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Identities on Bell polynomials and Sheffer sequences

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Abstract

In this paper, we study exponential partial Bell polynomials and Sheffer sequences. Two new characterizations of Sheffer sequences are presented, which indicate the relations between Sheffer sequences and Riordan arrays. Several general identities involving Bell polynomials and Sheffer sequences are established, which reduce to some elegant identities for associated sequences and cross sequences.

Keywords: Associated sequences; Bell polynomials; Cross sequences; Riordan arrays; Sheffer sequences; Combinatorial identities

1. Introduction

The Bell polynomials, or more explicitly, the exponential partial Bell polynomials, are defined as follows [3, pp. 133 and 134]:

Definition 1.1. The *exponential partial Bell polynomials* are the polynomials

$$B_{n,k} = B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$$

in an infinite number of variables $x_1, x_2, \ldots$, defined by the formal double series expansion:

$$\begin{aligned}\Phi = \Phi(t,u) &:= \exp\left(u \sum_{m\geq 1} x_m \frac{t^m}{m!}\right) = \sum_{n,k\geq 0} B_{n,k} \frac{t^n}{n!} u^k \\ &= 1 + \sum_{n\geq 1} \frac{t^n}{n!} \left\{ \sum_{k=1}^{n} u^k B_{n,k}(x_1, x_2, \ldots) \right\},\end{aligned} \tag{1.1}$$

or by the series expansion:

$$\Phi_k(t) := \frac{1}{k!} \left(\sum_{m\geq 1} x_m \frac{t^m}{m!} \right)^k = \sum_{n\geq k} B_{n,k} \frac{t^n}{n!}, \quad k = 0, 1, 2, \ldots. \tag{1.2}$$

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Their exact expression is

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{c_1!c_2!\cdots(1!)^{c_1}(2!)^{c_2}\cdots} x_1^{c_1} x_2^{c_2} \cdots, \tag{1.3}$$

where the summation takes place over all integers $c_1, c_2, c_3, \ldots \geq 0$, such that $c_1 + 2c_2 + 3c_3 + \cdots = n$ and $c_1 + c_2 + c_3 + \cdots = k$.

The Bell polynomials are quite general polynomials and they have been found in many applications in combinatorics. Comtet [3] devoted much to a thorough presentation of the Bell polynomials in the chapter on identities and expansions. For more results, the reader is referred to [2, Chapter 11] and [9, Chapter 5].

It is well-known that many special combinatorial sequences can be obtained from the Bell polynomials by appropriate choice of the variables $x_1, x_2, \ldots$. For instance, the Bell polynomials include as particular cases the Stirling numbers of both kinds, the Lah numbers, as well as the idempotent numbers (see [3, p. 135, Theorem B]):

$$B_{n,k}(1, 1, 1, \ldots) = S(n, k), \quad \text{(Stirling numbers of the second kind)},$$
$$B_{n,k}(1!, 2!, 3!, \ldots) = \binom{n-1}{k-1}\frac{n!}{k!}, \quad \text{(Lah numbers)},$$
$$B_{n,k}(0!, 1!, 2!, \ldots) = s(n, k), \quad \text{(unsigned Stirling numbers of the first kind)},$$
$$B_{n,k}(1, 2, 3, \ldots) = \binom{n}{k} k^{n-k}, \quad \text{(idempotent numbers)}.$$

Recently, Abbas and Bouroubi [1] studied the Bell polynomials and proposed two general identities by two different methods, one based on the Lagrange inversion formula and the other based on the binomial sequences. For convenience, let us list the second identity [1, Theorem 6]:

$$B_{n,k}(p_0(1), 2p_1(1), 3p_2(1), \ldots) = \binom{n}{k} p_{n-k}(k), \tag{1.4}$$

where $p_n(x)$ is a sequence of binomial type which satisfies

$$p_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} p_{n-k}(x) p_k(y). \tag{1.5}$$

Yang [17] further generalized (1.4) and obtained some other identities involving Bell polynomials and binomial sequences.

A sequence is of binomial type if and only if it is an associated sequence [10, p. 26]. Since an associated sequence is only a special Sheffer sequence, it will be instructive and interesting to do some research on Bell polynomials and Sheffer sequences with similar methods to those used in [1] and [17].

The main contributions of this article are in proposing two new characterizations for Sheffer sequences and giving some general identities involving Bell polynomials and Sheffer sequences.

This article is organized as follows. The definitions of some sequences are introduced at the end of this section. In Section 2, we demonstrate new characterizations of Sheffer sequences. The relations between Sheffer sequences and Riordan arrays are also briefly discussed there. Section 3 is devoted to general identities on Bell polynomials and Sheffer sequences, which generalize some results presented in [1,17]. Moreover, from the general identities, we obtain two elegant identities for cross sequences. In Section 4, we give some applications, and finally, in Section 5, we give some further remarks.

Now, let us introduce some definitions. If the formal power series $f(t)$ has a multiplicative inverse, denoted by $f(t)^{-1}$ or $1/f(t)$, then we call $f(t)$ an *invertible series*. If the series $f(t)$ has a compositional inverse, denoted by $\bar{f}(t)$ and satisfying $f(\bar{f}(t)) = \bar{f}(f(t)) = t$, then we call $f(t)$ a *delta series*.

