

Identities involving Narayana polynomials and Catalan numbers

Toufik Mansour^b, Yidong Sun^{a,*}

^a Department of Mathematics, Dalian Maritime University, 116026 Dalian, PR China

^b Department of Mathematics, University of Haifa, 31905 Haifa, Israel

ARTICLE INFO

Article history:

Received 7 June 2008

Received in revised form 30 November 2008

Accepted 4 December 2008

Available online 21 January 2009

Keywords:

Narayana polynomials

Legendre polynomials

Inverse relations

Catalan numbers

ABSTRACT

We first establish the result that the Narayana polynomials can be represented as the integrals of the Legendre polynomials. Then we represent the Catalan numbers in terms of the Narayana polynomials by three different identities. We give three different proofs for these identities, namely, two algebraic proofs and one combinatorial proof. Some applications are also given which lead to many known and new identities.

Crown Copyright © 2008 Published by Elsevier B.V. All rights reserved.

1. Introduction

The *Catalan numbers* [24, Sequence A000108] are defined by $C_n = \frac{1}{n+1} \binom{2n}{n}$, for all $n \geq 0$. The *Narayana polynomials* $N_n(q)$ and the associated *Narayana polynomials* $\mathcal{N}_n(q)$ [2] are defined by

$$N_n(q) = \sum_{k=1}^n \frac{1}{n} \binom{n}{k-1} \binom{n}{k} q^k \quad \text{and} \quad \mathcal{N}_n(q) = q^n N_n(q^{-1}) = N_n(q)/q,$$

for $n \geq 1$, with the initial values $N_0(q) = \mathcal{N}_0(q) = 1$. The coefficients $N_{n,k} = \frac{1}{n} \binom{n}{k-1} \binom{n}{k}$ with $N_{0,0} = 1$ are called the *Narayana numbers*, and it is well known that the sequence $\{N_n(1)\}_{n \geq 0}$ is the sequence of the Catalan numbers, while the sequence $\{N_n(2)\}_{n \geq 0}$ is the sequence of the large Schröder numbers [24, Sequence A006318]. The Narayana polynomials and associated Narayana polynomials have been considered by several authors, see [2,8,20,22,21,27]. For instance, Bonin, Shapiro and Simion [2] showed that the polynomial $N_n(1+q)$ is a q -analog of the n th large Schröder numbers. Coker [8] provided several different expressions:

$$\sum_{k=1}^n \frac{1}{n} \binom{n}{k-1} \binom{n}{k} q^{k-1} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2k} C_k q^k (1+q)^{n-2k-1}, \quad (1.1)$$

$$\sum_{k=1}^n \frac{1}{n} \binom{n}{k-1} \binom{n}{k} q^{2(k-1)} (1+q)^{2(n-k)} = \sum_{k=0}^{n-1} \binom{n-1}{k} C_{k+1} q^k (1+q)^k. \quad (1.2)$$

Identity (1.1) was studied by Simion and Ullman [22] and proved combinatorially by Chen, Deng and Du [4]. Later, Chen, Yan and Yang [7] proved (1.1) and (1.2) combinatorially in terms of weighted 2-Motzkin paths. Recently, Mansour and Sun [14] presented another expression for Narayana polynomials

* Corresponding author.

E-mail addresses: toufik@math.haifa.ac.il (T. Mansour), sydmath@yahoo.com.cn (Y. Sun).

$$N_n(q) = \sum_{k=0}^n \frac{1}{n} \binom{n}{k-1} \binom{n}{k} q^k = \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{k} \binom{2n-k}{n} (q-1)^k, \quad (1.3)$$

whose generalized version is appeared in [15, Example 2.13]. Chen and Pang [6] independently deduced (1.3) combinatorially. Pak [17, Example 8.5 and Exercise 8.15] also obtained (1.3) implicitly in his book.

The main result of this paper can be formulated as follows.

Theorem 1.1. *For any integer $n \geq 0$, there hold*

$$C_n = \sum_{k=0}^n \frac{2k+1}{2n+1} \binom{2n+1}{n-k} N_k(q) (1-q)^{n-k}, \quad (1.4)$$

$$q^{\frac{n}{2}+1} C_{\frac{n}{2}} = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} N_{k+1}(q) (1+q)^{n-k}, \quad (1.5)$$

$$q^{n+2} C_{n+1} = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} N_{k+1}(q^2) (1-q)^{2(n-k)}, \quad (1.6)$$

where $C_{\frac{n}{2}}$ is zero if n is odd.

In this paper, we first establish the result that the Narayana polynomials can be represented as the integrals of the Legendre polynomials [9] in Section 2. Then we give three different proofs for Theorem 1.1, see Sections 3–5, including two algebraic proofs and one combinatorial proof. Some applications are also given which lead to many known and new identities.

2. Narayana polynomials

Recall that the Legendre polynomials $P_n(x)$ [9,19], which are most familiar in the form

$$P_n(x) = 2^{-n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} \binom{2n-2k}{n-k} x^{n-2k},$$

have an alternate expression, namely,

$$P_n(x) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \left(\frac{x-1}{2}\right)^k = \sum_{k=0}^n \binom{n+k}{n-k} \binom{2k}{k} \left(\frac{x-1}{2}\right)^k,$$

so that

$$P_n(2x-1) = \sum_{k=0}^n \binom{n+k}{n-k} \binom{2k}{k} (x-1)^k. \quad (2.1)$$

Note that an equivalent form of (1.3) is

$$\sum_{k=0}^n \frac{1}{n} \binom{n}{k-1} \binom{n}{k} q^k = \sum_{k=0}^n \binom{n+k}{n-k} \frac{1}{k+1} \binom{2k}{k} (q-1)^{n-k}. \quad (2.2)$$

Then (2.1) and (2.2) generate the following result.

