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Some identities involving Fibonacci, Lucas polynomials and their applications

by
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Abstract

The main purpose of this paper is to study some sums of powers of Fibonacci polynomials and Lucas polynomials, and give several interesting identities. Finally, using these identities we shall prove a conjecture proposed by R. S. Melham in [4].

Key Words: Fibonacci polynomials, Lucas polynomials, combinatorial method, identity, R. S. Melham's conjectures.
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1 Introduction

For any variable quantity x, the Fibonacci polynomial $F_n(x)$ and the Lucas polynomial $L_n(x)$ are defined as $F_0(x) = 0$, $F_1(x) = 1$, and $F_{n+1}(x) = xF_n(x) + F_{n-1}(x)$ for all $n \geq 1$; $L_0(x) = 2$, $L_1(x) = x$, and $L_{n+1}(x) = xL_n(x) + L_{n-1}(x)$ for all $n \geq 1$. If $x = 1$, then $F_n(1) = F_n$ and $L_n(1) = L_n$, the famous Fibonacci sequence and Lucas sequence, respectively. It is clear that these two polynomial sequences are the second-order linear recurrence sequences. Letting $\alpha = \frac{x+\sqrt{x^2+4}}{2}$, $\beta = \frac{x-\sqrt{x^2+4}}{2}$, then from the properties of the second-order linear recurrence sequences, we have

$$F_n(x) = \frac{\alpha^n-\beta^n}{\alpha-\beta} \quad \text{and} \quad L_n(x) = \alpha^n + \beta^n.$$

Concerning $F_n(x)$ and $L_n(x)$, various authors studied them and obtained many interesting results. For example, E. Lucas's classical work [3] first studied the arithmetical properties of L_n, and obtained many important results. Y. Yuan and Z. Wenpeng [6] proved some identities involving $F_n(x)$.

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Recently, several authors studied the sums of powers of Fibonacci numbers and Lucas numbers, and obtained a series important identities, see [1], [2] and [5]. At the same time, R. S. Melham [4] also proposed the following two conjectures:

Conjecture 1. Let $m \geq 1$ be an integer. Then the sum

$$L_1 L_3 L_5 \cdots L_{2m+1} \sum_{k=1}^{n} F_{2k}^{2m+1}$$

can be expressed as $(F_{2n+1}-1)^2 R_{2m-1}(F_{2n+1})$, where $R_{2m-1}(x)$ is a polynomial of degree $2m-1$ with integer coefficients.

Conjecture 2. Let $m \geq 0$ be an integer. Then the sum

$$L_1 L_3 L_5 \cdots L_{2m+1} \sum_{k=1}^{n} L_{2k}^{2m+1}$$

can be expressed as $(L_{2n+1}-1) Q_{2m}(L_{2n+1})$, where $Q_{2m}(x)$ is a polynomial of degree $2m$ with integer coefficients.

The main purpose of this paper is to obtain some identities involving Fibonacci polynomials and Lucas polynomials. As applications, we use these identities to prove that the above Conjecture 2 is true. That is, we shall prove the following conclusions:

Theorem 1. For any positive integers h and n, we have the identities

(a). $$\sum_{m=0}^{h} F_{2m+1}^{2n}(x) = \frac{1}{(x^2+4)^n} \left\{ (h+1)\frac{(2n)!}{(n!)^2} + \sum_{k=1}^{n} \binom{2n}{n-k} \frac{F_{4k(h+1)}(x)}{F_{2k}(x)} \right\};$$

(b). $$\sum_{m=0}^{h} L_{2m+1}^{2n}(x) = (h+1)(-1)^n \frac{(2n)!}{(n!)^2} + \sum_{k=1}^{n} \binom{2n}{n-k} (-1)^{n-k} \frac{F_{4k(h+1)}(x)}{F_{2k}(x)};$$

(c). $$\sum_{m=0}^{h} F_{2m+1}^{2n+1}(x) = \frac{1}{(x^2+4)^n} \sum_{k=0}^{n} \binom{2n+1}{n-k} \frac{F_{2(2k+1)(h+1)}(x)}{L_{2k+1}(x)};$$

