

Research Article

Arithmetic Identities Involving Genocchi and Stirling Numbers

Guodong Liu

Department of Mathematics, Huizhou University, Huizhou, Guangdong 516015, China

Correspondence should be addressed to Guodong Liu, gdlou@pub.huizhou.gd.cn

Received 18 June 2009; Accepted 12 August 2009

Recommended by Leonid Berezansky

An explicit formula, the generalized Genocchi numbers, was established and some identities and congruences involving the Genocchi numbers, the Bernoulli numbers, and the Stirling numbers were obtained.

Copyright © 2009 Guodong Liu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

The Genocchi numbers G_n and the Bernoulli numbers B_n ($n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$) are defined by the following generating functions (see [1]):

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} \quad (|t| < \pi), \quad (1.1)$$

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad (|t| < 2\pi), \quad (1.2)$$

respectively. By (1.1) and (1.2), we have

$$G_{2n+1} = B_{2n+1} = 0, \quad (n \in \mathbb{N}) \quad G_n = 2(1 - 2^n)B_n, \quad (1.3)$$

with \mathbb{N} being the set of positive integers.

The Genocchi numbers G_n satisfy the recurrence relation

$$G_0 = 0, \quad G_1 = 1, \quad G_n = -\frac{1}{2} \sum_{k=1}^{n-1} \binom{n}{k} G_k \quad (n \geq 2), \quad (1.4)$$

so we find $G_2 = -1$, $G_4 = 1$, $G_6 = -3$, $G_8 = 17$, $G_{10} = -155$, $G_{12} = 2073$, $G_{14} = -38227, \dots$

The Stirling numbers of the first kind $s(n, k)$ can be defined by means of (see [2])

$$(x)_n = x(x-1) \dots (x-n+1) = \sum_{k=0}^n s(n, k) x^k, \quad (1.5)$$

or by the generating function

$$(\log(1+x))^k = k! \sum_{n=k}^{\infty} s(n, k) \frac{x^n}{n!}. \quad (1.6)$$

It follows from (1.5) or (1.6) that

$$s(n, k) = s(n-1, k-1) - (n-1)s(n-1, k), \quad (1.7)$$

with $s(n, 0) = 0$ ($n > 0$), $s(n, n) = 1$ ($n \geq 0$), $s(n, 1) = (-1)^{n-1} (n-1)!$ ($n > 0$), $s(n, k) = 0$ ($k > n$ or $k < 0$).

Stirling numbers of the second kind $S(n, k)$ can be defined by (see [2])

$$x^n = \sum_{k=0}^n S(n, k) (x)_k \quad (1.8)$$

or by the generating function

$$(e^x - 1)^k = k! \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!}. \quad (1.9)$$

It follows from (1.8) or (1.9) that

$$S(n, k) = S(n-1, k-1) + kS(n-1, k), \quad (1.10)$$

with $S(n, 0) = 0$ ($n > 0$), $S(n, n) = 1$ ($n \geq 0$), $S(n, 1) = 1$ ($n > 0$), $S(n, k) = 0$ ($k > n$ or $k < 0$).

The study of Genocchi numbers and polynomials has received much attention; numerous interesting (and useful) properties for Genocchi numbers can be found in many books (see [1, 3–16]). The main purpose of this paper is to prove an explicit formula for the generalized Genocchi numbers (cf. Section 2). We also obtain some identities congruences involving the Genocchi numbers, the Bernoulli numbers, and the Stirling numbers. That is, we will prove the following main conclusion.

Theorem 1.1. Let $n \geq k$ ($n, k \in \mathbb{N}$), then

$$\sum_{\substack{v_1, \dots, v_k \in \mathbb{N} \\ v_1 + \dots + v_k = n}} \frac{G_{v_1} \cdots G_{v_k}}{(v_1 \cdots v_k)v_1! \cdots v_k!} = (-1)^{n-k} \frac{2^k k!}{n!} \sum_{j=k}^n \frac{1}{2^j} S(n, j) s(j, k). \quad (1.11)$$

Remark 1.2. Setting $k = 1$ in (1.11), and noting that $s(j, 1) = (-1)^{j-1} (j-1)!$, we obtain

$$G_n = 2n \sum_{j=1}^n (-1)^{n-j} \frac{(j-1)!}{2^j} S(n, j) \quad (n \in \mathbb{N}). \quad (1.12)$$

Remark 1.3. By (1.11) and (1.3), we have

$$\sum_{\substack{v_1, \dots, v_k \in \mathbb{N} \\ v_1 + \dots + v_k = n}} \frac{(2^{v_1} - 1)B_{v_1} \cdots (2^{v_k} - 1)B_{v_k}}{(v_1 \cdots v_k)v_1! \cdots v_k!} = (-1)^n \frac{k!}{n!} \sum_{j=k}^n \frac{1}{2^j} S(n, j) s(j, k). \quad (1.13)$$

