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On Bernoulli identities and applications

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Part I

Abstract. Bernoulli numbers appear as special values of zeta functions at integers and identities relating the Bernoulli numbers follow as a consequence of properties of the corresponding zeta functions. The most famous example is that of the special values of the Riemann zeta function and the Bernoulli identities due to Euler. In this paper we introduce a general principle for producing Bernoulli identities and apply it to zeta functions considered by Shintani, Zagier and Eie. Our results include some of the classical results of Euler and Ramanujan. Kummer's congruences play important roles in the investigation of p-adic interpolation of the classical Riemann zeta function. It asserts congruence relations among Bernoulli numbers, *i.e.*

$$(1-p^{m-1})\,\frac{B_m}{m} \equiv (1-p^{n-1})\,\frac{B_n}{n} \pmod{p^{N+1}}$$

if $m \equiv n \pmod{(p-1)\,p^N}$ and $(p-1)$ is not a divisor of m. In the second part of this paper, we use a simple Bernoulli identity to prove that

$$\begin{aligned}(1-p^{m-1})\,&\frac{B_m}{m}\\ &\equiv \frac{p^{-(N+1)}}{m} \sum_{\substack{(j,p)=1\\ 1\le j<p^{N+1}}} j^m - \frac{1}{2} \sum_{\substack{(j,p)=1\\ 1\le j<p^{N+1}}} j^{m-1} \pmod{p^{N+1}}\,.\end{aligned}$$

We then deduce from this Kummer's congruence by using von Staudt's theorem and Euler's generalization of Fermat's theorem

$$a^m \equiv a^n \pmod{p^{N+1}},$$

if a is relative prime to p and $m \equiv n \pmod{(p-1)\,p^N}$. Our argument can be applied to derive congruences among Bernoulli polynomials and in general the special values at negative integers of zeta functions associated with rational functions considered by Eie.

1. Introduction.

Let $m_1, \ldots, m_r$ be positive integers and $P(T)$ be a polynomial in T with complex coefficients of degree less than $m_1 + \cdots + m_r$. For $|T| < 1$, we let

$$F(T) = \frac{P(T)}{(1-T^{m_1})\cdots(1-T^{m_r})} = \sum_{k=0}^{\infty} a(k)\,T^k.$$

Such functions occur as generating functions of partition numbers (*cf.* Hardy and Wright [5, Chapter XIX]) and dimensions of spaces of automorphic forms – *e.g.* if we let $a(k)$ be the dimension of the space of Siegel modular forms of genus 2 and weight k, then

$$\sum_{k=0}^{\infty} a(k)\,T^k = \frac{1+T^{35}}{(1-T^4)\,(1-T^6)\,(1-T^{10})\,(1-T^{12})}$$

(*cf.* Igusa [6]). The value of $a(k)$ is determined by F via the residue theorem as

$$a(k) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{F(z)\,dz}{z^{k+1}},$$

where $\mathcal{C}$ is a sufficiently small circle centered at the origin going counterclockwise.

The generating function of the numbers $a(k)$ is the Dirichlet series

$$Z_F(s) = \sum_{k=1}^{\infty} a(k)\,k^{-s}$$

(*cf.* Hardy and Wright [5, Chapter XVII]). This zeta function is related to $F(T)$ via a Mellin transform

$$Z_F(s)\,\Gamma(s) = \int_0^{\infty} t^{s-1}\,(F(e^{-t}) - F(0))\,dt,$$

for $\operatorname{Re} s$ sufficiently large. Our underlying principle is to evaluate $F(T)$ in two ways, yielding a Bernoulli identity, with special values of the zeta functions of Shintani [8], Zagier [9] and Eie [2], [3] on the one hand, the special values of classical zeta functions of Riemann and Hurwitz and sums of residues on the other. One gets easily this way Euler's identity: if $n \geq 2$,

$$\sum_{k=1}^{n-1} \frac{(2\,n)!}{(2\,k)!\,(2\,n-2\,k)!}\, B_{2k}\, B_{2n-2k} = -(2\,n+1)\, B_{2n}\ ,$$

(*cf.* [1, Part I, p. 122]) and Ramanujan's identities ($\alpha, \beta > 0$ with $\alpha\,\beta = \pi^2$),