(d). $$\sum_{m=0}^{h} L_{2m+1}^{2n+1}(x) = \sum_{k=0}^{n} \binom{2n+1}{n-k} (-1)^{n-k} \frac{L_{2(2k+1)(h+1)}(x)}{L_{2k+1}(x)}.$$

Theorem 2. For any positive integers h and n, we have the identities

(A). $$\sum_{m=1}^{h} L_{2m}^{2n}(x) = h\frac{(2n)!}{(n!)^2} + \sum_{k=1}^{n} \binom{2n}{n-k} \frac{F_{2k(2h+1)}(x) - F_{2k}(x)}{F_{2k}(x)};$$

$$\text{(B)}.\ \sum_{m=1}^{h} F_{2m}^{2n}(x) = \frac{(-1)^n}{(x^2+4)^n}\left[h\frac{(2n)!}{(n!)^2} + \sum_{k=1}^{n}\binom{2n}{n-k}(-1)^k \frac{F_{2k(2h+1)}(x) - F_{2k}(x)}{F_{2k}(x)}\right];$$

$$\text{(C)}.\quad \sum_{m=1}^{h} L_{2m}^{2n+1}(x) = \sum_{k=0}^{n}\binom{2n+1}{n-k}\frac{L_{(2k+1)(2h+1)}(x) - L_{2k+1}(x)}{L_{2k+1}(x)};$$

$$\text{(D)}.\ \sum_{m=1}^{h} F_{2m}^{2n+1}(x) = \frac{1}{(x^2+4)^n}\sum_{k=0}^{n}\binom{2n+1}{n-k}(-1)^{n-k}\frac{F_{(2k+1)(2h+1)}(x) - F_{2k+1}(x)}{L_{2k+1}(x)}.$$

As several applications of Theorem 2, we can deduce the following:

Corollary 1. Let $h \geq 1$ and $n \geq 0$ be two integers. Then the sum

$$L_1(x)L_3(x)L_5(x)\cdots L_{2n+1}(x)\sum_{m=1}^{h} L_{2m}^{2n+1}(x)$$

can be expressed as $(L_{2h+1}(x) - x)\,Q_{2n}\left(x,\ L_{2h+1}(x)\right)$, where $Q_{2n}(x,y)$ is a polynomial in two variables x and y with integer coefficients and degree $2n$ of y.

Corollary 2. Let $h \geq 1$ and $n \geq 0$ be two integers. Then the sum

$$L_1(x)L_3(x)L_5(x)\cdots L_{2n+1}(x)\sum_{m=1}^{h} F_{2m}^{2n+1}(x)$$

can be expressed as $(F_{2h+1}(x) - 1)\,H_{2n}\left(x,\ F_{2h+1}(x)\right)$, where $H_{2n}(x,y)$ is a polynomial in two variables x and y with integer coefficients and degree $2n$ of y.

Taking $x = 1$ in Corollary 1 and Corollary 2, then we have the following conclusions for Fibonacci and Lucas numbers:

Corollary 3. Let $h \geq 1$ be a positive integer. Then the sum

$$L_1L_3L_5\cdots L_{2n+1}\sum_{k=1}^{h} F_{2k}^{2n+1}$$

can be expressed as $(F_{2h+1} - 1)\,H_{2n}\left(F_{2h+1}\right)$, where $H_{2n}(x)$ is a polynomial of degree $2n$ with integer coefficients.

Corollary 4. Let $h \geq 1$ be an integer. Then the sum

$$L_1L_3L_5\cdots L_{2n+1}\sum_{k=1}^{h} L_{2k}^{2n+1}$$

can be expressed as $(L_{2h+1}-1)\,Q_{2n}\,(L_{2h+1})$, where $Q_{2n}(x)$ is a polynomial of degree $2n$ with integer coefficients.

Taking $x = 2$ in Corollary 1, note that $L_n(2) = P_n$, the nth Pell number, $P_0 = 0$, $P_1 = 1$ and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$. Then we also have the following:

Corollary 5. Let $h \geq 1$ be an integer. Then the sum

$$P_1P_3P_5\cdots P_{2n+1}\sum_{k=1}^{h} P_{2k}^{2n+1}$$

can be expressed as $(P_{2h+1}-2)\,R_{2n}\,(P_{2h+1})$, where $R_{2n}(x)$ is a polynomial of degree $2n$ with integer coefficients.