Theorem 1.4. Let $n, k \in \mathbb{N}$, then

$$\sum_{j=0}^n \frac{(-1)^j (k+j-1)!}{2^j} S(n, j) = 2^{k-1} \sum_{j=0}^{k-1} (-1)^j s(k, k-j) \frac{G_{n+k-j}}{n+k-j}. \quad (1.14)$$

Remark 1.5. Setting $k = 1, 2, 3, 4$ in (1.14), we get

$$\begin{aligned} \sum_{j=0}^n \frac{(-1)^j j!}{2^j} S(n, j) &= \frac{1}{n+1} G_{n+1}, \\ \sum_{j=0}^n \frac{(-1)^j (j+1)!}{2^j} S(n, j) &= \frac{2}{n+1} G_{n+1} + \frac{2}{n+2} G_{n+2}, \\ \sum_{j=0}^n \frac{(-1)^j (j+2)!}{2^j} S(n, j) &= \frac{8}{n+1} G_{n+1} + \frac{12}{n+2} G_{n+2} + \frac{4}{n+3} G_{n+3}, \\ \sum_{j=0}^n \frac{(-1)^j (j+3)!}{2^j} S(n, j) &= \frac{48}{n+1} G_{n+1} + \frac{88}{n+2} G_{n+2} + \frac{48}{n+3} G_{n+3} + \frac{8}{n+4} G_{n+4}. \end{aligned} \quad (1.15)$$

Theorem 1.6. Let $n \in \mathbb{N}$, $m \in \mathbb{N}_0$, then

$$\frac{2^n}{n+1} G_{n+1} \equiv 2^{n-m} \sum_{j=0}^m \binom{m}{j} j^n \pmod{m+1}. \quad (1.16)$$

Remark 1.7. Setting $m = p - 1$ in (1.16), we have

$$\frac{1}{n+1}G_{n+1} \equiv \sum_{j=0}^{p-1} (-1)^j j^n \pmod{p}, \quad (1.17)$$

where p is any odd prime.

2. Definition and Lemma

Definition 2.1. For a real or complex parameter x , we have the generalized Genocchi numbers $G_n^{(x)}$, which are defined by

$$\left(\frac{2}{e^{2t} + 1} \right)^x = \sum_{n=0}^{\infty} G_n^{(x)} \frac{t^n}{n!} \quad \left(|t| < \frac{\pi}{2}; 1^x := 1 \right). \quad (2.1)$$

By (1.1) and (2.1), we have

$$nG_{n-1}^{(1)} = 2^{n-1}G_n. \quad (2.2)$$

Remark 2.2. For an integer x , the higher-order Euler numbers $E_{2n}^{(x)}$ are defined by the following generating functions (see [17]):

$$\left(\frac{2}{e^t + e^{-t}} \right)^x = \sum_{n=0}^{\infty} E_{2n}^{(x)} \frac{t^{2n}}{(2n)!} \quad \left(|t| < \frac{\pi}{2} \right). \quad (2.3)$$

Then we have

$$G_n^{(x)} = (-1)^n \sum_{k=0}^{[n/2]} \binom{n}{2k} E_{2k}^{(x)} x^{n-2k}, \quad (2.4)$$

where $[n/2]$ denotes the greatest integer not exceeding $n/2$.

Lemma 2.3. Let $n \geq k$ ($n, k \in \mathbb{N}$), then

$$G_n^{(x)} = \sum_{k=1}^n \omega(n, k) x^k, \quad (2.5)$$

where

$$\omega(n, k) = (-1)^k \sum_{j=k}^n 2^{n-j} S(n, j) s(j, k). \quad (2.6)$$

Proof. By (2.1), (1.5), and (1.9) we have

$$\begin{aligned}
\sum_{n=0}^{\infty} G_n^{(x)} \frac{t^n}{n!} &= \left(\frac{2}{e^{2t} + 1} \right)^x = \left(\frac{1}{1 + (1/2)(e^{2t} - 1)} \right)^x \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j}{2^j} \binom{x+j-1}{j} (e^{2t} - 1)^j \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j}{2^j} \binom{x+j-1}{j} j! \sum_{n=j}^{\infty} 2^n S(n, j) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^n (-1)^j j! 2^{n-j} \binom{x+j-1}{j} S(n, j) \frac{t^n}{n!},
\end{aligned} \tag{2.7}$$

which readily yields

$$\begin{aligned}
G_n^{(x)} &= \sum_{j=0}^n (-1)^j j! 2^{n-j} \binom{x+j-1}{j} S(n, j) \\
&= \sum_{j=0}^n (-1)^j 2^{n-j} S(n, j) (x+j-1)(x+j-2) \cdots (x+1)x \\
&= \sum_{j=0}^n (-1)^j 2^{n-j} S(n, j) \sum_{k=1}^j (-1)^{j-k} s(j, k) x^k \\
&= \sum_{k=1}^n (-1)^k \sum_{j=k}^n 2^{n-j} S(n, j) s(j, k) x^k = \sum_{k=1}^n \omega(n, k) x^k.
\end{aligned} \tag{2.8}$$

