [4] studied the Fibonacci numbers of the forms $px^2 \pm 1$ and $px^3 \pm 1$, where p is a prime. The second author [5] obtained some identities involving the Fibonacci numbers. The main purpose of this paper is to study how to calculate the summation involving the Fibonacci polynomials:

$$\sum_{a_1+a_2+\dots+a_k=n} F_{a_1+1}(x) \cdot F_{a_2+1}(x) \cdot \dots \cdot F_{a_k+1}(x), \tag{2}$$

where the summation is over all k -dimension nonnegative integer coordinates (a_1, a_2, \dots, a_k) such that $a_1 + a_2 + \dots + a_k = n$, and k is any positive integer.

Regarding (2), it seems that it has not been studied yet, at least I have not seen expressions like (2) before. The problem is interesting because it is a generalization of [5], and it can also help us to find some new convolution properties for $F(x)$. In this paper we use the generating function of the sequence $F(x)$ and its partial derivative to study the evaluation of (2), and give an interesting identity for any fixed positive integers k and n . That is, we shall prove the following proposition.

Proposition: Let $F(x) = \{F_n(x)\}$ be defined by (1). Then, for any positive integers k and n , we have the calculating formula

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$$\sum_{a_1+a_2+\dots+a_k=n} F_{a_1+1}(x) \cdot F_{a_2+1}(x) \cdot \dots \cdot F_{a_k+1}(x) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+k-1-m}{m} \binom{n+k-1-2m}{k-1} \cdot x^{n-2m},$$

where $\binom{m}{n} = \frac{m!}{n!(m-n)!}$, and $\lfloor z \rfloor$ denotes the greatest integer not exceeding z .

From this proposition, we may immediately deduce the following several corollaries.

Corollary 1: For any positive integers k and n , we have the identity

$$\sum_{a_1+a_2+\dots+a_k=n+k} F_{a_1} \cdot F_{a_2} \cdot \dots \cdot F_{a_k} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+k-1-m}{m} \binom{n+k-1-2m}{k-1}.$$

Corollary 2: For any positive integers k and n , we have

$$\sum_{a_1+\dots+a_k=n+k} F_{2a_1} \cdot F_{2a_2} \cdot \dots \cdot F_{2a_k} = 3^k \cdot 5^{\frac{n-k}{2}} \cdot \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\binom{n+k-1-m}{m} \binom{n+k-1-2m}{k-1}}{5^m}.$$

Corollary 3: The identity

$$\sum_{a_1+\dots+a_k=n+k} F_{3a_1} \cdot F_{3a_2} \cdot \dots \cdot F_{3a_k} = 2^{2n+k} \cdot \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\binom{n+k-1-m}{m} \binom{n+k-1-2m}{k-1}}{16^m}$$

holds for all positive integers k and n .

Corollary 4: Let k and n be positive integers. Then

$$\sum_{a_1+\dots+a_k=n+k} F_{4a_1} \cdot F_{4a_2} \cdot \dots \cdot F_{4a_k} = 3^n \cdot 7^k \cdot 5^{\frac{n-k}{2}} \cdot \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\binom{n+k-1-m}{m} \binom{n+k-1-2m}{k-1}}{45^m}.$$

Corollary 5: Let k and n be positive integers. Then

$$\sum_{a_1+\dots+a_k=n+k} F_{5a_1} \cdot F_{5a_2} \cdot \dots \cdot F_{5a_k} = 5^k \cdot 11^n \cdot \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\binom{n+k-1-m}{m} \binom{n+k-1-2m}{k-1}}{121^m}.$$

In fact, for any positive integer m , using the proposition, we can give an exact calculating formula for

$$\sum_{a_1+a_2+\dots+a_k=n+k} F_{ma_1} \cdot F_{ma_2} \cdot \dots \cdot F_{ma_k}.$$