1) if $n > 1$,

$$\alpha^n \sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{2\alpha k}-1} - (-\beta)^n \sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{2\beta k}-1} = (\alpha^n - (-\beta)^n)\, \frac{B_{2n}}{4\,n}\ ,$$

2) if $n \in \mathbb{Z}$,

$$\alpha^{-n}\Big(\frac{1}{2}\,\zeta(2\,n+1) + \sum_{k=1}^{\infty} \frac{k^{-2n-1}}{e^{2\alpha k}-1}\Big) - (-\beta)^{-n}\Big(\frac{1}{2}\,\zeta(2\,n+1) + \sum_{k=1}^{\infty} \frac{k^{-2n-1}}{e^{2\beta k}-1}\Big) = -2^{2n} \sum_{k=0}^{n+1} (-1)^k\, \frac{B_{2k}}{(2\,k)!}\, \frac{B_{2n+2-2k}}{(2\,n+2-2\,k)!}\, \alpha^{n+1-k}\, \beta^k\ ,$$

3) if $n \geq 1$,

$$\alpha^{-n} \sum_{k=1}^{\infty} (-1)^{k+1}\, \frac{\operatorname{csch}(\alpha\, k)}{k^{2n+1}} - (-\beta)^{-n} \sum_{k=1}^{\infty} (-1)^{k+1}\, \frac{\operatorname{csch}(\beta\, k)}{k^{2n+1}} = 2^{2n+1} \sum_{k=0}^{n+1} (-1)^k\, \frac{B_{2k}\Big(\frac{1}{2}\Big)}{(2\,k)!}\, \frac{B_{2n+2-2k}\Big(\frac{1}{2}\Big)}{(2\,n+2-2\,k)!}\, \alpha^{n+1-k}\, \beta^k\ ,$$

(*cf.* [1, Part II, Chapter 14]).

In the first part of this paper we present some new Bernoulli identities. In view of the current motivic interest in special values of zeta functions, one cannot help from wondering if there is an abstract framework giving a unified explanation of these identities as in the case of polylogarithms (*cf.* Zagier [10]).

In the second part of the paper the Bernoulli identities are used to give new proofs of classical Kummer congruences. The Bernoulli numbers B_n $(n = 0, 1, 2, \ldots)$ and Bernoulli polynomials $B_n(x)$ $(n = 0, 1, 2, \ldots)$ are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n\, t^n}{n!}\,, \qquad |t| < 2\pi\,,$$

and

$$\frac{t\, e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)\, t^n}{n!}\,, \qquad |t| < 2\pi\,.$$

Suppose that m, n are positive even integers, p is an odd prime with $p-1$ not a divisor of m and N is a non-negative integer. Kummer's congruences asserted that if

$$m \equiv n \pmod{(p-1)\, p^N}\,,$$

then

$$(1 - p^{m-1})\, \frac{B_m}{m} \equiv (1 - p^{n-1})\, \frac{B_n}{n} \pmod{p^{N+1}}\,.$$

Kummer's congruences play important roles in the p-adic interpolation of the classical Riemman zeta function. Indeed if we consider the function

$$\zeta_p(s) = (1 - p^{-s})\, \zeta(s) = \sum_{\substack{n=1 \\ (n,p)=1}} n^{-s}\,, \qquad \operatorname{Re} s > 1\,.$$

Then the congruences tell us that $\zeta_p(s)$ is a continuous function on the ring of p-adic integers $\mathbb{Z}_p$, *i.e.*,

$$\zeta_p(1-m) \equiv \zeta_p(1-n) \pmod{p^{N+1}}\,,$$

if $m \equiv n \pmod{(p-1)\, p^N}$.

One can construct a p-adic measure μ on $\mathbb{Z}_p$ and express $\zeta_p(1-m)$ as a constant multiple of the p-adic integration

$$\int x^{m-1}\, d\mu(x)\,,$$

where the integration is over $\mathbb{Z}_p^*$ (see for example Koblitz [9]). Note that for $x \in (\mathbb{Z}/p^{N+1}\mathbb{Z})^*$, the set of invertible elements of the quotent ring $\mathbb{Z}/p^{N+1}\mathbb{Z}$, one has

$$x^{m-1} \equiv x^{n-1} \pmod{p^{N+1}},$$

if

$$m \equiv n \pmod{(p-1)\,p^N}.$$

So that Kummer's congruences follow as easy consequences by a simple argument (*cf.* [6]).