Definition 1.2 (*[10, Theorem 2.3.4]*). Let $g(t)$ be an invertible series and let $f(t)$ be a delta series; we say that the sequence $s_n(x)$ is the *Sheffer sequence* for the pair $(g(t), f(t))$ if and only if

$$\sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!} = \frac{1}{g(\bar{f}(t))}\mathrm{e}^{x\bar{f}(t)}, \tag{1.6}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

Definition 1.3 (*[10, p. 17]*). The Sheffer sequence for $(1, f(t))$ is the *associated sequence* for $f(t)$. If $s_n(x)$ is associated to $f(t)$, then $\sum_{n=0}^{\infty} s_n(x)t^n/n! = \exp(x\bar{f}(t))$. The Sheffer sequence for $(g(t), t)$ is the *Appell sequence* for $g(t)$. If $s_n(x)$ is Appell for $g(t)$, then $\sum_{n=0}^{\infty} s_n(x)t^n/n! = \exp(xt)/g(t)$.

For example, the Laguerre polynomials $L_n^{(\alpha)}(x)$ are Sheffer for $((1-t)^{-\alpha-1}, t/(t-1))$ and the Poisson–Charlier polynomials $c_n(x; a)$ are Sheffer for $(\mathrm{e}^{a(\mathrm{e}^t-1)}, a(\mathrm{e}^t-1))$. Moreover, the sequence x^n is associated to $f(t) = t$; the Hermite polynomials $H_n^{(\nu)}(x)$ are Appell for $\mathrm{e}^{\nu t^2/2}$; the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ and the generalized Euler polynomials $E_n^{(\alpha)}(x)$ are Appell for $\left(\frac{\mathrm{e}^t-1}{t}\right)^{\alpha}$ and $\left(\frac{\mathrm{e}^t+1}{2}\right)^{\alpha}$, respectively. More Sheffer sequences can be found in [10, Chapter 4].

Besides the generating function, there are several other ways to characterize Sheffer sequences (see [10, Section 2.3]). We now introduce an algebraic one.

Theorem 1.4 (*[10, Theorem 2.3.9]*). *A sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$, for some invertible $g(t)$, if and only if*

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_{n-k}(x)p_k(y), \tag{1.7}$$

where $p_n(x)$ is the associated sequence for $f(t)$. Particularly, if $s_n(x)$ is itself an associated sequence, then (1.7) will reduce to (1.5).

Next, we will give the definition of cross sequences.

Definition 1.5 (*[10, Theorem 5.3.1]*). A sequence $p_n^{(\lambda)}(x)$ is a *cross sequence* if and only if

$$p_n^{(\lambda+\mu)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_{n-k}^{(\lambda)}(x)p_k^{(\mu)}(y) \tag{1.8}$$

for all $n \geq 0$ and all real numbers λ and μ.

For example, the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$, the generalized Euler polynomials $E_n^{(\alpha)}(x)$, the Hermite polynomials $H_n^{(\nu)}(x)$, and the actuarial polynomials $a_n^{(\beta)}(x)$, are all cross sequences. Additionally, from Theorem 1.4 or the proof of [10, Theorem 5.3.1], we know that if $p_n^{(\lambda)}(x)$ is a cross sequence, then for each λ it is a Sheffer sequence with $p_n^{(0)}(x)$ the corresponding associated sequence.

The reader is referred to [3, Section 1.12] and [8] for more details on formal power series and to [10–12] for more results on special types of polynomial sequences.

2. New characterizations of Sheffer sequences

In this section, we will establish two new characterizations of Sheffer sequences, which generalize the results of Roman [10, Section 4.1.8] and Yang [17, Lemma 2]. Furthermore, it can be found that these two characterizations indicate the relationships between Sheffer sequences and exponential Riordan arrays.

Theorem 2.1. *A sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$, for some invertible $g(t)$, if and only if*

$$s_n(x+z) = \sum_{j=0}^{n} \binom{n}{j} s_{n-j}(z) \sum_{k=0}^{j} x^k B_{j,k}(\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_{j-k+1}), \tag{2.1}$$

where $\bar{f}_m = m![t^m]\bar{f}(t)$, that is, $\bar{f}(t) = \sum_{m=1}^{\infty} \bar{f}_m t^m/m!$.

Proof. If $s_n(x)$ is Sheffer for $(g(t), f(t))$, then by means of (1.1) and (1.6), we have

$$\sum_{n=0}^{\infty} s_n(x+z)\frac{t^n}{n!} = \frac{1}{g(\bar{f}(t))}\mathrm{e}^{z\bar{f}(t)}\mathrm{e}^{x\bar{f}(t)} = \sum_{i=0}^{\infty} s_i(z)\frac{t^i}{i!}\cdot\sum_{j=0}^{\infty}\sum_{k=0}^{j} x^k B_{j,k}(\bar{f}_1, \bar{f}_2, \ldots)\frac{t^j}{j!}$$
$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j} s_{n-j}(z)\sum_{k=0}^{j} x^k B_{j,k}(\bar{f}_1, \bar{f}_2, \ldots)\frac{t^n}{n!},$$

which yields (2.1) at once. It should be noticed that, for associated sequence $p_n(x)$ and for $z = 0$, (2.1) will reduce to

$$p_n(x) = \sum_{k=0}^{n} x^k B_{n,k}(\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_{n-k+1}). \tag{2.2}$$

This identity comes from the fact that the associated sequence $p_n(x)$ satisfies $p_0(x) = 1$ and $p_n(0) = 0$ for $n > 0$ (see [10, Theorem 2.4.5]). Thus, if a sequence $s_n(x)$ satisfies (2.1), then it also satisfies (1.7). By Theorem 1.4, $s_n(x)$ is a Sheffer sequence. □

In the proof of the next result, we will make use of the *potential polynomials* $P_n^{(r)}$, which are defined for each complex number r by

$$\sum_{n=0}^{\infty} P_n^{(r)}\frac{t^n}{n!} = \left(1 + \sum_{i=1}^{\infty} h_i \frac{t^i}{i!}\right)^r. \tag{2.3}$$

According to [3, p. 141, Theorem B], $P_n^{(r)} = P_n^{(r)}(h_1, h_2, \ldots, h_n) = \sum_{k=0}^{n}(r)_k B_{n,k}(h_1, h_2, \ldots)$.

