ON CERTAIN IDENTITIES INVOLVING FIBONACCI AND LUCAS NUMBERS

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In a recent article [3], Jennings established the following three theorems expressing $F_{(2q+1)n}$ as a polynomial in F_n as well as expressing F_{mn}/F_n as a polynomial in L_n, where F_n and L_n are, respectively, the n^{th} Fibonacci and Lucas numbers.

Theorem 1: $F_{(2q+1)n} = F_n \sum_{k=0}^{q} (-1)^{n(q+k)} \frac{2q+1}{q+k+1} 5^k \binom{q+k+1}{2k+1} F_n^{2k}, \quad n, q \geq 0.$

Theorem 2: $F_{(2q+1)n} = F_n \sum_{k=0}^{q} (-1)^{(n+1)(q+k)} \binom{q+k}{2k} L_n^{2k}, \quad n, q \geq 0.$

Theorem 3: $F_{2qn} = F_n \sum_{k=1}^{q} (-1)^{(n+1)(q+k)} \binom{q+k-1}{2k-1} L_n^{2k-1}, \quad n \geq 0, q \geq 1.$

In a later article [2], Filipponi derived Theorem 1 very simply by letting $X = \alpha^n$, $Y = -\beta^n$, and $m = (2q+1)$ in Waring's formula, given by

$$X^m + Y^m = \sum_{k=0}^{[m/2[} (-1)^k \frac{m}{m-k} \binom{m-k}{k} (XY)^k (X+Y)^{m-2k}, \quad m \geq 1. \tag{1}$$

By letting $m = 2q$ in the above formula, he also established the following theorem, which expresses L_{2qn} as a polynomial in F_n.

Theorem 4: $L_{2qn} = \sum_{k=0}^{q} (-1)^{n(q+k)} \frac{2q}{q+k} \binom{q+k}{2k} 5^k F_n^{2k}, \quad n, q \geq 0.$

In the same article, Filipponi derived another result by letting $X = \alpha^n$ and $Y = \beta^n$ in the identity given by (1). This result, which expresses L_{mn} in powers of L_n, is given by the following two theorems, wherein the notation of Jennings has been used:

Theorem 5: $L_{2qn} = \sum_{k=0}^{q} (-1)^{(n+1)(q+k)} \frac{2q}{q+k} \binom{q+k}{2k} L_n^{2k}, \quad n, q \geq 0.$

Theorem 6: $L_{(2q+1)n} = L_n \sum_{k=0}^{q} (-1)^{(n+1)(q+k)} \frac{2q+1}{q+k+1} \binom{q+k+1}{2k+1} L_n^{2k}, \quad n, q \geq 0.$

In this short article, we will first derive Theorems 2 and 3 of Jennings in a very simple manner by utilizing the following identity, which has been used by Carlitz in 1963 (see [1]) to obtain some results concerning certain Fibonacci arrays:

$$\frac{X^m - Y^m}{X-Y} = \sum_{k=0}^{[(m-1)/2]} (-1)^k \binom{m-k-1}{k} (XY)^k (X+Y)^{m-2k-1}, \quad m \geq 1. \tag{2}$$

Using this identity, we will establish two other theorems that express $L_{(2q+1)n} / L_n$ and F_{2qn} / L_n in powers of F_n.

Now, letting $X = \alpha^n$ and $Y = \beta^n$ in identity (2), we obtain $X + Y = \alpha^n + \beta^n = L_n$ and $XY = (\alpha\beta)^n = (-1)^n$. Thus, we have

$$\alpha^{mn} - \beta^{mn} = (\alpha^m - \beta^m) \sum_{k=0}^{[(m-1)/2]} (-1)^k \binom{m-k-1}{k} (-1)^{nk} L_n^{m-2k-1}, \quad n \geq 0, m \geq 1,$$

or

$$F_{mn} = F_n \sum_{k=0}^{[(m-1)/2]} (-1)^{(n+1)k} \binom{m-k-1}{k} L_n^{m-2k-1}, \quad n \geq 0, m \geq 1. \tag{3}$$

Setting $m = 2q+1$,

$$F_{(2q+1)n} = F_n \sum_{k=0}^{q} (-1)^{(n+1)k} \binom{2q-k}{k} L_n^{2q-2k}, \quad n, q \geq 0.$$

Changing k to $q-k$, we may rewrite the above as

$$F_{(2q+1)n} = F_n \sum_{k=0}^{q} (-1)^{(n+1)(q+k)} \binom{q+k}{2k} L_n^{2k}, \quad n, q \geq 0,$$

which proves Theorem 2. Similarly, by setting $m = 2q$ in (3), we establish Theorem 3.

We now state and prove the following two new theorems.

Theorem 7: $F_{2qn} = F_n L_n \sum_{k=0}^{q-1} (-1)^{n(q+k+1)} \binom{q+k}{2k+1} 5^k F_n^{2k}, \quad n, q \geq 0.$

Proof: Let $X = \alpha^n$, $Y = -\beta^n$, and $m = 2q$ in (2). Then we have $X + Y = \alpha^n - \beta^n = \sqrt{5}\, F_n$, $XY = -(\alpha\beta)^n = (-1)^{n+1}$, and

$$\alpha^{2qn} - \beta^{2qn} = (\alpha^n + \beta^n) \sum_{k=0}^{q-1} (-1)^k \binom{2q-k-1}{k} (-1)^{(n+1)k} 5^{(2q-2k-1)/2} F_n^{2q-2k-1}, \quad n, q \geq 0.$$

Thus

$$F_{2qn} = L_n \sum_{k=0}^{q-1} (-1)^{nk} \binom{2q-k-1}{k} 5^{q-k-1} F_n^{2q-2k-1}, \quad n, q \geq 0.$$

Changing k to $q-1-k$, we may rewrite the above as

$$F_{2qn} = F_n L_n \sum_{k=0}^{q-1} (-1)^{n(q+k+1)} \binom{q+k}{2k+1} 5^k F_n^{2k}, \quad n, q \geq 0,$$

and hence the theorem.

Theorem 8: $L_{(2q+1)n} = L_n \sum_{k=0}^{q} (-1)^{n(q+k)} \binom{q+k}{2k} 5^k F_n^{2k}, \quad n, q \geq 0.$

This theorem may be established by letting $X = \alpha^n$, $Y = -\beta^n$, and $m = (2q+1)$ in (2), and following the same steps used to prove Theorem 7.

Finally, it may be mentioned that similar results can be established for the Pell and Pell-Lucas numbers P_n and Q_n using the identities given in (1) and (2).

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