Theorem 2.1. *For any integer $n \geq 1$, there holds*

$$N_n(q) = (q-1)^{n+1} \int_0^{\frac{q}{q-1}} P_n(2x-1) dx.$$

Proof. It is clear that $N_n(0) = 0$ for all $n \geq 1$. Then we have

$$\begin{aligned} (q-1)^{n+1} \int_0^{\frac{q}{q-1}} P_n(2x-1) dx &= (q-1)^{n+1} \sum_{k=0}^n \binom{n+k}{n-k} \binom{2k}{k} \int_0^{\frac{q}{q-1}} (x-1)^k dx \\ &= \sum_{k=0}^n \binom{n+k}{n-k} \frac{1}{k+1} \binom{2k}{k} (q-1)^{n-k} \\ &\quad + (q-1)^{n+1} \sum_{k=0}^n \binom{n+k}{n-k} \frac{1}{k+1} \binom{2k}{k} (-1)^k \\ &= N_n(q) - N_n(0)(1-q)^{n+1} = N_n(q), \end{aligned}$$

which completes the proof. \square

Theorem 2.1 signifies that many classical sequences such as Catalan numbers and Schröder numbers can be represented as the integrals of Legendre polynomials.

Example 2.2. (i) Let $q = -1$. Using the parity identity [8,28]

$$N_n(-1) = \begin{cases} 0 & \text{if } n = 2r, \\ (-1)^{r+1}C_r & \text{if } n = 2r + 1, \end{cases} \tag{2.3}$$

we have for $n \geq 0$,

$$C_n = 2^{2n+1} \int_0^1 P_{2n+1}(x - 1)dx.$$

(ii) Let $q = 2$, we have that the large Schröder numbers $N_n(2)$ satisfy

$$N_n(2) = \int_0^2 P_n(2x - 1)dx.$$

Remark 2.3. Simons [23] established the following identity

$$\sum_{k=0}^n \frac{(-1)^{n-k}(n+k)!(1+x)^k}{(n-k)!k!^2} = \sum_{k=0}^n \frac{(n+k)!x^k}{(n-k)!k!^2},$$

or equivalently

$$\sum_{k=0}^n (-1)^{n-k} \binom{n+k}{n-k} \binom{2k}{k} (1+x)^k = \sum_{k=0}^n \binom{n+k}{n-k} \binom{2k}{k} x^k, \tag{2.4}$$

which was proved by Chapman [3], Prodinger [18], Wang and Sun [29]. It has been pointed out by Hirschhorn [13] that (2.4) is a special case of the Pfaff identity [12]. Recently, Munarini [16] gave a generalization of (2.4).

Obviously, (2.1) and (2.4) generate that

$$(-1)^n P_n(-2x - 1) = P_n(2x + 1),$$

which can be easily derived by the generating function of Legendre polynomials [9],

$$\sum_{n \geq 0} P_n(x)t^n = \frac{1}{\sqrt{1 - 2xt + t^2}}.$$

3. Proof of Theorem 1.1 and inverse relations

In this section, using three well-known inverses relations, we present our first proof for Theorem 1.1. The Legendre inverse relation reads [19]

$$A_n = \sum_{k=0}^n \binom{n+k}{n-k} B_k \iff B_n = \sum_{k=0}^n (-1)^{n-k} \frac{2k+1}{2n+1} \binom{2n+1}{n-k} A_k, \tag{3.1}$$

and the left-inversion formula [10] reads

$$A_n = \sum_{k=0}^{\lfloor \frac{n}{s} \rfloor} \binom{n+p}{sk+p} B_k \implies B_n = \sum_{k=0}^{sn} (-1)^{sn-k} \binom{sn+p}{k+p} A_k, \tag{3.2}$$

which, in the case $s = 1, p = 0$, implies the binomial inverse relation

$$A_n = \sum_{k=0}^n \binom{n}{k} B_k \iff B_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} A_k. \tag{3.3}$$

Now we are ready to present the proof of Theorem 1.1.

3.1. Proof of (1.4)

Rewriting (2.2), we have

$$\frac{N_n(q)}{(q-1)^n} = \sum_{k=0}^n \binom{n+k}{n-k} C_k(q-1)^{-k},$$

and using (3.1), we obtain an expressions for the Catalan numbers,

$$C_n = \sum_{k=0}^n (-1)^{n-k} \frac{2k+1}{2n+1} \binom{2n+1}{n-k} N_k(q)(q-1)^{n-k},$$

which completes the proof of (1.4).

3.2. Proof of (1.5)

Rewriting (1.1) in another form after replacing n by $n + 1$,

$$\frac{N_{n+1}(q)}{(1+q)^n} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k q^{k+1} (1+q)^{-2k},$$

and using (3.2) in the case $s = 2, p = 0$, we deduce another expression for Catalan numbers,

$$q^{n+1} C_n = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} N_{k+1}(q)(1+q)^{2n-k}, \tag{3.4}$$

which motivates us to consider the following related summation

$$f_n(q) = \sum_{i=1}^{2n+2} f_i q^i = \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} N_{k+1}(q)(1+q)^{2n+1-k}. \tag{3.5}$$

Lemma 3.1. For all $n \geq 0, f_n(q) = 0$.

Proof. Comparing the coefficients of two sides in (3.5), we have

$$f_m = \sum_{j=0}^m \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} N_{k+1,j} \binom{2n+1-k}{m-j}.$$

Noting that $N_{k+1,j} \binom{2n+1-k}{m-j}$ is a polynomial on k with degree $m + j - 2$, which does not exceed $2n$ when $1 \leq m \leq n + 1$. According to the well-known difference formula

$$\sum_{k=0}^n (-1)^k \binom{n}{k} (x-k)^r = \begin{cases} 0 & \text{if } 0 \leq r < n, \\ n! & \text{if } r = n, \end{cases}$$

we can derive that each inner sum is zero in f_m for $1 \leq m \leq n + 1$. Note that $f_n(q) = q^{2n+3} f_n(q^{-1})$ by $q^{n+1} N_n(q^{-1}) = N_n(q)$, which implies that $f_m = 0$ for $n + 2 \leq m \leq 2n + 2$. Hence, $f_n(q) = 0$ for $n \geq 0$, as claimed. \square

By combining (3.4) and (3.5), using Lemma 3.1, we obtain (1.5).