Theorem 2.2. *A sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$, for some invertible $g(t)$, if and only if*

$$s_n(xy+z) = \sum_{j=0}^{n}\binom{n}{j} s_{n-j}(z)\sum_{k=0}^{j}(x)_k B_{j,k}(p_1(y), p_2(y), \ldots, p_{j-k+1}(y)), \tag{2.4}$$

where $(x)_k$ is the falling factorial defined by $(x)_0 := 1$, $(x)_k := x(x-1)\cdots(x-k+1)$ and $p_n(x)$ is the associated sequence for $f(t)$.

Proof. The associated sequence $p_n(y)$ has the generating function $\sum_{i=0}^{\infty} p_i(y)t^i/i! = \mathrm{e}^{y\bar{f}(t)}$. Since $p_0(y) = 1$, then Eq. (2.3) indicates that

$$\left(\mathrm{e}^{y\bar{f}(t)}\right)^x = \left(1 + \sum_{i=1}^{\infty} p_i(y)\frac{t^i}{i!}\right)^x = \sum_{n=0}^{\infty} P_n^{(x)}(p_1(y), p_2(y), \ldots)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(x)_k B_{n,k}(p_1(y), p_2(y), \ldots)\frac{t^n}{n!}.$$

Hence, for the Sheffer sequence $s_n(x)$, we have

$$\sum_{n=0}^{\infty} s_n(xy+z)\frac{t^n}{n!} = \frac{1}{g(\bar{f}(t))}\mathrm{e}^{z\bar{f}(t)}\left(\mathrm{e}^{y\bar{f}(t)}\right)^x = \sum_{i=0}^{\infty} s_i(z)\frac{t^i}{i!}\cdot\sum_{j=0}^{\infty}\sum_{k=0}^{j}(x)_k B_{j,k}(p_1(y), p_2(y), \ldots)\frac{t^j}{j!}$$
$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j} s_{n-j}(z)\sum_{k=0}^{j}(x)_k B_{j,k}(p_1(y), p_2(y), \ldots)\frac{t^n}{n!}.$$

Identifying the coefficients of $t^n/n!$ in the last equation gives (2.4). Analogous to the proof of Theorem 2.1, it can be found that for the associated sequence $p_n(x)$ and for $z = 0$, Eq. (2.4) will reduce to

$$p_n(xy) = \sum_{k=0}^{n}(x)_k B_{n,k}(p_1(y), p_2(y), \ldots, p_{n-k+1}(y)). \tag{2.5}$$

Thus, if a sequence $s_n(x)$ satisfies (2.4), then it also satisfies (1.7). This means $s_n(x)$ is a Sheffer sequence. □

In the above, we obtain two identities for the associated sequence $p_n(x)$, i.e., (2.2) and (2.5). Identity (2.2) can be found in [10, Section 4.1.8] and [17, Lemma 2, Eq. (9)]; while identity (2.5) is a generalization of [17, Lemma 2, Eq. (10)].

Our results also indicate the relations between Sheffer sequences and Riordan arrays. Because He et al. [5] and the authors [16] have systematically studied such relations, respectively, in the present paper, only a brief discussion will be given.

An *exponential Riordan array* is a pair $(g(t), f(t))$ of formal power series, where $g(t)$ is an invertible series and $f(t)$ is a delta series. The Riordan array $(g(t), f(t))$ defines an infinite, lower triangular array $(d_{n,k})_{0\le k\le n<\infty}$ according to the rule:

$$d_{n,k} = \left[\frac{t^n}{n!}\right] g(t)\frac{(f(t))^k}{k!}. \tag{2.6}$$

The product of two Riordan arrays is still a Riordan array, i.e.,

$$(g(t), f(t)) * (h(t), l(t)) = (g(t)h(f(t)), l(f(t))).$$

Moreover, the Riordan array $(g(t), f(t))$ has inverse $(1/g(\bar{f}(t)), \bar{f}(t))$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$. The reader may consult the papers by Shapiro et al. [13] and Sprugnoli [14,15] for more results of the theory of Riordan arrays; while for some recent developments, the reader may be referred to [4–7,16].

Now, let $s_n(x)$ be the Sheffer sequence for $(g(t), f(t))$ and define $s_n(x) := \sum_{k=0}^{n} s_{n,k}x^k$. From (2.1), we have

$$s_n(x) = \sum_{k=0}^{n}\left\{\sum_{j=k}^{n}\binom{n}{j} s_{n-j}(0)B_{j,k}(\bar{f}_1, \bar{f}_2, \ldots)\right\} x^k, \tag{2.7}$$

which gives the explicit expression for the coefficients $s_{n,k}$. The generating function of $s_{n,k}$ is

$$\begin{aligned}\sum_{n=k}^{\infty} s_{n,k}\frac{t^n}{n!} &= \sum_{n=k}^{\infty}\sum_{j=k}^{n}\binom{n}{j} s_{n-j}(0)B_{j,k}(\bar{f}_1, \bar{f}_2, \ldots)\frac{t^n}{n!}\\ &= \sum_{i=0}^{\infty} s_i(0)\frac{t^i}{i!}\cdot\sum_{j=k}^{\infty} B_{j,k}(\bar{f}_1, \bar{f}_2, \ldots)\frac{t^j}{j!} = \frac{1}{g(\bar{f}(t))}\cdot\frac{(\bar{f}(t))^k}{k!}.\end{aligned}$$

