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Combinatorial Identities and Inverse Binomial Coefficients

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In this paper, we present a method for obtaining a wide class of combinatorial identities. We give several examples; some of them have already been considered previously, and others are new. © 2002 Elsevier Science (USA)

1. INTRODUCTION

In 1981, Rockett [R, Theorem 1] (see also [Pl]) proved the following. For any nonnegative integer n,

$$\sum_{k=0}^{n}\binom{n}{k}^{-1} = \frac{n+1}{2^{n+1}}\sum_{k=1}^{n+1}\frac{2^k}{k}. \tag{1}$$

In 1999, Trif [T] proved the above result using the Beta function. The present paper can be regarded as a far-reaching generalization of the ideas presented in [T]. Our main result, in its simplest form, can be stated as follows.

THEOREM 1.1. *Let $r, n \geq k$ be any nonnegative integer numbers, and let $f(n,k)$ be given by*

$$f(n,k) = \frac{(n+r)!}{n!}\int_{u_1}^{u_2} p^k(t)q^{n-k}(t)\,dt,$$

where $p(t)$ and $q(t)$ are two functions defined on $[u_1, u_2]$. Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be any two sequences, and let $A(x), B(x)$ be the corresponding ordinary generating functions. Then

$$\sum_{n\geq 0}\left[\sum_{k=0}^{n} f(n,k)a_k b_{n-k}\right]x^n = \frac{d^r}{dx^r}\left[x^r\int_{u_1}^{u_2} A(xp(t))B(xq(t))\,dt\right].$$

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As an easy consequence of Theorem 1.1, we get a family of identities, including the one presented above.

EXAMPLE 1.2 (see [JS]). Let $a_n = a^n$ and $b_n = b^n$ for all $n \geq 0$, and let $a + b \neq 0$. So the corresponding generating functions are $A(x) = (1 - ax)^{-1}$ and $B(x) = (1 - bx)^{-1}$.

It is easy to see that

$$\binom{s}{r}^{-1} = (s+1)\int_0^1 t^r(1-t)^{s-r}\,dt, \tag{2}$$

for all nonnegative real numbers s and r such that $s \geq r$.

By Theorem 1.1 and (2),

$$\begin{aligned}\sum_{n\geq 0} x^n \sum_{k=0}^{n} a^k b^{n-k}\binom{n}{k}^{-1} &= \frac{d}{dx}\left(x\int_0^1 \frac{1}{(1-axt)(1-bx+bxt)}\,dt\right)\\ &= \frac{d}{dx}\left(\frac{-\ln(1-ax)-\ln(1-bx)}{a+b-abx}\right),\end{aligned}$$

and after simple transformations, we get

$$\sum_{k=0}^{n} a^k b^{n-k}\binom{n}{k}^{-1} = \frac{n+1}{(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)^{n+1}}\sum_{k=1}^{n+1}\frac{(a^k+b^k)\left(\frac{1}{a}+\frac{1}{b}\right)^k}{k}$$

for any nonnegative integer n. In particular, for $a = b = 1$, we get (1).

EXAMPLE 1.3. Let us define $a_n = n$, $b_n = 1$ for $n \geq 0$. By Theorem 1.1 and (2), it is easy to see that

$$\sum_{n\geq 0}\left[\sum_{k=0}^{n} k\binom{n}{k}^{-1}\right]x^n = \frac{-2x\ln(1-x)}{(2-x)^3} - \frac{x(3x-4)}{(2-x)^2(1-x)^2}.$$

Hence, for any nonnegative integer n,

$$\sum_{k=0}^{n} k\binom{n}{k}^{-1} = \frac{1}{2^n}\left[(n+1)(2^n-1) + \sum_{k=0}^{n-2}\frac{(n-k)(n-k-1)2^{k-1}}{k+1}\right].$$

In the rest of the paper, we prove Theorem 1.1 and generalize it to functions represented by integrals over a real d-dimensional domain. We present several examples; some of them have been considered previously, and others are new. For combinatorial identities yields from integral representation in the complex domain, see [E].

2. ONE-DIMENSIONAL CASE

First of all, let us prove Theorem 1.1. Let $f(n,k)$ be as in the statement of the theorem. Then

$$\sum_{k=0}^{n} f(n,k)a_n b_{n-k} = \frac{(n+r)!}{n!}\int_{u_1}^{u_2}\sum_{k=0}^{n} a_k p^k(t) b_{n-k} q^{n-k}(t)\,dt,$$

which means that

$$\sum_{n\geq 0} x^n \sum_{k=0}^{n} f(n,k)a_n b_{n-k} = \sum_{n\geq 0}\left[\frac{(n+r)!x^n}{n!}\int_{u_1}^{u_2}\sum_{k=0}^{n} a_k p^k(t) b_{n-k} q^{n-k}(t)\right] dt.$$

Let $A(x)=\sum_{n\geq 0} a_n x^n$, $B(x)=\sum_{n\geq 0} b_n x^n$; hence

$$\sum_{n\geq 0}\sum_{k=0}^{n} f(n,k)a_k b_{n-k}x^n = \frac{d^r}{dx^r}\left[x^r\int_{u_1}^{u_2} A(xp(t))B(xq(t))\,dt\right],$$

which means that Theorem 1.1 holds. ∎

Now, we present other applications of Theorem 1.1.