3.3. Proof of (1.6)

Rewriting (1.2) in another form after replacing n by $n + 1$,

$$N_{n+1} \left(\frac{q^2}{(1+q)^2} \right) (1+q)^{2n+2} = \sum_{k=0}^n \binom{n}{k} C_{k+1} q^{k+2} (1+q)^k,$$

and using (3.3), we deduce that

$$q^{n+2} (1+q)^n C_{n+1} = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} N_{k+1} \left(\frac{q^2}{(1+q)^2} \right) (1+q)^{2k+2}.$$

Replacing q by $\frac{q}{1-q}$, after simplification, we get (1.6).

3.4. Applications

Theorem 1.1 can produce numerical known or new identities. For instance,

- The case $q = -1$ in (1.4) together with (2.3), lead to a new identity

$$(2^n - 1)C_n = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^r \frac{4r+3}{2n+1} \binom{2n+1}{n-2r-1} 2^{n-2r-1} C_r.$$

- Taking the coefficient of q^n in both sides of (1.4), we get another parity identity

$$\sum_{k=0}^n (-1)^k \frac{2k+1}{2n+1} \binom{2n+1}{n-k} = 0, \quad (n \geq 1), \tag{3.6}$$

which has been proved by Chen, Li and Shapiro [5].

- The case $q = 1$ in (1.5) leads to a new identity

$$C_n = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} C_{k+1} 2^{2n-k}.$$

- The case $q = -1$ in (1.6) leads to a known identity [1,8,19]

$$C_{n+1} = \sum_{k=0}^n (-1)^k \binom{n}{k} C_{k+1} 4^{n-k}.$$

and the case $q = \sqrt{-1}$ in (1.6) leads to the Touchard identity [1,8]

$$C_{n+1} = \sum_{k=0}^n \binom{n}{2k} C_k 2^{n-2k}.$$

- Let $q = \sqrt{2}$ in (1.6), by the relation $(1 - \sqrt{2})^n = (P_n + P_{n-1}) - P_n \sqrt{2}$, where P_n is the n th Pell number (defined by the recurrence relation $P_{n+1} = 2P_n + P_{n-1}$ with $P_{-1} = 1, P_0 = 0$), we have new identities involving Catalan numbers, large Schröder numbers, and Pell numbers

$$2^{n+1} C_{2n+1} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} N_{k+1}(2) P_{4n-2k-1},$$

$$2^{n+1} C_{2n+2} = \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} N_{k+1}(2) P_{4n-2k+2}.$$

- Let $q = \sqrt{5}$ in (1.6), by the relation $(\frac{1-\sqrt{5}}{2})^{n+1} = \frac{L_n - F_n \sqrt{5}}{2}$, where L_n and F_n are respectively the n th Lucas number and the n th Fibonacci number (defined by the same recurrence relation $G_{n+1} = G_n + G_{n-1}$ with $G_{-1} = 2, G_0 = 1$ for L_n and $G_{-1} = 0, G_0 = 1$ for F_n), we have new identities involving Catalan numbers, Lucas numbers, and Fibonacci numbers

$$5^{n+1} C_{2n+1} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} N_{k+1}(5) L_{4n-2k-1} 2^{4n-2k-1},$$

$$5^{n+1} C_{2n+2} = \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} N_{k+1}(5) F_{4n-2k+1} 2^{4n-2k+1}.$$

4. Proof of Theorem 1.1 and generating functions

In this section we present our second proof for Theorem 1.1 which is based on generating function techniques.

Recall that $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n}$, which satisfies the relation $C(x) = 1 + xC(x)^2 = \frac{1}{1-xC(x)}$. By the Lagrange inversion formula [30], one can deduce that

$$[x^{n-k}]C(x)^{2k+1} = \frac{2k+1}{2n+1} \binom{2n+1}{n-k}. \tag{4.1}$$

Define $\Omega(q; x) = \sum_{n \geq 0} N_n(q)x^n$, then $\Omega(q; x)$ has the explicit expression [11]

$$\Omega(q; x) = \frac{1 + x - qx - \sqrt{1 - 2x + x^2 - 2qx - 2qx^2 + q^2x^2}}{2x},$$

which can be rewritten as

$$\Omega(q; x) = \frac{1}{1+x-qx} C\left(\frac{x}{(1+x-qx)^2}\right) = 1 + \frac{qx}{1-x-qx} C\left(\frac{qx^2}{(1-x-qx)^2}\right).$$

Identity (4.1) and the identity involving $\Omega(q; x)$ imply at once that

$$\begin{aligned} \sum_{n \geq 0} \left[\sum_{k=0}^n \frac{2k+1}{2n+1} \binom{2n+1}{n-k} N_k(q)(1-q)^{n-k} \right] x^n &= \sum_{k \geq 0} \frac{N_k(q)}{(1-q)^k} \left[\sum_{n \geq k} \frac{2k+1}{2n+1} \binom{2n+1}{n-k} (1-q)^n x^n \right] \\ &= \sum_{k \geq 0} \frac{N_k(q)}{(1-q)^k} (1-q)^k x^k C((1-q)x)^{2k+1} \\ &= C((1-q)x) \sum_{k \geq 0} N_k(q) (xC((1-q)x)^2)^k \\ &= C((1-q)x) \Omega(q; xC((1-q)x)^2) \\ &= \frac{C((1-q)x)}{1 + (1-q)x C((1-q)x)^2} C\left(\frac{xC((1-q)x)^2}{(1 + (1-q)x C((1-q)x)^2)^2}\right) \\ &= C(x), \end{aligned}$$

which is equivalent to (1.4).