2. PROOF OF THE PROPOSITION

In this section we shall complete the proof of the proposition. First, note that

$$F_n(x) = \frac{1}{\sqrt{x^2+4}} \left[\left(\frac{x+\sqrt{x^2+4}}{2} \right)^n - \left(\frac{x-\sqrt{x^2+4}}{2} \right)^n \right],$$

so we can easily deduce that the generating function of $F(x)$ is

$$G(t, x) = \frac{1}{1 - xt - t^2} = \frac{1}{(\alpha - t)(\beta - t)} \tag{3}$$

$$= \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} (\alpha^{n+1} - \beta^{n+1}) \cdot t^n = \sum_{n=0}^{\infty} F_{n+1}(x) \cdot t^n.$$

Let $\frac{\partial G^k(t, x)}{\partial x^k}$ denote the k^{th} partial derivative of $G(t, x)$ for x , and $F_n^{(k)}(x)$ denote the k^{th} derivative of $F_n(x)$. Then from (3) we have

$$\frac{\partial G(t, x)}{\partial x} = \frac{t}{(1 - xt - t^2)^2} = \sum_{n=0}^{\infty} F_{n+1}^{(1)}(x) \cdot t^n,$$

$$\frac{\partial G^2(t, x)}{\partial x^2} = \frac{2! \cdot t^2}{(1 - xt - t^2)^3} = \sum_{n=0}^{\infty} F_{n+1}^{(2)}(x) \cdot t^n, \tag{5}$$

...

$$\frac{\partial G^{k-1}(t, x)}{\partial x^{k-1}} = \frac{(k-1)! \cdot t^{k-1}}{(1 - xt - t^2)^k} = \sum_{n=0}^{\infty} F_{n+1}^{(k-1)}(x) \cdot t^n = \sum_{n=0}^{\infty} F_{n+1}^{(k-1)}(x) \cdot t^{n+k-1},$$

where we have used the fact that $F_{n+1}(x)$ is a polynomial of degree n .

For any two absolutely convergent power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, note that

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} \left(\sum_{u+v=n} a_u b_v \right) x^n.$$

So from (5) we obtain

$$\sum_{n=0}^{\infty} \left(\sum_{a_1 + \dots + a_k = n} F_{a_1+1}(x) \cdot F_{a_2+1}(x) \cdot \dots \cdot F_{a_k+1}(x) \right) \cdot t^n = \left(\sum_{n=0}^{\infty} F_{n+1}(x) \cdot t^n \right)^k \tag{6}$$

$$= \frac{1}{(1 - xt - t^2)^k} = \frac{1}{(k-1)! \cdot t^{k-1}} \frac{\partial G^{k-1}(t, x)}{\partial x^{k-1}} = \frac{1}{(k-1)!} \sum_{n=0}^{\infty} F_{n+k}^{(k-1)}(x) \cdot t^n.$$

Equating the coefficients of t^n on both sides of equation (6), we obtain the identity

$$\sum_{a_1 + a_2 + \dots + a_k = n} F_{a_1+1}(x) \cdot F_{a_2+1}(x) \cdot \dots \cdot F_{a_k+1}(x) = \frac{1}{(k-1)!} \cdot F_{n+k}^{(k-1)}(x). \tag{7}$$

On the other hand, note that from the combinatorial identity

$$\binom{n-m+1}{m} = \binom{n-m}{m} + \binom{n-m}{m-1}, \tag{8}$$

the recurrence formula $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$, and by mathematical induction, we can easily deduce

$$F_{n+1}(x) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-m}{m} \cdot x^{n-2m}. \tag{9}$$

In fact, from the definition of $F_n(x)$, we know that (9) is true for $n=0$ and $n=1$. Assume (9) is true for all integers $0 \leq n \leq k$. Then, for $n = k + 1$, applying (8) and the inductive hypothesis we immediately obtain

$$\begin{aligned} \sum_{m=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1-m}{m} \cdot x^{k+1-2m} &= 1 + \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-m}{m+1} \cdot x^{k-1-2m} \\ &= 1 + \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1-m}{m+1} \cdot x^{k-1-2m} + \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1-m}{m} \cdot x^{k-1-2m} \\ &= \sum_{m=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k-m}{m} \cdot x^{k+1-2m} + \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1-m}{m} \cdot x^{k-1-2m} \\ &= x \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-m}{m} \cdot x^{k-2m} + \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1-m}{m} \cdot x^{k-1-2m} \\ &= xF_{k+1}(x) + F_k(x) = F_{k+2}(x), \end{aligned}$$

where we have used $\binom{k-m}{m} = 0$ if $m > \frac{k}{2}$. So by induction we know that (9) is true for all non-negative integer n .