This implies that $s_{n,k}$ are the elements of the exponential Riordan array $(1/g(\bar{f}(t)), \bar{f}(t))$. Define

$$S[x] = (s_0(x), s_1(x), s_2(x), \ldots)^{\mathrm{T}} \quad \text{and} \quad X = (1, x, x^2, \ldots)^{\mathrm{T}},$$

then

$$S[x] = \left(\frac{1}{g(\bar{f}(t))}, \bar{f}(t)\right) * X \quad \text{and} \quad X = (g(t), f(t)) * S[x]. \tag{2.8}$$

The above discussion can be summarized as the following theorem, which can also be found in [5, Eq. (1.6) and Theorem 3.3] and [16, Theorem 3.2 and Corollary 6.5].

Theorem 2.3. *If the sequence $s_n(x) = \sum_{k=0}^{n} s_{n,k}x^k$ is Sheffer for $(g(t), f(t))$, then the coefficients $s_{n,k}$ are the elements of the exponential Riordan array $(1/g(\bar{f}(t)), \bar{f}(t))$. If $x^n = \sum_{k=0}^{n} a_{n,k}s_k(x)$, then $a_{n,k}$ are the elements of the exponential Riordan array $(g(t), f(t))$.*

Let $p_n(x) := \sum_{k=0}^{n} p_{n,k}x^k$ be the associated sequence for $f(t)$, then the coefficients $p_{n,k}$ are the elements of the exponential Riordan array $(1, \bar{f}(t))$. According to (1.2) and (2.6), $p_{n,k} = B_{n,k}(\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_{n-k+1})$. This gives the interpretation of Eq. (2.2). It should be noticed that the array $(1, \bar{f}(t))$ is just the *iteration matrix* [3, p. 145] for $\bar{f}(t)$ and $\Omega_n = 1/n!$.

By [3, p. 206, Theorem A], the Stirling numbers of the second kind $S(n, k)$ constitute the exponential Riordan array $(1, \mathrm{e}^t - 1)$. Let $\hat{X} = (1, (x)_1, (x)_2, \ldots)^{\mathrm{T}}$. Because $x^n = \sum_{k=0}^{n} S(n, k)(x)_k$ (see [3, p. 207, Theorem B]), we have $X = (1, \mathrm{e}^t - 1) * \hat{X}$. Combining this matrix equation with (2.8) gives

$$S[x] = \left(\frac{1}{g(\bar{f}(t))}, \bar{f}(t)\right) * (1, \mathrm{e}^t - 1) * \hat{X} = \left(\frac{1}{g(\bar{f}(t))}, \mathrm{e}^{\bar{f}(t)} - 1\right) * \hat{X}.$$

The generic element of the Riordan array $\left(\frac{1}{g(\bar{f}(t))}, \mathrm{e}^{\bar{f}(t)} - 1\right)$ is

$$\begin{aligned}\left[\frac{t^n}{n!}\right] \frac{1}{g(\bar{f}(t))} \frac{\left(\mathrm{e}^{\bar{f}(t)} - 1\right)^k}{k!} &= \left[\frac{t^n}{n!}\right] \sum_{i=0}^{\infty} s_i(0) \frac{t^i}{i!} \cdot \sum_{j=k}^{\infty} B_{j,k}(p_1(1), p_2(1), \ldots) \frac{t^j}{j!} \\ &= \sum_{j=k}^{n} \binom{n}{j} s_{n-j}(0) B_{j,k}(p_1(1), p_2(1), \ldots),\end{aligned}$$

which leads us to the following expression of $s_n(x)$:

$$s_n(x) = \sum_{k=0}^{n} \left\{ \sum_{j=k}^{n} \binom{n}{j} s_{n-j}(0) B_{j,k}(p_1(1), p_2(1), \ldots) \right\} (x)_k. \tag{2.9}$$

It is easy to see that (2.9) is a special case of (2.4).

3. General identities

In this section, we will generalize the results of Abbas–Bouroubi [1] and Yang [17] by substituting Sheffer sequences for the variables $x_1, x_2, \ldots$ in the Bell polynomials.

Let us first consider the power of the generating function

$$\Psi(x, t) := \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!} = \frac{1}{g(\bar{f}(t))} \mathrm{e}^{x \bar{f}(t)},$$

where $s_n(x)$ is Sheffer for $(g(t), f(t))$. For simplicity, suppose that $s_0(x) = 1$ (see the remarks given in Section 5). Since $1/g(\bar{f}(t)) = \Psi(0, t) = 1 + \sum_{n=1}^{\infty} s_n(0) t^n / n!$, then by the definition of the potential polynomials, we have

$$\begin{aligned}(\Psi(x, t))^k &= \left(\frac{1}{g(\bar{f}(t))}\right)^{k-1} \Psi(kx, t) = \sum_{i=0}^{\infty} P_i^{(k-1)}(s_1(0), s_2(0), \ldots) \frac{t^i}{i!} \cdot \sum_{j=0}^{\infty} s_j(kx) \frac{t^j}{j!} \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} P_i^{(k-1)}(s_1(0), s_2(0), \ldots) s_{n-i}(kx) \frac{t^n}{n!}\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}(\Psi(x, t))^k &= \left(\frac{1}{g(\bar{f}(t))}\right)^{k} \mathrm{e}^{kx \bar{f}(t)} = \sum_{i=0}^{\infty} P_i^{(k)}(s_1(0), s_2(0), \ldots) \frac{t^i}{i!} \cdot \sum_{j=0}^{\infty} p_j(kx) \frac{t^j}{j!} \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} P_i^{(k)}(s_1(0), s_2(0), \ldots) p_{n-i}(kx) \frac{t^n}{n!},\end{aligned} \tag{3.2}$$

where $p_n(x)$ is the associated sequence for $f(t)$. Now, the following theorem holds.