This completes the proof of Lemma 2.3. \square

Remark 2.4. From (1.7), (1.10), and Lemma 2.3 we know that $G_n^{(x)}$ is a polynomial of x with integral coefficients. For example, setting $n = 1, 2, 3, 4$ in Lemma 2.3, we get

$$\begin{aligned}
G_1^{(x)} &= -x, & G_2^{(x)} &= -x + x^2, & G_3^{(x)} &= 3x^2 - x^3, \\
G_4^{(x)} &= 2x + 3x^2 - 6x^3 + x^4.
\end{aligned} \tag{2.9}$$

Remark 2.5. Let $n, m \in \mathbb{N}$, then by (2.5), we have

$$\sum_{k=1}^n \omega(n, k) = \frac{2^n}{n+1} G_{n+1}. \tag{2.10}$$

Therefore, if $q \in \mathbb{N}$ is odd, then by (2.10) we get

$$G_{2^k q} \equiv 0 \pmod{q}, \quad (2.11)$$

where $k \in \mathbb{N}$.

3. Proof of the Theorems

Proof of Theorem 1.1. By applying Lemma 2.3, we have

$$k! \omega(n, k) = \frac{d^k}{dx^k} G_n^{(x)} \Big|_{x=0}. \quad (3.1)$$

On the other hand, it follows from (2.1) that

$$\sum_{n=k}^{\infty} \frac{d^k}{dx^k} G_n^{(x)} \Big|_{x=0} \frac{t^n}{n!} = \left(\log \frac{2}{e^{2t} + 1} \right)^k, \quad (3.2)$$

where $\log(2/(e^{2t} + 1))$ is the principal branch of logarithm of $2/(e^{2t} + 1)$.

Thus, by (3.1) and (3.2), we have

$$k! \sum_{n=k}^{\infty} \omega(n, k) \frac{t^n}{n!} = \left(\log \frac{2}{e^{2t} + 1} \right)^k. \quad (3.3)$$

Now note that

$$\frac{d}{dt} \log \frac{2}{e^{2t} + 1} = \frac{2}{e^{2t} + 1} - 2 = \sum_{n=0}^{\infty} G_n^{(1)} \frac{t^n}{n!} - 2 = \sum_{n=0}^{\infty} \frac{2^n G_{n+1}}{n+1} \frac{t^n}{n!} - 2, \quad (3.4)$$

whence by integrating from 0 to t , we deduce that

$$\log \frac{2}{e^{2t} + 1} = \sum_{n=1}^{\infty} \frac{2^{n-1} G_n}{n} \frac{t^n}{n!} - 2t = \sum_{n=1}^{\infty} (-1)^n \frac{2^{n-1} G_n}{n} \frac{t^n}{n!}. \quad (3.5)$$

Since $G_{2n+1} = 0$ ($n \in \mathbb{N}$). Substituting (3.5) in (3.3) we get

$$\omega(n, k) = (-1)^n \frac{n! 2^{n-k}}{k!} \sum_{\substack{v_1, \dots, v_k \in \mathbb{N} \\ v_1 + \dots + v_k = n}} \frac{G_{v_1} \cdots G_{v_k}}{(v_1 \cdots v_k) v_1! \cdots v_k!}. \quad (3.6)$$

By (3.6) and (2.6), we may immediately obtain Theorem 1.1. This completes the proof of Theorem 1.1. \square

Proof of Theorem 1.4. By (2.1) and note the identity

$$2\left(\frac{2}{e^{2t}+1}\right)^x + \frac{1}{x} \frac{d}{dt} \left(\frac{2}{e^{2t}+1}\right)^x = \left(\frac{2}{e^{2t}+1}\right)^{x+1}, \quad (3.7)$$

we have

$$G_n^{(x+1)} = 2G_n^{(x)} + \frac{1}{x} G_{n+1}^{(x)}. \quad (3.8)$$

By (3.8), (1.7), and note that $G_n^{(1)} = 2^n / (n+1) G_{n+1}$, we obtain

$$\begin{aligned} G_n^{(k)} &= \frac{1}{(k-1)!} \sum_{j=0}^{k-1} (-1)^j 2^j s(k, k-j) G_{n+k-1-j}^{(1)} \\ &= \frac{2^{n+k-1}}{(k-1)!} \sum_{j=0}^{k-1} (-1)^j s(k, k-j) \frac{G_{n+k-j}}{n+k-j}. \end{aligned} \quad (3.9)$$