Here we shall develop another elementary proof of Kummer's congruences by a simple identity among Riemann zeta function and Hurwitz zeta functions,

$$\text{(I)} \qquad (1-p^{-s})\,\zeta(s) = p^{-(N+1)s} \sum_{\substack{(j,p)=1 \\ 1 \le j < p^{N+1}}} \zeta\Big(s; \frac{j}{p^{N+1}}\Big),$$

where the Hurwitz zeta function is defined as

$$\zeta(s;\delta) = \sum_{n=0}^{\infty} (n+\delta)^{-s}, \qquad \operatorname{Re} s > 1,\ \delta > 0.$$

Such an identity follows easily from the consideration of zeta functions associated with rational functions of the form

$$F(T) = \frac{P(T)}{(1-T^{m_1})\cdots(1-T^{m_r})}$$

(see Part I).

Note that both the Riemann zeta function $\zeta(s)$ and Hurwitz zeta function $\zeta(s;\delta)$ have analytic continuations in the whole complex plane. Moreover, their special values at non-positive integers are given by Bernoulli numbers and Bernoulli polynomials, respectively. Specifically, one has

$$\zeta(1-m) = (-1)^{m-1}\,\frac{B_m}{m} \qquad \text{and} \qquad \zeta(1-m;\delta) = -\frac{B_m(\delta)}{m}.$$

Set $s = 1-m$ in the identity (I), we get

$$\text{(II)} \qquad (1-p^{m-1})\,\frac{B_m}{m} = \frac{1}{m} \sum_{\substack{(j,p)=1 \\ 1 \le j < p^{N+1}}} \sum_{l=0}^{m} \binom{m}{l} B_l\, j^{m-l}\, p^{(N+1)(l-1)}.$$

Here

$$\binom{m}{l} = \frac{m!}{l!\,(m-l)!}$$

is the binomial coefficient.

On the other hand, von Staudt's theorem ([2, Chapter 5, Theorem 4]) implies that pB_l is alway p-integral, *i.e.* it contains no divisor of p in the denominator of pB_l. So after modulo p^{N+1}, we get

$$\text{(III)}\qquad \begin{aligned}(1-p^{m-1})\,\frac{B_m}{m} \equiv \frac{1}{m} & \sum_{\substack{(j,p)=1\\ 1\le j<p^{N+1}}} j^m\, p^{-(N+1)} \\ & - \frac{1}{2} \sum_{\substack{(j,p)=1\\ 1\le j<p^{N+1}}} j^{m-1} \pmod{p^{N+1}}\,.\end{aligned}$$

Next we evaluate the sum

$$\sum_{\substack{(j,p)=1\\ 1\le j<p^{N+1}}} j^m\,,$$

in the multiplicative group $(\mathbb{Z}/p^{N+1}\,\mathbb{Z})^*$ by decomposing it into a direct product of finite cyclic groups and we obtain Kummer's congruences by assuming von Staudt's Theorem; finally we give a proof of von Staudt's theorem by using the Bernoulli identity (II) with $N=0$.

At the end of the paper we extend Kummer's congruences on Bernoulli numbers to congruences on Bernoulli polynomials.

2. Special values of zeta functions.

2.1. Bernoulli numbers and Bernoulli polynomials.

We recall some results on special values of zeta functions.

For the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}\,, \qquad \operatorname{Re} s > 1$$

and the Hurwitz zeta function

$$\zeta(s;\delta)=\sum_{n=0}^{\infty}(n+\delta)^{-s}\,, \qquad \delta>0\,,\ \mathrm{Re}\, s>1\,,$$

it is well known that for an integer $m\ge 0$,

$$\zeta(-m)=(-1)^m\,\frac{B_{m+1}}{m+1} \qquad \text{and} \qquad \zeta(-m;\delta)=-\frac{B_{m+1}(\delta)}{m+1}\,.$$

2.2. Zeta functions associated with linear forms.

Let $\beta=(\beta_1,\ldots,\beta_r)$ be an r-tuple of nonnegative integers and $L(x)=a_1\,x_1+\cdots+a_r\,x_r+\delta$ be a linear form with

$$\mathrm{Re}\, a_j>0 \qquad \text{and} \qquad \mathrm{Re}\,\Big(\delta+\sum_{j=1}^{r}a_j\Big)>0\,.$$