Theorem 3.1. *If $s_n(x)$ is the Sheffer sequence for $(g(t), f(t))$ and satisfies $s_0(x) = 1$, then*

$$B_{n,k}(s_0(x), 2s_1(x), 3s_2(x), \ldots) = \binom{n}{k} \sum_{i=0}^{n-k} \binom{n-k}{i} s_{n-k-i}(kx) P_i^{(k-1)}(s_1(0), s_2(0), \ldots) \tag{3.3}$$

$$= \binom{n}{k} \sum_{i=0}^{n-k} \binom{n-k}{i} p_{n-k-i}(kx) P_i^{(k)}(s_1(0), s_2(0), \ldots), \tag{3.4}$$

where $p_n(x)$ is the associated sequence for $f(t)$.

Proof. On the one hand,

$$\frac{1}{k!}(t\Psi(x,t))^k = \frac{1}{k!}\left(t\sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!}\right)^k = \frac{1}{k!}\left(\sum_{n=1}^{\infty} ns_{n-1}(x)\frac{t^n}{n!}\right)^k$$
$$= \sum_{n=k}^{\infty} B_{n,k}(s_0(x), 2s_1(x), 3s_2(x), \ldots)\frac{t^n}{n!}.$$

On the other hand, in view of (3.1), we have

$$\frac{1}{k!}(t\Psi(x,t))^k = \frac{1}{k!}t^k(\Psi(x,t))^k = \frac{1}{k!}t^k\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}P_i^{(k-1)}(s_1(0), s_2(0), \ldots)s_{n-i}(kx)\frac{t^n}{n!}$$
$$= \sum_{n=k}^{\infty}\binom{n}{k}\sum_{i=0}^{n-k}\binom{n-k}{i}s_{n-k-i}(kx)P_i^{(k-1)}(s_1(0), s_2(0), \ldots)\frac{t^n}{n!}.$$

Equating the coefficients of $t^n/n!$ gives the first identity of the theorem. Analogously, by appealing to (3.2), we can also obtain the second identity. □

Corollary 3.2 (*[17, Theorem 1]*). *If $p_n(x)$ is an associated sequence, then*

$$B_{n,k}(p_0(x), 2p_1(x), 3p_2(x), \ldots) = \binom{n}{k}p_{n-k}(kx). \tag{3.5}$$

Proof. According to [10, Theorem 2.4.5], the associated sequence $p_n(x)$ satisfies $p_n(0) = 0$ for $n > 0$. Therefore, $P_i^{(k-1)}(p_1(0), p_2(0), p_3(0), \ldots) = \delta_{0,i}$ and we obtain from (3.3) the identity (3.5). It can be verified that (3.4) will give the same result. □

Corollary 3.2 is one of the main results of [17], which also generalizes the identity (1.4). Next, it will be shown that there is a similar identity for cross sequences.

Corollary 3.3. *If $p_n^{(\lambda)}(x)$ is a cross sequence and satisfies $p_0^{(\lambda)}(x) = 1$, then*

$$B_{n,k}(p_0^{(\lambda)}(x), 2p_1^{(\lambda)}(x), 3p_2^{(\lambda)}(x), \ldots) = \binom{n}{k}p_{n-k}^{(k\lambda)}(kx).$$

Proof. Let $\Psi(x,t;\lambda) := \sum_{n=0}^{\infty} p_n^{(\lambda)}(x)t^n/n!$. By Definition 1.5, we have

$$\Psi(x,t;\lambda)\Psi(y,t;\mu) = \Psi(x+y,t;\lambda+\mu).$$

Thus,

$$\sum_{i=0}^{\infty}P_i^{(k-1)}(p_1^{(\lambda)}(0), p_2^{(\lambda)}(0), \ldots)\frac{t^i}{i!} = \left(\sum_{n=0}^{\infty}p_n^{(\lambda)}(0)\frac{t^n}{n!}\right)^{k-1}$$
$$= (\Psi(0,t;\lambda))^{k-1} = \Psi(0,t;(k-1)\lambda) = \sum_{i=0}^{\infty}p_i^{((k-1)\lambda)}(0)\frac{t^i}{i!}.$$

Equating the coefficients of $t^i/i!$ gives

$$P_i^{(k-1)}(p_1^{(\lambda)}(0), p_2^{(\lambda)}(0), \ldots) = p_i^{((k-1)\lambda)}(0),$$

which, combined with (3.3), leads us to the corollary at once. Analogously, we can obtain the same result from (3.4). □

Theorem 3.4. *If $s_n(x)$ is the Sheffer sequence for $(g(t), f(t))$ and satisfies $s_0(x) = 1$, then*

$$B_{n,k}(s_1(x), s_2(x), s_3(x), \ldots) = \sum_{i=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\frac{1}{k!}\binom{k}{j}\binom{n}{i} s_{n-i}(jx)P_i^{(j-1)}(s_1(0), s_2(0), \ldots) \tag{3.6}$$

$$= \sum_{i=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\frac{1}{k!}\binom{k}{j}\binom{n}{i} p_{n-i}(jx)P_i^{(j)}(s_1(0), s_2(0), \ldots), \tag{3.7}$$

where $p_n(x)$ is the associated sequence for $f(t)$.