EXAMPLE 2.1. Immediately, by (2) and Theorem 1.1, we get, for any nonnegative integer numbers c and d,

$$\sum_{n\geq 0} x^{cn}\sum_{k=0}^{n}\binom{cn}{dk}^{-1} = \frac{d}{dx}\int_0^1 \frac{x\cdot dt}{(1-(1-t)^c x^c)(1-t^d(1-t)^{c-d}x^c)}.$$

For $c=d=2$, it is easy to get, for any nonnegative integer n,

$$\sum_{k=0}^{n}\binom{2n}{2k}^{-1} = \frac{n(2n+1)}{2^{2n+2}}\sum_{k=0}^{2n+1}\frac{2^k}{k+1}.$$

THEOREM 2.2. *Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be two sequences, let $A(x)$ and $B(x)$ be the corresponding ordinary generating functions, and let μ be the differential operator of the first order defined by $\mu(f)=\frac{d}{dx}(x\cdot f)$. Then, for any positive integer m,*

$$\sum_{n\geq 0}\left[\sum_{k=0}^{n}\binom{n}{k}^{-m} a_k b_{n-k}\right]x^n$$
$$=\mu^m\left[\underbrace{\int_0^1\int_0^1\cdots\int_0^1}_{m\text{ times}} A(xt_1t_2\cdots t_m)B((1-t_1)(1-t_2)\cdots(1-t_m)x)\,dt_1\,dt_2\cdots dt_m\right].$$

Proof. Using (2), we get

$$\binom{n}{k}^{-m} = (n+1)^m \left[\int_0^1 t^k (1-t)^{n-k}\, dt \right]^m,$$

which means that

$$\binom{n}{k}^{-m} = (n+1)^m \underbrace{\int_0^1 \cdots \int_0^1}_{m \text{ times}} (t_1 t_2 \cdots t_m)^k ((1-t_1)(1-t_2)\cdots(1-t_m))^{n-k}\, dt_1 \cdots dt_m.$$

So, similarly to the proof of Theorem 1.1, this theorem holds. ∎

Now let us find another representation for $\binom{n}{k}^{-m}$.

PROPOSITION 2.3. *For any nonnegative integers n, m,*

$$\sum_{k=0}^{n} \binom{n}{k}^{-m} = (n+1)^m \sum_{k=0}^{n} \left[\sum_{i=0}^{k} \frac{(-1)^i}{n-k+1+i} \binom{k}{i} \right]^m.$$

Proof. By (2), we get, for all positive integer m,

$$\binom{n}{k}^{-m} = (n+1)^m \left(\int_0^1 t^k (1-t)^{n-k}\, dt \right)^m,$$

which means that

$$\binom{n}{k}^{-m} = (n+1)^m \left[\int_0^1 \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} t^{k+i}\, dt \right]^m,$$

hence the proposition holds. ∎

The above proposition and (1) yield the following.

COROLLARY 2.4. *For any nonnegative integer n,*

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k}^{-1} &= (n+1) \sum_{k=0}^{n} \frac{1}{(n+1-k)2^k} \\ &= (n+1) \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{(-1)^j}{n-k+1+j} \binom{k}{j}.\end{aligned}$$

COROLLARY 2.5. *For any nonnegative integer n,*

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k}^{-2} &= (n+1)^2 \sum_{k=0}^{n} \left[\sum_{i=0}^{k} \frac{(-1)^i}{n-k+1+i} \binom{k}{i} \right]^2 \\ &= (n+1)^2 \sum_{k=0}^{n} \frac{2}{n-k+1} \sum_{j=0}^{k} \frac{(-1)^j}{n+2+i} \binom{k}{i}.\end{aligned}$$

Proof. By Proposition 2.3, the first equality holds. Now let us prove the second equality. By Theorem 2.2, we get

$$\sum_{n\geq 0} x^n \sum_{k=0}^{n} \binom{n}{k}^{-2} = \mu^2\left[\int_0^1\int_0^1 \frac{1}{(1-tux)(1-(1-t)(1-u)x)}\,du\,dt\right];$$

therefore,

$$\sum_{n\geq 0} x^n \sum_{k=0}^{n} \binom{n}{k}^{-2} = \mu^2\left[\int_0^1 \frac{-2\ln(1-tx)}{x(1-t(1-t)x)}\,dt\right].$$

Hence, since $\ln(1-tx) = \sum_{n\geq 1}(-t^n x^n)/n$ and $\frac{1}{1-t(1-t)x} = \sum_{n\geq 0} t^n(1-t)^n x^n$, the second equality holds. ∎

3. GENERALIZATION: d-DIMENSIONAL CASE

The following result, which is a generalization of Theorem 1.1, gives us a general method for obtaining combinatorial identities.