It is clear that our Corollary 1 proves a generalization of Melham's Conjecture. Our Corollary 3 make some substantial progress for the Melham's Conjecture 1. Corollary 5 give some new identities for the Pell numbers.

2. Proof of the theorems

In this section, we shall give the proofs of our Theorems. First we prove Theorem 1. In fact, for any positive integer n and real number $x \neq 0$, by using the familiar binomial expansion

$$\left(x+\frac{1}{x}\right)^n = \sum_{k=0}^{n}\tbinom{n}{k}\,x^{n-2k}$$

we get

$$\left(x+\frac{1}{x}\right)^{2n} = \frac{(2n)!}{(n!)^2} + \sum_{k=1}^{n}\tbinom{2n}{n-k}\left(x^{2k}+\frac{1}{x^{2k}}\right), \tag{1.1}$$

$$\left(x-\frac{1}{x}\right)^{2n} = (-1)^n\frac{(2n)!}{(n!)^2} + \sum_{k=1}^{n}\tbinom{2n}{n-k}\,(-1)^{n-k}\left(x^{2k}+\frac{1}{x^{2k}}\right), \tag{1.2}$$

$$\left(x+\frac{1}{x}\right)^{2n+1} = \sum_{k=0}^{n}\tbinom{2n+1}{n-k}\left(x^{2k+1}+\frac{1}{x^{2k+1}}\right), \tag{1.3}$$

and

$$\left(x-\frac{1}{x}\right)^{2n+1} = \sum_{k=0}^{n}\tbinom{2n+1}{n-k}\,(-1)^{n-k}\left(x^{2k+1}-\frac{1}{x^{2k+1}}\right). \tag{1.4}$$

Now taking $x = \alpha^{2m+1}$ in (1.1), (1.2), (1.3), and (1.4), then $\frac{1}{x} = -\beta^{2m+1}$. From the definitions of $F_n(x)$ and $L_n(x)$, we may immediately deduce the identities

$$F_{2m+1}^{2n}(x) = \frac{1}{(x^2+4)^n}\left[\frac{(2n)!}{(n!)^2} + \sum_{k=1}^{n}\binom{2n}{n-k} L_{2k(2m+1)}(x)\right], \tag{1.5}$$

$$L_{2m+1}^{2n}(x) = (-1)^n\frac{(2n)!}{(n!)^2} + \sum_{k=1}^{n}\binom{2n}{n-k}(-1)^{n-k} L_{2k(2m+1)}(x), \tag{1.6}$$

$$F_{2m+1}^{2n+1}(x) = \frac{1}{(x^2+4)^n}\sum_{k=0}^{n}\binom{2n+1}{n-k} F_{(2m+1)(2k+1)}(x), \tag{1.7}$$

and

$$L_{2m+1}^{2n+1}(x) = \sum_{k=0}^{n}\binom{2n+1}{n-k}(-1)^{n-k} L_{(2m+1)(2k+1)}(x). \tag{1.8}$$

Now taking $x=\alpha^{2m}$ in (1.1),(1.2), (1.3), and (1.4), we deduce the identities

$$L_{2m}^{2n}(x) = \frac{(2n)!}{(n!)^2} + \sum_{k=1}^{n}\binom{2n}{n-k} L_{4km}(x), \tag{1.9}$$

$$F_{2m}^{2n}(x) = \frac{1}{(x^2+4)^n}\left[(-1)^n\frac{(2n)!}{(n!)^2} + \sum_{k=1}^{n}\binom{2n}{n-k}(-1)^{n-k} L_{4km}(x)\right], \tag{1.10}$$

$$L_{2m}^{2n+1}(x) = \sum_{k=0}^{n}\binom{2n+1}{n-k} L_{2m(2k+1)}(x), \tag{1.11}$$

and

$$F_{2m}^{2n+1}(x) = \frac{1}{(x^2+4)^n}\sum_{k=0}^{n}\binom{2n+1}{n-k}(-1)^{n-k} F_{2m(2k+1)}(x). \tag{1.12}$$