For $\mathrm{Re}\, s>r+|\beta|$, define the zeta function associated with L as

$$\begin{aligned}Z(L,\beta,s)&=\sum_{n\in\mathbb{N}^r}n^{\beta}L(n)^{-s}\\&=\sum_{n_1=1}^{\infty}\cdots\sum_{n_r=1}^{\infty}n_1^{\beta_1}\cdots n_r^{\beta_r}\,(a_1\,n_1+\cdots+a_r\,n_r+\delta)^{-s}\,,\end{aligned}$$

where we use the notation $n^{\beta}=n_1^{\beta_1}\cdots n_r^{\beta_r}$.

These zeta functions were first considered in more general context by Eie in [2]. In particular, they have meromorphic continuations in the whole complex s-plane. Furthermore, their special values at non-positive integers are given explicitly there. Here we summarize the results we need from [3].

For any polynomial $f(x)$ of p variables and degree k

$$f(x)=\sum_{|\alpha|=0}^{k}a_{\alpha}\,x_1^{\alpha_1}\cdots x_p^{\alpha_p}\,,$$

we let

$$J^p(f(x))=\sum_{|\alpha|=0}^{k}a_{\alpha}\,\zeta(-\alpha_1)\cdots\zeta(-\alpha_p)=\sum_{|\alpha|=0}^{k}a_{\alpha}\prod_{j=1}^{p}\frac{(-1)^{\alpha_j}B_{\alpha_j+1}}{\alpha_j+1}\,,$$

where $\alpha = (\alpha_1, \ldots, \alpha_p)$ ranges over all p-tuples of non-negative integers and $|\alpha| = \alpha_1 + \cdots + \alpha_p$.

Also for any nonempty subset S of the index set $I = \{1, 2, \ldots, r\}$, we let

$$L_S(x) = \sum_{i \in I-S} a_i\, x_i + \delta = L(x) - \sum_{j \in S} a_j\, x_j$$

and $|S|$ be the cardinal number of S.

The following proposition is an immediate consequence of the main theorem in [3].

Proposition 1. *For any integer $m \geq 0$, the special value at $s = -m$ of $Z(L, \beta; s)$ is given by*

$$\begin{aligned} Z(L, \beta; -m) = J^r(x^\beta L^m(x)) \\ + \sum_S \Big(\prod_{j \in S} \frac{(-1)^{\beta_j + 1}\, \beta_j!}{a_j^{\beta_j + 1}} \Big) \frac{1}{\alpha(S)!}\, J^{r-|S|} \\ \cdot \Big(\prod_{i \notin S} x_i^{\beta_i}\, L_S^{\alpha(S)}(x) \Big), \end{aligned}$$

where S ranges over all non-empty subset of $I = \{1, 2, \ldots, r\}$ in the summation and

$$\alpha(S) = m + |S| + \sum_{j \in S} \beta_j\, .$$

Here we describe the analytic continuation of $Z(L, \beta; s)$. For $\operatorname{Re} s > r + |\beta|$, we have

$$\begin{aligned} Z(L, \beta; s)\, \Gamma(s) \\ &= \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} n_1^{\beta_1} \cdots n_r^{\beta_r} \int_0^\infty t^{s-1}\, e^{-(a_1 n_1 + \cdots + a_r n_r + \delta)t}\, dt \\ &= \int_0^\infty e^{-\delta t} \prod_{j=1}^{r} \Big(\sum_{n=1}^{\infty} n^{\beta_j}\, e^{-a_j n t} \Big)\, dt\, . \end{aligned}$$

Set

$$F_j(t) = \sum_{n=1}^{\infty} n^{\beta_j}\, e^{-a_j n t} \qquad \text{and} \qquad F(t) = e^{-\delta t} \prod_{j=1}^{r} F_j(t)\, .$$

A term by term differentiation of the identity

$$\sum_{n=1}^{\infty} e^{-a_j n t} = \frac{1}{e^{a_j t} - 1}\,, \qquad t > 0\,,$$

we get

$$F_j(t) = (-a_j)^{-\beta_j} \Big(\frac{d}{dt}\Big)^{\beta_j} \Big(\frac{1}{e^{a_j t} - 1}\Big)\,.$$