THEOREM 3.1. *Let X be a multiset of variables x_j, where $j = 1, 2, \ldots, d+1$, and let $X' = \{x_{i_1}, \ldots, x_{i_l}\}$ be the underlying set. Let $g(t)$ and $f_j(t)$, $j = 1, 2, \ldots, d$, be any $d+1$ functions such that $\phi(x_{i_1}, \ldots, x_{i_l}) = g(x_{d+1})\prod_{j=1}^{d} f_j(x_j)$ is a function defined on an l-dimensional domain D. Let r be a nonnegative integer number, and let $f(k_1, k_2, \ldots, k_d)$ be given by*

$$f(k_1, k_2, \ldots, k_d) = \frac{(k_1+\cdots+k_d+r)!}{(k_1+\cdots+k_d)!}\int_D \phi(x_{i_1}, \ldots, x_{i_l})\,dx_{i_1}\cdots dx_{i_l}.$$

Then for any sequences $\{a_n^{(j)}\}_{n\geq 0}$, $j = 1, 2, \ldots, d$,

$$\sum_{n\geq 0}\sum_{k_1+\cdots+k_d=n} f(k_1, k_2, \ldots, k_d)x^n \prod_{j=1}^{d} a_{k_i}^{(j)}$$

$$= \frac{d^r}{dx^r}\left[x^r \int_D g(x_{d+1})\prod_{j=1}^{d} A_j(xf_j(x_j))\,dx_{i_1}\cdots dx_{i_l}\right],$$

where $A_j(x)$ is the ordinary generating function of the sequence $\{a_n^{(j)}\}_{n\geq 0}$.

Another way to generalize Theorem 1.1 is the following. Let V be the hyperplane defined by $\sum_{i=1}^{d}(\frac{x_i}{a_i})^{p_i} = 1$, where $x_i \geq 0$ for all $i = 1, 2, \ldots, d$. If $p_i \geq 0$ for all i, then the *Dirichlet's integral* is defined by

$$\int_V \prod_{j=1}^{d} x_j^{\alpha_j - 1}\,dx_1\cdots dx_d = \frac{a_1^{\alpha_1}\cdots a_d^{\alpha_d}}{p_1\cdots p_d}\frac{\Gamma\big(\frac{\alpha_1}{p_1}\big)\cdots\Gamma\big(\frac{\alpha_d}{p_d}\big)}{\Gamma\big(1+\frac{\alpha_1}{p_1}+\cdots+\frac{\alpha_d}{p_d}\big)}. \tag{3}$$

So for $p_j = 1$, $a_j = 1$, and $\sum_{j=1}^{d} \alpha_j = n$, we obtain

$$\binom{n}{\alpha_1, \ldots, \alpha_d}^{-m} = \frac{(n+d-1)!^m}{n!^m} \left(\int_{x_1+\cdots+x_d=1} x_1^{\alpha_1} \cdots x_d^{\alpha_d}\, dx_1 \cdots dx_d \right)^m. \tag{4}$$

Hence, Theorem 3.1, Theorem 1.1, and (3) yield the following.

THEOREM 3.2. *Let $\{a_n^{(j)}\}_{n\geq 0}$ be any sequence for all $j = 1, 2, \ldots, d$, and let ν be the differential operator of the $(d-1)$th order defined by $\nu_d(f) = (d^{d-1}/dx^{d-1})(x^{d-1} f)$. Then*

$$\sum_{n\geq 0} x^n \sum_{\alpha_1+\cdots+\alpha_d=n} \binom{n}{\alpha_1, \ldots, \alpha_d}^{-m} \prod_{j=1}^{d} a_{\alpha_j}^{(j)}$$

$$= \nu_d^m \left[\underbrace{\int_V \cdots \int_V}_{m \text{ times}} \prod_{j=1}^{d} A_j(x x_{j,1} x_{j,2} \cdots x_{j,m}) \prod_{i=1,\, j=1}^{d,\, m} dx_{i,j} \right],$$

where V is the hyperplane defined by $x_1 + x_2 + \cdots + x_d = 1$, and $A_j(x)$ is the ordinary generating function of sequence $\{a_n^{(j)}\}_{n\geq 0}$, $j = 1, 2, \ldots, d$.

EXAMPLE 3.3 (see Carlson [C, Chapter 8]). Let $a_n^{(j)} = \binom{2n}{n}$ for $n \geq 0$, $j = 1, 2, \ldots, d$, and $m = 1$. By Theorem 3.2 and (4), it is easy to see that

$$\sum_{n\geq 0} x^n \sum_{\alpha_1+\cdots+\alpha_d=n} \binom{n}{\alpha_1, \ldots, \alpha_d}^{-1} \prod_{j=1}^{d} \binom{2\alpha_j}{\alpha_j}$$

$$= \frac{d^{d-1}}{dx^{d-1}} x^{d-1} \left[\int_{x_1+\cdots+x_d=1} \prod_{j=1}^{d} \frac{1}{\sqrt{1-4x x_j}} \prod_{j=1}^{d} dx_j \right].$$

As a numerical example, for $d = 2$, equating the coefficients at x^n, we get

$$\sum_{j=0}^{n} \binom{n}{j}^{-1} \binom{2j}{j} \binom{2n-2j}{n-j} = \sum_{j=0}^{n} 2^{n-j} \binom{2j}{j}.$$

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