Comparing (3.9) and (2.8), we immediately obtain Theorem 1.4. This completes the proof of Theorem 1.4. \square

Proof of Theorem 1.6. By Lemma 2.3, we have

$$G_n^{(m+x)} = \sum_{j=1}^n \omega(n, j) (m+x)^j \equiv \sum_{j=1}^n \omega(n, j) x^j = G_n^{(x)} \pmod{m}. \quad (3.10)$$

Therefore

$$G_n^{(k)} = G_n^{(m+k-m)} \equiv G_n^{(-m)} = 2^{n-m} \sum_{j=0}^m \binom{m}{j} j^n \pmod{m+k}. \quad (3.11)$$

Taking $k = 1$ in (3.11) and note that $G_n^{(1)} = 2^n / (n+1) G_{n+1}$, we immediately obtain Theorem 1.6. This completes the proof of Theorem 1.6. \square

Acknowledgments

The author would like to thank the anonymous referee for valuable suggestions. This work was supported by the Guangdong Provincial Natural Science Foundation (no. 8151601501000002).

References

- [1] F. T. Howard, "Applications of a recurrence for the Bernoulli numbers," *Journal of Number Theory*, vol. 52, no. 1, pp. 157–172, 1995.
- [2] J. Riordan, *Combinatorial Identities*, John Wiley & Sons, New York, NY, USA, 1968.
- [3] I. N. Cangul, H. Ozden, and Y. Simsek, "A new approach to q -Genocchi numbers and their interpolation functions," *Nonlinear Analysis: Theory, Methods & Applications*. In press.
- [4] M. Cenkci, M. Can, and V. Kurt, " q -extensions of Genocchi numbers," *Journal of the Korean Mathematical Society*, vol. 43, no. 1, pp. 183–198, 2006.
- [5] L. Jang and T. Kim, " q -Genocchi numbers and polynomials associated with fermionic p -adic invariant integrals on \mathbb{Z}_p ," *Abstract and Applied Analysis*, vol. 2008, Article ID 232187, 8 pages, 2008.
- [6] T. Kim, "A note on the q -Genocchi numbers and polynomials," *Journal of Inequalities and Applications*, vol. 2007, Article ID 71452, 8 pages, 2007.
- [7] T. Kim, "On the q -extension of Euler and Genocchi numbers," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 1458–1465, 2007.
- [8] T. Kim, L.-C. Jang, and H. K. Pak, "A note on q -Euler and Genocchi numbers," *Proceedings of the Japan Academy. Series A*, vol. 77, no. 8, pp. 139–141, 2001.
- [9] T. Kim, "On the multiple q -Genocchi and Euler numbers," *Russian Journal of Mathematical Physics*, vol. 15, no. 4, pp. 481–486, 2008.
- [10] G. D. Liu and R. X. Li, "Sums of products of Euler-Bernoulli-Genocchi numbers," *Journal of Mathematical Research and Exposition*, vol. 22, no. 3, pp. 469–475, 2002 (Chinese).
- [11] H. Ozden and Y. Simsek, "A new extension of q -Euler numbers and polynomials related to their interpolation functions," *Applied Mathematics Letters*, vol. 21, no. 9, pp. 934–939, 2008.
- [12] H. Ozden, I. N. Cangul, and Y. Simsek, "Multivariate interpolation functions of higher-order q -Euler numbers and their applications," *Abstract and Applied Analysis*, vol. 2008, Article ID 390857, 16 pages, 2008.
- [13] S.-H. Rim, K. H. Park, and E. J. Moon, "On Genocchi numbers and polynomials," *Abstract and Applied Analysis*, vol. 2008, Article ID 898471, 7 pages, 2008.
- [14] C. S. Ryoo, "A numerical computation on the structure of the roots of q -extension of Genocchi polynomials," *Applied Mathematics Letters*, vol. 21, no. 4, pp. 348–354, 2008.
- [15] C. S. Ryoo, "A numerical investigation on the structure of the roots of q -Genocchi polynomials," *Journal of Applied Mathematics and Computing*, vol. 26, no. 1-2, pp. 325–332, 2008.
- [16] Y. Simsek, I. N. Cangul, V. Kurt, and D. Kim, " q -Genocchi numbers and polynomials associated with q -Genocchi-type l -functions," *Advances in Difference Equations*, vol. 2008, Article ID 815750, 12 pages, 2008.
- [17] Y. Simsek, "Complete sums of products of (h, q) -extension of Euler numbers and polynomials," <http://arxiv.org/abs/0707.2849>.



Hindawi

Submit your manuscripts at
<http://www.hindawi.com>