From (9) we can deduce that the $(k-1)$ th derivative of $F_{n+k}(x)$ is

$$F_{n+k}^{(k-1)}(x) = \left(\sum_{m=0}^{\lfloor \frac{n+k-1}{2} \rfloor} \binom{n+k-1-m}{m} \cdot x^{n+k-1-2m} \right)^{(k-1)} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n+k-1-m)!}{m! \cdot (n-2m)!} x^{n-2m}. \tag{10}$$

Combining (7) and (10), we obtain the identity

$$\sum_{a_1 + \dots + a_k = n} F_{a_1+1}(x) \cdot F_{a_2+1}(x) \cdot \dots \cdot F_{a_k+1}(x) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+k-1-m}{m} \cdot \binom{n+k-1-2m}{k-1} \cdot x^{n-2m}.$$

This completes the proof of the Proposition.

Proof of the Corollaries: Taking $x = 1$ in the Proposition and noting that $F_0 = 0$, we have

$$\begin{aligned} \sum_{a_1 + a_2 + \dots + a_k = n} F_{a_1+1}(1) \cdot F_{a_2+1}(1) \cdot \dots \cdot F_{a_k+1}(1) &= \sum_{a_1+1+a_2+1+\dots+a_k+1=n+k} F_{a_1+1} \cdot F_{a_2+1} \cdot \dots \cdot F_{a_k+1} \\ &= \sum_{a_1+a_2+\dots+a_k=n+k} F_{a_1} \cdot F_{a_2} \cdot \dots \cdot F_{a_k} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+k-1-m}{m} \cdot \binom{n+k-1-2m}{k-1}. \end{aligned}$$

This proves Corollary 1.

Taking $x = -\sqrt{5}$, 4 , $-3\sqrt{5}$, and 11 , respectively, in the Proposition, and noting that

$$\begin{aligned} F_n(-\sqrt{5}) &= \frac{(-1)^{n+1}}{3} \left[\left(\frac{3+\sqrt{5}}{2} \right)^n - \left(\frac{3-\sqrt{5}}{2} \right)^n \right] = \frac{(-1)^{n+1}\sqrt{5}}{3} \cdot F_{2n}, \\ F_n(4) &= \frac{1}{2\sqrt{5}} \left[(2+\sqrt{5})^n - (2-\sqrt{5})^n \right] = \frac{1}{2\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{3n} - \left(\frac{1-\sqrt{5}}{2} \right)^{3n} \right] = \frac{1}{2} \cdot F_{3n}, \\ F_n(-3\sqrt{5}) &= \frac{(-1)^{n+1}}{7} \left[\left(\frac{7+3\sqrt{5}}{2} \right)^n - \left(\frac{7-3\sqrt{5}}{2} \right)^n \right] = \frac{(-1)^{n+1}\sqrt{5}}{7} \cdot F_{4n}, \end{aligned}$$

and

$$F_n(11) = \frac{1}{5\sqrt{5}} \left[\left(\frac{11+5\sqrt{5}}{2} \right)^n - \left(\frac{11-5\sqrt{5}}{2} \right)^n \right] = \frac{1}{5\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{5n} - \left(\frac{1-\sqrt{5}}{2} \right)^{5n} \right] = \frac{1}{5} \cdot F_{5n},$$

we may immediately deduce Corollary 2, Corollary 3, Corollary 4, and Corollary 5.

This completes the proof of the Corollaries.

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