The Fibonacci Quarterly 1968 (vol.6,4): 245-249

CONTINUOUS EXTENSIONS OF FIBONACCI IDENTITIES

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1. INTRODUCTION

Some attention has been given to extending the domain of definition of Fibonacci and Lucas numbers from the integers to the real numbers (see, for example, [1]). We give here what seems to be the most natural continuous extension from the point of view of recurrence relations. We then show how several familiar identities have quite natural continuous analogues, providing some support for our contention that these extensions are "the" continuous real extensions of the Fibonacci and Lucas numbers.

2. CONTINUOUS EXTENSIONS

We wish to find real-valued functions $U(x)$ satisfying the difference equation

(1.1) $$U(x) - c_1U(x-1) - c_2U(x-2) = 0 ,$$

where c_1 and c_2 are real constants. Let a and b denote the roots of the characteristic polynomial

$$x^2 - c_1x - c_2$$

of (1), where we assume a and b are nonzero real numbers. The quadratic formula gives

$$a = \frac{c_1 + \sqrt{c_1^2 + 4c_2}}{2}, \qquad b = \frac{c_1 - \sqrt{c_1^2 + 4c_2}}{2} .$$

Then

$$a^2 - c_1a - c_2 = 0 ,$$

so, for any real x, multiplying this by a^{x-2} gives

(Received September, 1966)

$$a^x - c_1 a^{x-1} - c_2 a^{x-2} = 0 ,$$

Similarly,

$$b^x - c_1 b^{x-1} - c_2 b^{x-2} = 0 .$$

Hence

$$U(x) = k_1 a^x + k_2 b^x ,$$

where k_1 and k_2 are any real constants, satisfies (1.1). If $a > 0$ and $b > 0$, then $U(x)$ is a continuous real function. However, if $a > 0$ and $b < 0$, as in the Fibonacci case, then b^x assumes imaginary values, so $U(x)$ does not immediately give us the real-valued continuous extension we seek. But since c_1 and c_2 are real, we see

$$V(x) = \mathrm{Re}(U(x))$$

is a real function satisfying (1.1). This $V(x)$ will have the nice properties we are looking for.

Let us make these ideas explicit for the Fibonacci and Lucas case. Here then we let $c_1 = c_2 = 1$, so that

$$a = \tfrac{1}{2}(1 + \sqrt{5}) > 0, \qquad b = \tfrac{1}{2}(1 - \sqrt{5}) < 0 .$$

Letting

$$a^{-1} = \beta = -b > 0 ,$$

we see since $e^{\pi i} = -1$,

$$b^x = (-1)^x \beta^x = e^{\pi i x}\beta^x = \beta^x(\cos \pi x + i \sin \pi x) .$$

To find the continuous Fibonacci extension $F(x)$, we use the initial conditions $F(0) = 0$, $F(1) = 1$ to produce the system

$$0 = k_1 + k_2, \qquad 1 = k_1 a + k_2 b \ ,$$

which has the solution $k_1 = -k_2 = 1/\sqrt{5}$. Then

$$\text{(2.1)} \qquad F(x) = \operatorname{Re}[(a^x - b^x)/\sqrt{5}] = \frac{a^x - \beta^x \cos \pi x}{\sqrt{5}} \ .$$

Similar consideration for the Lucas extension $L(x)$ obeying (1.1) with the initial conditions $L(0) = 2$, $L(1) = 1$ give

$$\text{(2.2)} \qquad L(x) = a^x + \beta^x \cos \pi x \ .$$

Note that if n is an integer, it follows from the recurrence relation and the chosen initial conditions that

$$F(n) = F_n, \quad L(n) = L_n \ ,$$

where F_n and L_n denote the usual Fibonacci and Lucas numbers, respectively. Hence $F(x)$ and $L(x)$ are continuous (indeed, infinitely differentiable) real-valued extensions of the Fibonacci and Lucas numbers.

3. CONTINUOUS IDENTITIES

We give in this section the continuous analogues of some familiar Fibonacci and Lucas identities. It follows immediately from (2.1) and (2.2) that

$$\text{(3.1)} \qquad F(x + 1) + F(x - 1) = L(x) \ .$$

By multiplying out the left side, and using $\beta = a^{-1}$, one can easily verify that

$$\text{(3.2)} \qquad F(x + 1)F(x - 1) - F^2(x) = \cos \pi x \ .$$

This is a particularly neat generalization of the Fibonacci identity

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n \ .$$

Similarly, one can show

(3.3) $$L(x)^2 - L(x+1)L(x-1) = 5\cos \pi x,$$

which generalizes

$$L_n^2 - L_{n+1}L_{n-1} = 5(-1)^n .$$

Equations (2.1) and (2.2) can be solved for a^x to give

(3.4) $$a^x = \tfrac{1}{2}\{L(x) + \sqrt{5}F(x)\},$$

which leads to the deMoivre-type formula

(3.5) $$\left(\frac{L(x) + \sqrt{5}F(x)}{2}\right)^n = \frac{L(nx) + \sqrt{5}F(nx)}{2} .$$

Slightly less satisfying is the easily checked formula

(3.6) $$F(x)L(x) = \tfrac{1}{2}\{F(2x) + (a^{2x} - \beta^{2x})/\sqrt{5}\},$$

which reduces to $F_{2n} = F_nL_n$ for n integral. Similarly,

(3.7) $$F(x+1)^2 + F(x)^2 = \tfrac{1}{2}\{F(2x+1) + (a^{2x+1} + \beta^{2x+1})/\sqrt{5}\},$$

which generalizes

$$F_{n+1}^2 + F_n^2 = F_{2n+1},$$

and also

(3.8) $$F(x+1)^2 - F(x-1)^2 = [F(x+1) - F(x-1)][F(x+1) + F(x-1)] = F(x)L(x).$$

We have indicated here how one might continuously extend most Fibonacci and Lucas identities. The functions $F(x)$ and $L(x)$ can be differentiated and integrated using standard formulas, but the results are not particularly simple. Finally, we note that the above ideas may be carried out to extend general second-order recurring sequences to continuous functions, as indicated in Section 2. However, because of increased complexity, we do not state the more general results here.

REFERENCES

1. Eric Halsey, "The Fibonacci Number F_u where u is Not an Integer," Fibonacci Quarterly, 3 (1965), pp. 147-152.

2. F. D. Parker, "A Fibonacci Function," Fibonacci Quarterly, Feb. 1968, pp. 1-2.

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[Continued from p. 244.]

It is well known that the number of protons Z in the lightest stable nuclei is, as a rule, equal to the number of neutrons N. When the atomic number Z increases, the proton-neutron ratio in the nucleus Z/N decreases gradually from 1.0 to about 0.63.

The ratio of Z/N in the heaviest practical stable nucleus ($_{92}U^{238}$) — found in nature — reaches already the value 0.620, but with the still heavier hypothetical element 114 this ratio (114/184 = 0.6195) would yield (if this element could eventually be created) one of the best approximations to the "g. r."-value found in the world of atoms.

It is interesting to note that the ratio of protons of fission-fragments in above nuclear reaction (70/114 = 0.6140) also lies in the range of the "g. r."-value and differs from this value by 0.0040 only.

REFERENCES

1. "Onward to Element 126," Scientific American, Vol. 217 (October 1967), p. 50.
2. G. T. Seaborg, "Zukunftsaspekte der Transuranforschung"* Physikalische Blätter, Heft 8 (August 1967), pp. 354-361.

*This is an abstract from the statement that the Nobel-Prize-winning chemist G. T. Seaborg made on the occasion of receiving the Willard-Gibbs-medal on 20 May 1966 in Chicago.

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