For the second identity (1.5), consider the second identity for $\Omega(q; x)$ rewritten as

$$\sum_{k \geq 0} N_{k+1}(q)x^k = \frac{\Omega(q; x) - 1}{x} = \frac{q}{1 - (1+q)x} C\left(\frac{qx^2}{(1 - (1+q)x)^2}\right).$$

Then

$$\begin{aligned} \sum_{n \geq 0} \left[\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} N_{k+1}(q)(1+q)^{n-k} \right] x^n &= \sum_{k \geq 0} \frac{N_{k+1}(q)}{(1+q)^k} \left[\sum_{n \geq k} (-1)^{n-k} \binom{n}{k} (1+q)^n x^n \right] \\ &= \sum_{k \geq 0} \frac{N_{k+1}(q)}{(1+q)^k} \frac{(1+q)^k x^k}{(1+x(1+q))^{k+1}} \\ &= \frac{1}{1+x(1+q)} \frac{\Omega(q; \frac{x}{1+x(1+q)}) - 1}{\frac{x}{1+x(1+q)}} = qC(qx^2), \end{aligned}$$

which is equivalent to (1.5).

For the last identity (1.6), we have

$$\begin{aligned} \sum_{n \geq 0} \left[\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} N_{k+1}(q^2)(1-q)^{2(n-k)} \right] x^n &= \sum_{k \geq 0} \frac{N_{k+1}(q^2)}{(1-q)^{2k}} \left[\sum_{n \geq k} (-1)^{n-k} \binom{n}{k} (1-q)^{2n} x^n \right] \\ &= \sum_{k \geq 0} \frac{N_{k+1}(q^2)}{(1-q)^{2k}} \frac{(1-q)^{2k} x^k}{(1+x(1-q)^2)^{k+1}} \\ &= \frac{1}{1+x(1-q)^2} \frac{\Omega(q^2; \frac{x}{1+x(1-q)^2}) - 1}{\frac{x}{1+x(1-q)^2}} \\ &= \frac{q^2}{1-2qx} C\left(\frac{q^2 x^2}{(1-2qx)^2}\right) = \sum_{k \geq 0} C_k \frac{q^{2k+2} x^{2k}}{(1-2qx)^{2k+1}} \\ &= \sum_{n \geq 0} x^n q^{n+2} \sum_{k=0}^n \binom{n}{2k} C_k 2^{n-2k} = \sum_{n \geq 0} x^n q^{n+2} C_{n+1}, \end{aligned}$$

which is equivalent to (1.6). Note that the last equation follows by the well-known Touchard identity, which can also be derived by setting $q = 1$ in (1.1) after replacing n by $n + 1$.

5. Combinatorial Proof of Theorem 1.1

In order to give the combinatorial proof of (1.4), we need the following definitions. A *Dyck path* of length $2n$ is a lattice path from $(0, 0)$ to $(2n, 0)$ in the first quadrant of xy -plane, consisting of up-steps $u = (1, 1)$ and down-steps $d = (1, -1)$, which never passes below the x -axis. We will refer to n as the semilength of the path. It is well known that the set of Dyck paths of semilength n is counted by the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$. A *peak* in a Dyck path is an occurrence of ud . By the *height* of a step we mean the ordinate of its endpoint. By a *return step* we mean a down-step ending at height zero. Dyck paths that have exactly one return step are said to be *primitive*. If D_1 and D_2 are Dyck paths, we define $D_1 D_2$ to be the *concatenation* of D_1 and D_2 .

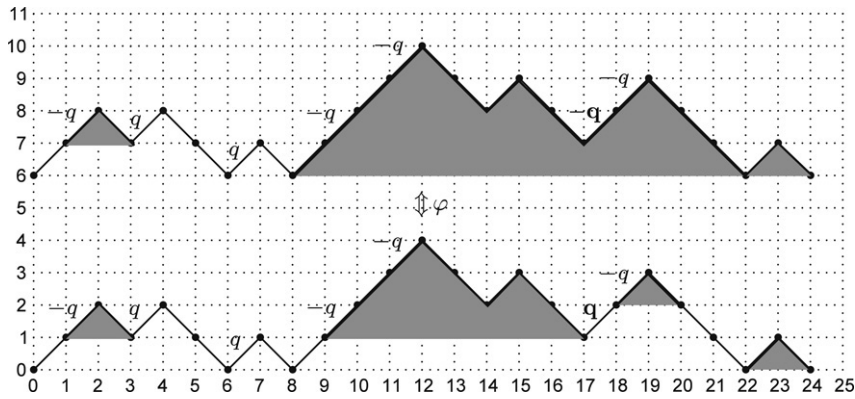


Fig. 1. The involution φ on $\mathcal{D}_{12} \setminus \tilde{\mathcal{D}}_{12}$, where the weight 1 of up-steps is unlabeled. The first Dyck path D constructed from $D^{(0)} = u^2d^2ud$, $D^{(1)} = \emptyset$, $D^{(2)} = ud$, $D^{(3)} = D^{(4)} = D^{(5)} = D^{(6)} = \emptyset$, $D^{(7)} = u^4d^2ud^2u^2d^3ud$, and $\varphi(D)$ constructed from $D^{(0)} = u^2d^2udu^2d^2$, $D^{(1)} = \emptyset$, $D^{(2)} = ud$, $D^{(3)} = D^{(4)} = D^{(5)} = D^{(6)} = D^{(7)} = \emptyset$, $D^{(8)} = u^3d^2ud^2$, $D^{(9)} = ud$, $D^{(10)} = \emptyset$, $D^{(11)} = ud$.

A *weighted Dyck path* is a Dyck path D for which every up-step is endowed with a weight. The weight of a Dyck path D is the product of the weights of its up-steps, the weight of a set S of Dyck paths means the sum of the weights of D in S .

Let $\mathcal{D}_{n,k}$ denote the set of weighted Dyck paths of length $2n$ constructed from an ordered sequence of $2k + 2$ Dyck paths $D^{(0)}, D^{(1)}, \dots, D^{(2k+1)}$ by the following steps:

- $D^{(0)}$ is a Dyck path of length $2k$ for $0 \leq k \leq n$ with an up-step in each peak weighted by q and other up-steps weighted by 1. It should be noticed that $D^{(0)}$ could be an empty path;
- There are totally $n - k$ up-steps in the rest $2k + 1$ Dyck paths, and all up-steps of each $D^{(i)}$ are weighted by 1 or $-q$ for $1 \leq i \leq 2k + 1$, i.e., such up-steps can be regarded to be weighted by $(1 - q)$;
- Each $D^{(i)}$ is inserted into the i th endpoint of $D^{(0)}$ including the beginning point for $1 \leq i \leq 2k + 1$.