Proof. On the one hand,

$$\frac{1}{k!}(\Psi(x,t) - 1)^k = \frac{1}{k!}\left(\sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!} - 1\right)^k = \frac{1}{k!}\left(\sum_{n=1}^{\infty} s_n(x)\frac{t^n}{n!}\right)^k$$
$$= \sum_{n=k}^{\infty} B_{n,k}(s_1(x), s_2(x), s_3(x), \ldots)\frac{t^n}{n!}.$$

On the other hand, making use of (3.1), we have

$$\frac{1}{k!}(\Psi(x,t) - 1)^k = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\Psi(x,t))^j$$
$$= \sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\frac{1}{k!}\binom{k}{j}\binom{n}{i} s_{n-i}(jx)P_i^{(j-1)}(s_1(0), s_2(0), \ldots)\frac{t^n}{n!}.$$

Therefore, the first identity of the theorem can be established by identifying the coefficients. In an analogous way, by (3.2), we can obtain the second identity. □

Similar to the case of Theorem 3.1, the identities given in Theorem 3.4 will reduce to simpler ones for associated sequences and cross sequences.

Corollary 3.5 (*[17, Theorem 2]*). *If $p_n(x)$ is an associated sequence, then*

$$B_{n,k}(p_1(x), p_2(x), p_3(x), \ldots) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} p_n(jx).$$

Corollary 3.6. *If $p_n^{(\lambda)}(x)$ is a cross sequence and satisfies $p_0^{(\lambda)}(x) = 1$, then*

$$B_{n,k}(p_1^{(\lambda)}(x), p_2^{(\lambda)}(x), p_3^{(\lambda)}(x), \ldots) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} p_n^{(j\lambda)}(jx).$$

Corollary 3.5 is another main result of [17]. Besides the specializations for associated sequences and cross sequences, some other corollaries of Theorems 3.1 and 3.4 may also be interesting. For example, we can obtain from (3.3) and (3.6) the following results.

Corollary 3.7. *If $s_n(x)$ is a Sheffer sequence and satisfies $s_0(x) = 1$, then*

$$B_{n,2}(s_0(x), 2s_1(x), 3s_2(x), \ldots) = \binom{n}{2}\sum_{j=0}^{n-2}\binom{n-2}{j} s_{n-2-j}(2x)s_j(0), \tag{3.8}$$

$$B_{n,2}(s_1(x), s_2(x), s_3(x), \ldots) = \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j} s_{n-j}(2x)s_j(0) - s_n(x) + \frac{1}{2}\delta_{0,n} \tag{3.9}$$

$$= \frac{1}{(n+1)(n+2)}B_{n+2,2}(s_0(x), 2s_1(x), 3s_2(x), \ldots) - s_n(x) + \frac{1}{2}\delta_{0,n}. \tag{3.10}$$

Proof. To establish (3.8), it suffices to note that for $j \geq 1$,

$$P_j^{(1)}(s_1(0), s_2(0), \ldots) = \sum_{l=0}^{j} (1)_l B_{j,l}(s_1(0), s_2(0), \ldots) = s_j(0); \tag{3.11}$$

while to verify (3.9), we should do more computation. Actually, when $k = 2$, the right side of (3.6) will reduce to

$$\frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} s_{n-j}(0) P_j^{(-1)}(s_1(0), s_2(0), \ldots) - \sum_{j=0}^{n} \binom{n}{j} s_{n-j}(x) P_j^{(0)}(s_1(0), s_2(0), \ldots)$$
$$+ \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} s_{n-j}(2x) P_j^{(1)}(s_1(0), s_2(0), \ldots).$$

The first term equals $\delta_{0,n}/2$ because

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} s_{n-j}(0) P_j^{(-1)}(s_1(0), s_2(0), \ldots) \frac{t^n}{n!}$$
$$= \sum_{j=0}^{\infty} P_j^{(-1)}(s_1(0), s_2(0), \ldots) \frac{t^j}{j!} \cdot \sum_{i=0}^{\infty} s_i(0) \frac{t^i}{i!} = \left(1 + \sum_{n=1}^{\infty} s_n(0) \frac{t^n}{n!}\right)^{-1} \cdot \sum_{i=0}^{\infty} s_i(0) \frac{t^i}{i!} = 1.$$

The second term equals $-s_n(x)$ because $P_j^{(0)}(s_1(0), s_2(0), \ldots) = \delta_{0,j}$. Thus, combining with (3.11), we obtain (3.9). Finally, by an evident substitution, we obtain (3.10). □

Corollary 3.8. *If $s_n(x)$ is a Sheffer sequence and satisfies $s_0(x) = 1$, then*

$$B_{n,3}(s_0(x), 2s_1(x), 3s_2(x), \ldots) = \binom{n}{3} \sum_{j=0}^{n-3} \binom{n-3}{j} s_{n-3-j}(3x) \sum_{i=0}^{j} \binom{j}{i} s_{j-i}(0) s_i(0),$$

$$B_{n,3}(s_1(x), s_2(x), s_3(x), \ldots) = -\frac{1}{6} \delta_{0,n} + \frac{1}{2} s_n(x)$$
$$-\frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} s_{n-j}(2x) s_j(0) + \frac{1}{6} \sum_{j=0}^{n} \binom{n}{j} s_{n-j}(3x) \sum_{i=0}^{j} \binom{j}{i} s_{j-i}(0) s_i(0).$$

Proof. It is similar to the proof of Corollary 3.7 but to derive the final results, Eq. (3.9) has to be made use of. □

When $k = 2$ or $k = 3$, Eqs. (3.4) and (3.7) will lead us to the same results. Additionally, because $B_{n,n}(x_1, x_2, \ldots) = x_1^n$ and $B_{n,n-1}(x_1, x_2, \ldots) = \frac{1}{2} n(n-1) x_1^{n-2} x_2$, we can obtain from the above theorems and corollaries some other identities, which are left to the interested readers.