For any integer $h > 0$, we sum on m in (1.5),

$$\begin{aligned}
&\sum_{m=0}^{h} F_{2m+1}^{2n}(x) = \frac{1}{(x^2+4)^n}\left[(h+1)\frac{(2n)!}{(n!)^2} + \sum_{k=1}^{n}\binom{2n}{n-k}\sum_{m=0}^{h} L_{2k(2m+1)}(x)\right] \\
= \quad & \frac{h+1}{(x^2+4)^n}\left\{\frac{(2n)!}{(n!)^2} + \sum_{k=1}^{n}\frac{\binom{2n}{n-k}}{h+1}\left[\frac{\alpha^{2k}\left(\alpha^{4k(h+1)}-1\right)}{\alpha^{4k}-1} + \right.\right. \\
& \left.\left.\frac{\beta^{2k}\left(\beta^{4k(h+1)}-1\right)}{\beta^{4k}-1}\right]\right\} \\
= \quad & \frac{h+1}{(x^2+4)^n}\left\{\frac{(2n)!}{(n!)^2} + \right. \\
& \left.\sum_{k=1}^{n}\frac{\binom{2n}{n-k}}{h+1}\frac{\alpha^{4kh+2k}-\alpha^{4kh+6k}+\beta^{4kh+2k}-\beta^{4kh+6k}}{2-\alpha^{4k}-\beta^{4k}}\right\}. \qquad (1.13)
\end{aligned}$$

Note that the identities

$$\begin{aligned}
& \alpha^{4kh+2k}-\alpha^{4kh+6k}+\beta^{4kh+2k}-\beta^{4kh+6k} \\
= \quad & -\left(\alpha^{4kh+4k}-\beta^{4kh+4k}\right)\left(\alpha^{2k}-\beta^{2k}\right) = -(x^2+4)F_{4kh+4k}F_{2k}
\end{aligned}$$

and

$$2-\alpha^{4k}-\beta^{4k} = -\left(\alpha^{2k}-\beta^{2k}\right)^2 = -(x^2+4)F_{2k}^2,$$

from (1.13) we may immediately deduce the identity

$$\sum_{m=0}^{h} F_{2m+1}^{2n}(x) = \frac{1}{(x^2+4)^n}\left\{(h+1)\frac{(2n)!}{(n!)^2} + \sum_{k=1}^{n}\binom{2n}{n-k}\frac{F_{4k(h+1)}(x)}{F_{2k}(x)}\right\}.$$

This proves the identity (a) of Theorem 1.

Similarly, from formulae (1.6), (1.7) and (1.8) we can deduce the other three identities of Theorem 1.

Now we prove Theorem 2. From (1.9), we have

$$\begin{aligned}
&\sum_{m=1}^{h} L_{2m}^{2n}(x) = h\frac{(2n)!}{(n!)^2} + \sum_{k=1}^{n} \binom{2n}{n-k} \sum_{m=1}^{h} \left(\alpha^{4km} + \beta^{4km}\right) \\
= \quad & h\frac{(2n)!}{(n!)^2} + \sum_{k=1}^{n} \binom{2n}{n-k} \left(\frac{\alpha^{4k(h+1)} - \alpha^{4k}}{\alpha^{4k} - 1} + \frac{\beta^{4k(h+1)} - \beta^{4k}}{\beta^{4k} - 1} \right) \\
= \quad & h\frac{(2n)!}{(n!)^2} + \sum_{k=1}^{n} \binom{2n}{n-k} \frac{\alpha^{4kh} - \alpha^{4k(h+1)} - 2 + \alpha^{4k} + \beta^{4kh} - \beta^{4k(h+1)} + \beta^{4k}}{2 - \alpha^{4k} - \beta^{4k}} \\
= \quad & h\frac{(2n)!}{(n!)^2} + \sum_{k=1}^{n} \binom{2n}{n-k} \frac{\left(\alpha^{4kh+2k} - \beta^{4kh+2k}\right)\left(\alpha^{2k} - \beta^{2k}\right) - \left(\alpha^{2k} - \beta^{2k}\right)^2}{\left(\alpha^{2k} - \beta^{2k}\right)^2} \\
= \quad & h\frac{(2n)!}{(n!)^2} + \sum_{k=1}^{n} \binom{2n}{n-k} \frac{F_{2k(2h+1)}(x) - F_{2k}(x)}{F_{2k}(x)}.
\end{aligned}$$

This proves the identity (A) of Theorem 2.