Let $\mathcal{D}_n = \bigcup_{k=0}^n \mathcal{D}_{n,k}$. For any $D \in \mathcal{D}_n$, denote by $w(D)$ the weight of D . Let $\tilde{\mathcal{D}}_n$ denote the subset of Dyck paths $D \in \mathcal{D}_n$ such that all up-steps in D are weighted by 1, such Dyck paths only appear in $\mathcal{D}_{n,0}$.

Theorem 5.1. *There exists a sign reversing involution φ on the set $\mathcal{D}_n \setminus \tilde{\mathcal{D}}_n$.*

Proof. For any $D \in \mathcal{D}_n \setminus \tilde{\mathcal{D}}_n$, it can be uniquely written as $D = D_1D_2 \cdots D_m$ for some $1 \leq m \leq n$, where D_i 's are weighted primitive Dyck paths. Obviously, there exists at least a D_i such that D_i has an up-step weighted by q or $-q$. Now we can recursively construct the involution φ as follows. First find the maximum i for $1 \leq i \leq m$ such that D_i has an up-step weighted by q or $-q$, then define $\varphi(D) = D_1 \cdots D_{i-1}\varphi(D_i)D_{i+1} \cdots D_m$. Note that $D_i = uD_i^*d$, where $D_i^* \in \mathcal{D}_j$ for some $0 \leq j \leq n - 1$.

- If the first up-step in $D_i = uD_i^*d$ has weight q or $-q$, then D_i^* has no up-steps with weight q , otherwise it will contradict with the definition of $D \in \mathcal{D}_n$. Let D'_i denote the weighted Dyck path obtained from D_i by changing the sign of the weight of the first up-step. Then define $\varphi(D_i) = D'_i$;
- If the first up-step in $D_i = uD_i^*d$ has weight 1, then define $\varphi(D_i) = u\varphi(D_i^*)d$.

It is clear that the φ is a sign reversing involution on the set $\mathcal{D}_n \setminus \tilde{\mathcal{D}}_n$. See Fig. 1 for an illustration. \square

Proof of (1.4)

It is clear that the weight of $\tilde{\mathcal{D}}_n$ is the n th Catalan number C_n . According to the definition of $D \in \mathcal{D}_{n,k}$, it is easy to derive the weight of $\mathcal{D}_{n,k}$. On the one hand, it is well known that the total weight for $D^{(0)}$ is the Narayana polynomial $N_k(q)$ [25]. On the other hand, the total product of the weights of $D^{(1)}, D^{(2)}, \dots, D^{(2k+1)}$ is just the coefficient of x^{n-k} in $(1 - q)^{n-k}C(x)^{2k+1}$. Then by (4.1) we have

$$w(\mathcal{D}_{n,k}) = N_k(q)(1 - q)^{n-k}[x^{n-k}]C(x)^{2k+1} = \frac{2k + 1}{2n + 1} \binom{2n + 1}{n - k} N_k(q)(1 - q)^{n-k}.$$

By Theorem 5.1, we have $w(\tilde{\mathcal{D}}_n) = \sum_{k=0}^n w(\mathcal{D}_{n,k})$, which completes the proof. \square

Remark 5.2. Specially, let $\tilde{\mathcal{D}}_n$ denote the subset of Dyck paths $D \in \mathcal{D}_n$ such that all up-steps in D are weighted by q or $-q$. Such Dyck paths can only appear in $\mathcal{D}_{n,k}$ for $0 \leq k \leq n$ satisfying that (a) $D^{(0)} = (ud)^k$ with up-steps weighted by q , and (b) all up-steps in $D^{(i)}$ are weighted by $-q$ for $1 \leq i \leq k$. Let $\varphi_{\tilde{\mathcal{D}}_n}$ be the φ restricted to $\tilde{\mathcal{D}}_n$, it is clear that $\varphi_{\tilde{\mathcal{D}}_n}$ is a sign reversing involution on $\tilde{\mathcal{D}}_n$. Then (3.6) is followed similarly to the proof of Theorem 5.1 and the proof of (1.4) above.

In order to give the combinatorial proof of (1.5) and (1.6), we need the following definitions. A *plane tree* T can be defined recursively (see for example [26]) as a finite set of vertices such that a distinguished vertex u is called the *root* of T , and the

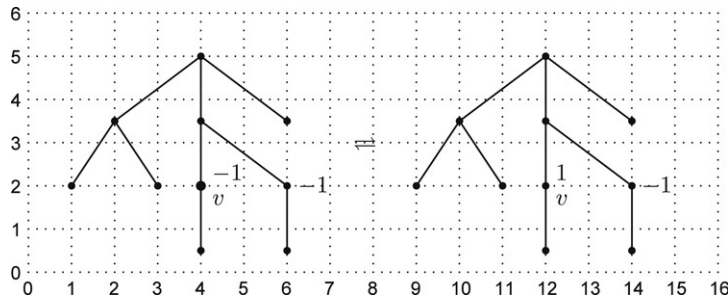


Fig. 2. The involution on \mathcal{P}_n^* .