4. Applications

In this section, we will give some applications of the general identities obtained in Section 3. The sequences we chose are all Appell sequences, cross sequences or Sheffer sequences. For associated sequences, the reader is referred to [1,17] or can deal with them by appealing to Corollaries 3.2 and 3.5.

Example 4.1. The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ [10, Section 4.2.2] form a cross sequence, then Corollaries 3.3 and 3.6 give

$$B_{n,k}(B_0^{(\alpha)}(x), 2B_1^{(\alpha)}(x), 3B_2^{(\alpha)}(x), \ldots) = \binom{n}{k} B_{n-k}^{(k\alpha)}(kx),$$

$$B_{n,k}(B_1^{(\alpha)}(x), B_2^{(\alpha)}(x), B_3^{(\alpha)}(x), \ldots) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} B_n^{(j\alpha)}(jx).$$

The generalized Euler polynomials $E_n^{(\alpha)}(x)$ [10, Section 4.2.3] also form a cross sequence, then we have

$$B_{n,k}(E_0^{(\alpha)}(x), 2E_1^{(\alpha)}(x), 3E_2^{(\alpha)}(x), \ldots) = \binom{n}{k} E_{n-k}^{(k\alpha)}(kx),$$

$$B_{n,k}(E_1^{(\alpha)}(x), E_2^{(\alpha)}(x), E_3^{(\alpha)}(x), \ldots) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} E_n^{(j\alpha)}(jx).$$

Moreover, the Hermite polynomials $H_n^{(\nu)}(x)$ [10, Section 4.2.1] and the actuarial polynomials $a_n^{(\beta)}(x)$ [10, Section 4.3.4] are both cross sequences, so they satisfy similar identities to those presented above.

Example 4.2. The Laguerre polynomials $L_n^{(\alpha)}(x)$ [10, Section 4.3.1] are Sheffer for $(g(t) = (1-t)^{-\alpha-1}, f(t) = t/(t-1))$ and satisfy

$$L_n^{(\alpha+\beta+1)}(x+y) = \sum_{k=0}^{n}\binom{n}{k} L_{n-k}^{(\alpha)}(x)L_k^{(\beta)}(y).$$

Therefore, by setting $\Psi(x, t; \alpha) := \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n/n!$, we have

$$\Psi(x, t; \alpha)\Psi(y, t; \beta) = \Psi(x + y, t; \alpha + \beta + 1).$$

Thus,

$$\sum_{i=0}^{\infty} P_i^{(k-1)}(L_1^{(\alpha)}(0), L_2^{(\alpha)}(0), \ldots)\frac{t^i}{i!} = \left(\sum_{n=0}^{\infty} L_n^{(\alpha)}(0)\frac{t^n}{n!}\right)^{k-1} = \Psi(0, t; (k-1)\alpha + k - 2),$$

which implies that $P_i^{(k-1)}(L_1^{(\alpha)}(0), L_2^{(\alpha)}(0), \ldots) = L_i^{((k-1)\alpha+k-2)}(0)$. Combining this with Eqs. (3.3) and (3.6), we obtain

$$B_{n,k}(L_0^{(\alpha)}(x), 2L_1^{(\alpha)}(x), 3L_2^{(\alpha)}(x), \ldots) = \binom{n}{k} L_{n-k}^{(k\alpha+k-1)}(kx),$$

$$B_{n,k}(L_1^{(\alpha)}(x), L_2^{(\alpha)}(x), L_3^{(\alpha)}(x), \ldots) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} L_n^{(j\alpha+j-1)}(jx).$$

Since the associated sequence for $f(t)$ is $L_n^{(-1)}(x)$, then Eqs. (3.4) and (3.7) will give the same identities.

Example 4.3. The Poisson–Charlier polynomials $c_n(x; a)$ [10, Section 4.3.3] form the Sheffer sequence for $(g(t) = \mathrm{e}^{a(\mathrm{e}^t-1)}, f(t) = a(\mathrm{e}^t - 1))$ and $a^{-n}(x)_n$ is the corresponding associated sequence for $f(t)$. Because

$$c_n(x; a) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}a^{-k}(x)_k,$$

we have $c_n(0; a) = (-1)^n$. Moreover, by Definition 1.1, $B_{i,l}((-1)^1, (-1)^2, \ldots) = (-1)^i S(i, l)$, where $S(i, l)$ are the Stirling numbers of the second kind. Then

$$P_i^{(k-1)}(c_1(0; a), c_2(0; a), \ldots) = \sum_{l=0}^{i}(k-1)_l(-1)^i S(i, l) = (1-k)^i.$$

In this case, Theorems 3.1 and 3.4 yield

$$\begin{aligned} B_{n,k}(c_0(x; a), 2c_1(x; a), 3c_2(x; a), \ldots) &= \binom{n}{k}\sum_{i=0}^{n-k}\binom{n-k}{i}(1-k)^i c_{n-k-i}(kx; a) \\ &= \binom{n}{k}\sum_{i=0}^{n-k}\binom{n-k}{i}(-k)^i a^{-n+k+i}(kx)_{n-k-i} \end{aligned}$$

and

$$B_{n,k}(c_1(x;a),c_2(x;a),c_3(x;a),\ldots) = \sum_{j=0}^{k}(-1)^{k-j}\frac{1}{k!}\binom{k}{j}\sum_{i=0}^{n}\binom{n}{i}(1-j)^i c_{n-i}(jx;a)$$
$$= \sum_{j=0}^{k}(-1)^{k-j}\frac{1}{k!}\binom{k}{j}\sum_{i=0}^{n}\binom{n}{i}(-j)^i a^{-n+i}(jx)_{n-i},$$

respectively.