Similarly, from formulae (1.10), (1.11) and (1.12) we can also deduce the other three identities of Theorem 2.

Now we use (C) of Theorem 2 to prove Corollary 1. It is clear that if $P(x) \in Z(x)$, then $a - b$ divides $P(a) - P(b)$. From this properties and note that the identity $L_{2k+1}\left(L_{2n+1}(x)\right) = L_{(2n+1)(2k+1)}(x)$ we can deduce

$$\left(L_{2h+1}(x) - x\right) \mid L_{2k+1}\left(L_{2h+1}(x)\right) - L_{2k+1}(x) =$$

$$= L_{(2h+1)(2k+1)}(x) - L_{2k+1}(x). \tag{1.14}$$

Combining (C) of Theorem 2, (1.14) and note that $\left(L_{2k+1}(x) - x,\ L_{2k+1}(x)\right) = 1$ we may immediately deduce the identity

$$\begin{aligned}
& L_1(x)L_3(x)L_5(x)\cdots L_{2n+1}(x) \sum_{m=1}^{h} L_{2m}^{2n+1}(x) \\
= \quad & L_1(x)L_3(x)L_5(x)\cdots L_{2n+1}(x) \left(\sum_{k=0}^{n} \binom{2n+1}{n-k} \frac{L_{(2h+1)(2k+1)}(x) - L_{2k+1}(x)}{L_{2k+1}(x)} \right) \\
= \quad & \left(L_{2h+1}(x) - x\right) Q_{2n}\left(x,\ L_{2h+1}(x)\right),
\end{aligned}$$

where $Q_{2n}(x, y)$ is a polynomial in two variables x and y with integer coefficients and degree $2n$ of y. This proves Corollary 1.

To prove Corollary 2, from (D) of Theorem 2 we know that we only to prove the polynomials $x^2 + 4$ and $\left(F_{2h+1}(x) - 1\right)$ satisfying $\left(F_{2h+1}(x) - 1, x^2 + 4\right) = 1$ and $\left(F_{2h+1}(x) - 1\right) \mid \left(F_{(2h+1)(2k+1)}(x) - F_{2k+1}(x)\right)$ for all integers $k \geq 0$.

First from the definition of $F_n(x)$ and binomial expansion we can easy to prove $(F_{2h+1}(x)-1, x^2+4) = 1$. Therefore, $\big(F_{2h+1}(x)-1, (x^2+4)^n\big) = 1$.

Next, we prove that the polynomial $(F_{2h+1}(x)-1)$ divide $(F_{(2h+1)(2k+1)}(x) - F_{2k+1}(x))$. In fact note the fact that

$$F_a(x) - F_b(x) = F_{(a-\epsilon b)/2}(x) L_{(a+\epsilon b)/2}(x)$$

valid for all $a \equiv b \pmod 2$ with $\epsilon \in \{1, -1\}$ given by $\epsilon = 1$ if $a \equiv b \pmod 4$ and $\epsilon = -1$ if $a - b \equiv 2 \pmod 4$. Take $a = 2h+1$, $b = 1$ so $a - b = 2h$ and $a_1 = (2k+1)a$, $b_1 = 2k+1$. Then $a_1 - b_1 = (2k+1)(a-b)$, so ϵ is the same for (a, b) as for (a_1, b_1) (namely it is 1 if h is even and -1 if h is odd). Thus,

$$F_{2h+1}(x) - 1 = F_{2h+1}(x) - F_1(x) = F_h(x)L_{h+1}(x) \text{ or } F_{h+1}(x)L_h(x)$$

according to whether h is even or odd, respectively, and also

$$F_{(2h+1)(2k+1)}(x) - F_{2k+1}(x) = F_{(2k+1)h}(x)L_{(2k+1)(h+1)}(x)$$

or

$$F_{(2k+1)(h+1)}(x)L_{(2k+1)h}(x)$$

again according to whether h is even or odd respectively. Now the claim follows from the fact that $F_u(x)|F_v(x)$ whenever $u|v$ and if additionally v/u is odd, then also $L_u(x)|L_v(x)$. This completes the proof of Corollary 2.

It seems that using our method we can not solve the Melham's Conjecture 1 completely. But we believe that it is true.

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