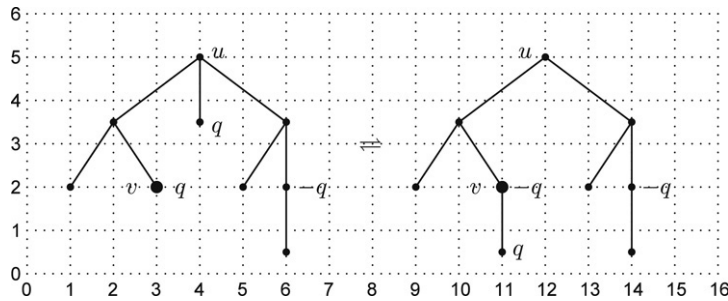


Fig. 3. The involution on $\mathcal{P}_n \setminus \tilde{\mathcal{P}}_n$.

remaining vertices are put into an ordered partition (T_1, T_2, \dots, T_m) of $m \geq 0$ disjoint non-empty sets, each of which is a plane tree called a subtree of u . The root u_i of T_i is called a child of u , and u is called the father of u_i . The out-degree of a vertex of T is the number of its subtrees. An internal vertex of T is a vertex of out-degree at least one. A vertex of out-degree zero is called a leaf of T . A complete binary tree is a plane tree such that each internal vertex has out-degree two.

A weighted plane tree is a plane tree for which every vertex is endowed with a weight. The weight of a plane tree T is the product of the weights of its vertices, the weight of a set S of plane trees means the sum of the weights of T in S .

Let $\mathcal{P}_{n,k}$ denote the set of weighted plane trees of $n + 2$ vertices such that

- The leaves have weight q ;
- There exist $n - k$ vertices of out-degree one, except for the root if it has out-degree one, with weight -1 or $-q$. In other words, such vertices can be regarded to be weighted by $-(1 + q)$;
- All other internal vertices have weight 1. There may exist vertices of out-degree one with weight 1.

Let $\mathcal{P}_n = \bigcup_{k=0}^n \mathcal{P}_{n,k}$. For any $T \in \mathcal{P}_n$, denote by $w(T)$ the weight of T . Let \mathcal{P}_n^* denote the subset of \mathcal{P}_n such that there are at least one vertex, except for the root, of out-degree one weighted by 1 or -1 , let $\tilde{\mathcal{P}}_n$ denote the subset of \mathcal{P}_n such that the root has out-degree one, and all other internal vertices have out-degree two. It is clear that $\mathcal{P}_n^* \cap \tilde{\mathcal{P}}_n = \emptyset$.

Theorem 5.3. *There exists a sign reversing involution ψ on the set $\mathcal{P}_n \setminus \tilde{\mathcal{P}}_n$.*

Proof. Note that a tree $T \in \mathcal{P}_n$ is in \mathcal{P}_n^* if and only if it contains a vertex, different from the root, of out-degree one with weight either 1 or -1 . Consider the first occurrence of such vertex, denoted by v , when traversing the weighted plane tree T in pre-order, (i.e., visiting the root first, then traversing its subtrees from left to right). Then replace the weight of v in T by $-w(v)$, we obtain another weighted plane tree T^* in \mathcal{P}_n^* , and then define $\psi(T) = T^*$. See Fig. 2 for an example.

For any $T \in \mathcal{P}_n \setminus (\mathcal{P}_n^* \cup \tilde{\mathcal{P}}_n)$, we can construct recursively the involution ψ as follows. First we should consider the following two cases:

- The root of T has out-degree one, in this case the unique subtree of the root has either a vertex of out-degree greater than two or a vertex of out-degree one with weight $-q$.
- The root of T has out-degree not less than two.

If the root u of T has $i (\geq 2)$ number of subtrees, denoted by T_1, T_2, \dots, T_i , when T_1 is a complete binary tree, then delete the subtree T_2 of the root u , regard T_2 as the subtree of the right-most leaf v of T_1 , and replace the weight q of v by $-q$, hence we obtain a new weighted plane tree T^* in $\mathcal{P}_n \setminus (\mathcal{P}_n^* \cup \tilde{\mathcal{P}}_n)$, and define $\psi(T) = T^*$; when T_1 is not a complete binary tree, consider the left-most component of T , that is the tree T'_1 with the root u and T_1 as its unique subtree, then $\psi(T)$ is obtained by adding the subtrees T_2, \dots, T_i right to the root u of $\psi(T'_1)$ step by step. See Fig. 3 for an example.

If the root u of T has a unique subtree, denoted by T' , let u' be the root of T' which is the only child of u . Note that u' is always an internal vertex.

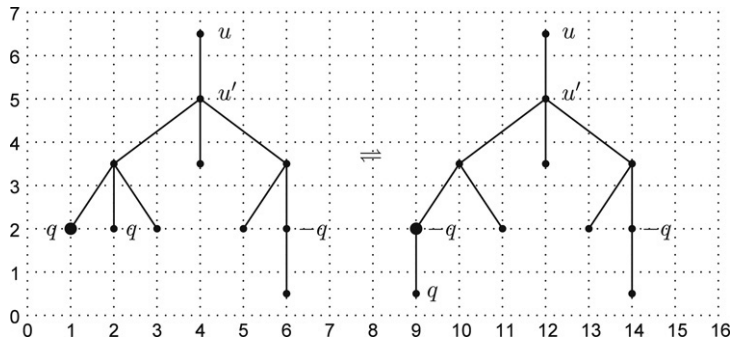


Fig. 4. The involution on $\mathcal{P}_n \setminus \tilde{\mathcal{P}}_n$.

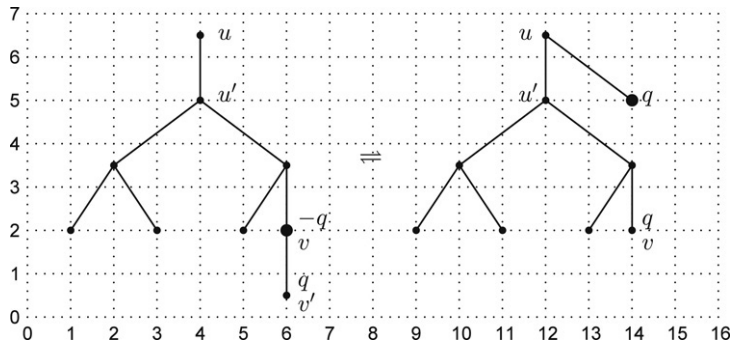


Fig. 5. The involution on $\mathcal{P}_n \setminus \tilde{\mathcal{P}}_n$.