5. Further remarks

Let $s_n(x)$ be the Sheffer sequence for $(g(t), f(t))$ and define $g_0 := g(0)$. Because

$$\sum_{n=0}^{\infty} s_n(0)\frac{t^n}{n!} = s_0(0) + \sum_{n=1}^{\infty} s_n(0)\frac{t^n}{n!} = \frac{1}{g(\bar{f}(t))},$$

we have $s_0(x) = 1/g_0$. At the beginning of Section 3, it is supposed that $s_0(x) = 1$, which is equivalent to $g(0) = g_0 = 1$. This assumption can simplify the derivations and the final results. It can be found that most of the Sheffer sequences presented in [10] satisfy the condition $s_0(x) = 1$, so Theorems 3.1 and 3.4 will not lose their generalities.

More general identities, without the restriction $s_0(x) = 1$, can be similarly derived. In fact, according to [3, p. 141, Eq. (5f)], we have

$$\left(\frac{1}{g(\bar{f}(t))}\right)^k = \left(\frac{1}{g_0} + \sum_{n=1}^{\infty} s_n(0)\frac{t^n}{n!}\right)^k$$
$$= \sum_{i=0}^{\infty}\frac{t^i}{i!}\left\{\sum_{l=0}^{i}(k)_l\left(\frac{1}{g_0}\right)^{k-l} B_{i,l}(s_1(0), s_2(0),\ldots)\right\},$$

from which we can obtain two new explicit expressions for the power $(\Psi(x,t))^k$. Thus, the identities related to

$$B_{n,k}(s_0(x), 2s_1(x), 3s_2(x),\ldots) \quad \text{and} \quad B_{n,k}(s_1(x), s_2(x), s_3(x),\ldots)$$

can be established. However, we chose not to list these identities.

There is a different way to derive more general identities without the restriction $s_0(x) = 1$. Based on the generating function (1.2), we have

$$\left(\frac{1}{g(\bar{f}(t))}\right)^k = \left(\sum_{n=0}^{\infty} s_n(0)\frac{t^n}{n!}\right)^k = t^{-k}\left(\sum_{n=1}^{\infty} n s_{n-1}(0)\frac{t^n}{n!}\right)^k$$
$$= t^{-k}k!\sum_{i=k}^{\infty} B_{i,k}(s_0(0), 2s_1(0), 3s_2(0),\ldots)\frac{t^i}{i!}$$
$$= \sum_{i=0}^{\infty}\binom{i+k}{k}^{-1} B_{i+k,k}(s_0(0), 2s_1(0), 3s_2(0),\ldots)\frac{t^i}{i!}. \tag{5.1}$$

Then we can compute the power $(\Psi(x,t))^k$ and obtain the corresponding identities. These are given below.

Theorem 5.1. *If $s_n(x)$ is the Sheffer sequence for $(g(t), f(t))$, then*

$$B_{n,k}(s_0(x), 2s_1(x), 3s_2(x),\ldots) = \frac{n}{k}\sum_{i=0}^{n-k}\binom{n-1}{n-k-i} s_{n-k-i}(kx) B_{i+k-1,k-1}(s_0(0), 2s_1(0), 3s_2(0),\ldots)$$
$$= \sum_{i=0}^{n-k}\binom{n}{n-k-i} p_{n-k-i}(kx) B_{i+k,k}(s_0(0), 2s_1(0), 3s_2(0),\ldots),$$

where $p_n(x)$ is the associated sequence for $f(t)$.

Theorem 5.2. *If $s_n(x)$ is the Sheffer sequence for $(g(t), f(t))$, then*

$$\begin{aligned} &B_{n,k}(s_1(x), s_2(x), s_3(x), \ldots) \\ &\quad = \sum_{i=0}^{n}\sum_{j=0}^{k}(-s_0(x))^{k-j}\frac{1}{k!}\binom{k}{j}\frac{(n)_i}{(i+j-1)_i}s_{n-i}(jx)B_{i+j-1,j-1}(s_0(0), 2s_1(0), 3s_2(0), \ldots) \\ &\quad = \sum_{i=0}^{n}\sum_{j=0}^{k}(-s_0(x))^{k-j}\frac{1}{k!}\binom{k}{j}\frac{(n)_i}{(i+j)_i}p_{n-i}(jx)B_{i+j,j}(s_0(0), 2s_1(0), 3s_2(0), \ldots), \end{aligned}$$

where $p_n(x)$ is the associated sequence for $f(t)$.

The potential polynomial $P_i^{(r)}(h_1, h_2, \ldots, h_i)$ for r a nonnegative integer is expressed by the exponential partial Bell polynomial as

$$\binom{i+r}{r}P_i^{(r)}(h_1, h_2, \ldots, h_i) = B_{i+r,r}(1, 2h_1, 3h_2, \ldots)$$

(see [3, p. 156, Exercise 4]), and so for $s_0(x) = 1$, Theorems 5.1 and 5.2 reduce to Theorems 3.1 and 3.4, respectively.

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