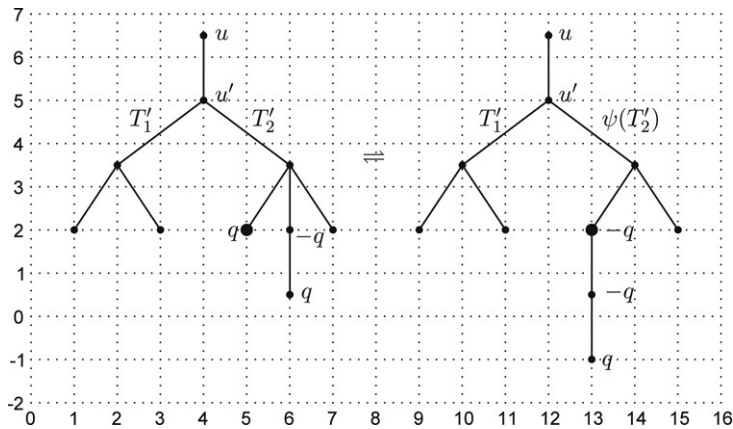


Fig. 6. The involution on $\mathcal{P}_n \setminus \tilde{\mathcal{P}}_n$.

(i) If the out-degree of u' is greater than two, then $\psi(T)$ is defined to be the tree T^* in $\mathcal{P}_n \setminus (\mathcal{P}_n^* \cup \tilde{\mathcal{P}}_n)$ which has the root u with a unique subtree $\psi(T')$; See Fig. 4 for an example.

(ii) If the out-degree of u' is one or two, find the right-most leaf v' ,

- if there exist vertices of weight $-q$ in the path $u'v'$, then choose the vertex v (if u' has out-degree one, then $v = u'$) which is first occurring in the path $u'v'$, denoted by T'' as the subtree of v , if deleting T'' in T' , the resulting tree is a complete binary tree. Then deleting the subtree T'' in T' , annexing it to the right of u , and changing the weight $-q$ of v to be q , we obtain a new weighted plane tree T^* in $\mathcal{P}_n \setminus (\mathcal{P}_n^* \cup \tilde{\mathcal{P}}_n)$, and define $\psi(T) = T^*$; See Fig. 5 for an example.
- if deleting T'' in T' , the resulting tree is not a complete binary tree or if there is no vertex of weight $-q$ in the path $u'v'$, then u' must have out-degree two. Let T_1, T_2 be the left and right subtrees of u' and T'_1, T'_2 be the left and right components of u' respectively. If T_1 is not a complete binary tree, then $\psi(T)$ is defined to be the tree T^* in $\mathcal{P}_n \setminus (\mathcal{P}_n^* \cup \tilde{\mathcal{P}}_n)$ by replacing T'_1 in T by $\psi(T'_1)$. If T_1 is a complete binary tree, so T_2 must not be a complete binary tree, then $\psi(T)$ is defined to be the tree T^* in $\mathcal{P}_n \setminus (\mathcal{P}_n^* \cup \tilde{\mathcal{P}}_n)$ by replacing T'_2 in T by $\psi(T'_2)$. See Fig. 6 for an example.

Clearly, the ψ just defined is indeed a sign reversing involution on the set $\mathcal{P}_n \setminus \tilde{\mathcal{P}}_n$. \square

Let $\mathcal{Q}_{n,k}$ denote the set of weighted plane trees of $n + 2$ vertices such that

- The leaves have weight q^2 ;
- There exist $n - k$ vertices of out-degree one, except for the root if it has out-degree one, with weight $-1, 2q$ or $-q^2$. In other words, such vertices can be regarded to be weighted by $-(1 - q)^2$;
- All other internal vertices have weight 1. There may exist vertices of out-degree one with weight 1.

Let $\mathcal{Q}_n = \bigcup_{k=0}^n \mathcal{Q}_{n,k}$, and \mathcal{Q}_n^* denote the subset of \mathcal{Q}_n such that there are at least one vertex, except for the root, of out-degree one weighted by 1 or -1 , let $\tilde{\mathcal{Q}}_n$ denote the subset of \mathcal{Q}_n such that the root has out-degree one, and all other internal vertices have either out-degree two or out-degree one with weight $2q$. It is clear that $\mathcal{Q}_n^* \cap \tilde{\mathcal{Q}}_n = \emptyset$. Similar to the proof of Theorem 5.3, a sign reversing involution can be constructed on $\mathcal{Q}_n \setminus \tilde{\mathcal{Q}}_n$, the details leave to the interested readers. Hence we have

Theorem 5.4. *There exists a sign reversing involution on the set $\mathcal{Q}_n \setminus \tilde{\mathcal{Q}}_n$.*

Now we can give the combinatorial proof of identities (1.5) and (1.6).

Proof of (1.5) and (1.6). For any $T \in \mathcal{P}_{k,k}$, namely, T is a weighted plane tree of $k + 2$ vertices with leaves weighted by q and all other internal vertices weighted by 1, inserting $n - k$ vertices of weight $-(1 + q)$ into the $k + 1$ edges of T (repetition allowed), we can obtain $\binom{n}{k}$ number of weighted plane trees in $\mathcal{P}_{n,k}$. It is well known that the weight of $\mathcal{P}_{k,k}$ is $N_{k+1}(q)$ [25], then $\mathcal{P}_{n,k}$ has the weight $\binom{n}{k} N_{k+1}(q)(-1 - q)^{n-k}$. Similarly, $\mathcal{Q}_{n,k}$ has the weight $(-1)^{n-k} \binom{n}{k} N_{k+1}(q^2)(1 - q)^{2(n-k)}$.

On the other hand, for any $T \in \tilde{\mathcal{P}}_n$, we know that the root of T has out-degree one and has only one subtree T' which is a weighted complete binary tree with $n + 1$ vertices, it is well known that the number of complete binary trees with $n + 1$ vertices is counted by Catalan number $C_{\frac{n}{2}}$ [25], where $C_{\frac{n}{2}} = 0$ if n is odd. So the weight of $\tilde{\mathcal{P}}_n$ is $q^{\frac{n}{2}+1} C_{\frac{n}{2}}$.

For any $T \in \tilde{\mathcal{Q}}_n$, let $\tilde{\mathcal{Q}}_{n,k}$ denote the subset of $\tilde{\mathcal{Q}}_n$ such that T has $n - 2k$ vertices, except for the root, of out-degree one with weight $2q$. For any $T \in \tilde{\mathcal{Q}}_{2k,k}$, we know that the root of T has out-degree one and has only one subtree T' which is a weighted complete binary tree with $2k + 1$ vertices, inserting $n - 2k$ vertices of weight $2q$ into the $2k + 1$ edges of T (repetition allowed), we can obtain $\binom{n}{2k}$ number of weighted plane trees in $\tilde{\mathcal{Q}}_{n,k}$. It is clear that $\tilde{\mathcal{Q}}_{2k,k}$ is counted by Catalan numbers C_k and has weight $q^{2k+2} C_k$, then $\tilde{\mathcal{Q}}_{n,k}$ has weight $(2q)^{n-2k} \binom{n}{2k} q^{2k+2} C_k = q^{n+2} \binom{n}{2k} C_k 2^{n-2k}$. Hence $\tilde{\mathcal{Q}}_n$ has weight $q^{n+2} \sum_{k=0}^n \binom{n}{2k} C_k 2^{n-2k}$, which is $q^{n+2} C_{n+1}$ by Touchard identity.

Using Theorems 5.3 and 5.4, one can easily obtain that the weight of \mathcal{P}_n (resp. \mathcal{Q}_n) equals that of $\tilde{\mathcal{P}}_n$ (resp. $\tilde{\mathcal{Q}}_n$), which completes the proof. \square

Acknowledgements

The authors are grateful to the anonymous referees for the helpful suggestions and comments. The second author was supported by The National Science Foundation of China (10801020).

References

- [1] M. Aigner, Catalan-like numbers and determinants, *J. Combin. Theory Ser. A* 87 (1999) 33–51.
- [2] J. Bonin, L. Shapiro, R. Simion, Some q -analogues of the Schröder numbers arising from combinatorial statistics on lattice paths, *J. Statist. Plann. Inference* 34 (1993) 35–55.
- [3] R. Chapman, A curious identity, revisited, *Math. Gazette* 87 (2003) 139–141.
- [4] W.Y.C. Chen, E.Y.P. Deng, R.R.X. Du, Reduction of m -regular noncrossing partitions, *European J. Combin.* 26 (2) (2005) 237–243.
- [5] W.Y.C. Chen, N.Y. Li, L. Shapiro, The butterfly decomposition of plane trees, *Discrete Appl. Math.* 155 (17) 2187–2201.
- [6] W.Y.C. Chen, S.X.M. Pang, On the combinatorics of the Pfaff identity, *Discrete Math.* (2008), in press (doi:10.1016/j.disc.2008.04.048).
- [7] W.Y.C. Chen, S.H.F. Yan, L.L.M. Yang, Identities from weighted 2-Motzkin paths, *Adv. Appl. Math.* 41 (3) (2008) 329–334.
- [8] C. Coker, Enumerating a class of lattice paths, *Discrete Math.* 271 (2003) 13–28.
- [9] L. Comtet, *Advanced Combinatorics*, D. Reidel, Dordrecht-Holland, 1970.
- [10] C. Corsani, D. Merlini, R. Sprugnoli, Left-inversion of combinatorial sums, *Discrete Math.* 180 (1–3) (1998) 107–122.
- [11] E. Deutsch, Dyck path enumeration, *Discrete Math.* 204 (1999) 167–202.
- [12] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, 2nd edition, in: *Encyclopedia of Mathematics and its Applications*, vol. 96, Cambridge University Press, 2004.
- [13] M.D. Hirschhorn, Comment on a curious identity, *Math. Gazette* 87 (2003) 528–530.
- [14] T. Mansour, Y. Sun, Dyck Paths and partial Bell polynomials, *Austr. J. Combin.* 42 (2008) 285–297.
- [15] T. Mansour, Y. Sun, Bell polynomials and k -generalized Dyck Paths, *Discrete Appl. Math.* 156 (12) (2008) 2279–2292.
- [16] E. Munarini, Generalization of a binomial identity of Simons, *Integer: Electron. J. Combin. Number Theory* 5 (2005) #A15.
- [17] I. Pak, Lectures on discrete and polyhedral geometry. <http://www.math.umn.edu/pak>.
- [18] H. Prodinger, A curious identity proved by Cauchy's integral formula, *Math. Gazette* 89 (2005) 266–267.
- [19] J. Riordan, *Combinatorial Identities*, John Wiley & Sons, Inc, New York, 1968.
- [20] D.G. Rogers, Rhyming schemes: Crossings and coverings, *Discrete Math.* 33 (1981) 67–77.
- [21] D.G. Rogers, L.W. Shapiro, Deques, trees and lattice paths, *Lect. Notes Math.* 884 (1981) 293–303.
- [22] R. Simion, D. Ullman, On the structure of the lattice of noncrossing partitions, *Discrete Math.* 98 (1991) 193–206.
- [23] S. Simons, A curious identity, *Math. Gazette* 85 (2001) 296–298.
- [24] N.J.A. Sloane, The on-line encyclopedia of integer sequences. www.research.att.com/jas/sequ-ences.
- [25] R. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge Univ. Press, Cambridge, 1999.
- [26] R. Stanley, *Enumerative Combinatorics*, vol. 1, Cambridge Univ. Press, Cambridge, 1997.
- [27] R. Sulanke, Counting lattice paths by Narayana polynomials, *Electron. J. Combin.* 2 (2000) #R40.
- [28] Y. Sun, The statistic “number of udu 's” in Dyck paths, *Discrete Math.* 287 (1–3) (2004) 177–186.
- [29] X. Wang, Y. Sun, A new proof of a curious identity, *Math. Gazette* 91 (502) (2007) 105–106.
- [30] H. Wilf, *Generatingfunctionology*, Academic Press, New York, 1990.