Thus around $t = 0$, $F_j(t)$ has the asymptotic expansion

$$\frac{\beta_j!}{(a_j\, t)^{\beta_j+1}} + (-1)^{\beta_j} \sum_{n_j \geq \beta_j+1} \frac{B_n (a_j\, t)^{n-\beta_j-1}}{n(n-\beta_j-1)!}\,.$$

It follows that at $t = 0$, $F(t)$ has an asymptotic expansion of the form

$$\sum_{n \geq -(|\beta|+r)} C_n\, t^n\,.$$

Consequently, the analytic continuation of $Z(L, \beta; s)$ and its special values at negative integers follow from Lemma 7 in Section 4.

When $\beta = 0$, we have the following

Corollary. *For any integer $m \geq r$, one has*

$$Z(L, 0; r-m) \\ = \sum_{|\alpha|=m} \frac{(-1)^{m-r-\alpha_{r+1}}\,(m-r)!}{\alpha_1! \cdots \alpha_r!\, \alpha_{r+1}!}\, B_{\alpha_1} \cdots B_{\alpha_r}\, a_1^{\alpha_1-1} \cdots a_r^{\alpha_r-1}\, \delta^{\alpha_{r+1}}\,.$$

2.3. Shintani zeta functions.

Next we consider another kind of zeta function which were investigated first by Shintani in [8] and then Eie in [3]. Here we reformulate the main result in [3].

Let $A = (a_1, \ldots, a_r)$ and $u = (u_1, \ldots, u_r)$ be r-tuples of complex numbers such that $\mathrm{Re}\, a_j > 0$ and $u_j > 0$. Define the zeta function

$$Z(A, u; s) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} (a_1(n_1+u_1) + \cdots + a_r(n_r+u_r))^{-s}\,,$$

where $\operatorname{Re} s > r$.

Proposition 2. *For any integer $m \geq r$, one has*

$$Z(A,u;r-m) \\ = (-1)^r \sum_{|p|=m} \frac{(m-r)!}{p_1!\cdots p_r!} B_{p_1}(u_1)\cdots B_{p_r}(u_r)\, a_1^{p_1-1}\cdots a_r^{p_r-1}\,.$$

Here the summation is over all p-tuples of non-negative integers such that and $|p| = p_1 + \cdots + p_r = m$.

3. Euler's Identity.

If we start from the fraction

$$F(T) = \frac{1}{(1-T)^2} = \sum_{k=0}^{\infty}(k+1)\,T^k\,,$$

we obtain the identity

$$\zeta(s-1)+\zeta(s) = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}(n_1+n_2)^{-s} + 2\,\zeta(s)\,,$$

from the Dirichlet series $Z_F(s)$. Setting $s = 2-2\,n$, we get Euler's identity

$$\sum_{k=1}^{n-1} \frac{(2\,n)!}{(2\,k)!\,(2\,n-2\,k)!} B_{2k}\,B_{2n-2k} = -(2\,n+1)\,B_{2n}\,, \qquad n \geq 2\,.$$

In this section we shall establish a new identity analogous to that of Euler and then as an illustration of our method we give an extension of the Euler identity to Bernoulli polynomials. We state a lemma.

Lemma 3. *Given*

$$P(T) = \sum_{j=0}^{m} b_j\,T^j$$

and

$$F(T) = \frac{P(T)}{(1-T^{m_1})\cdots(1-T^{m_r})}$$

Note that k is relative prime to p, so the mapping $x \longmapsto k\,x+\alpha$ is an one to one mapping from $\mathbb{Z}/p^{N+1}\,\mathbb{Z}$ into $\mathbb{Z}/p^{N+1}\,\mathbb{Z}$. Thus we have

$$\sum_{j\equiv\alpha \pmod k} j^m \equiv \sum_{\substack{1\le j<p^{N+1}\\ (j,p)=1}} j^m \pmod{p^{N+1}}.$$

Hence our congruences follow by the same argument as in Proposition 13.

REMARK. It is possible to construct another p-adic measure on the space $\mathbb{Z}_p$ so that the integration of the monomial x^{m-1} over $\mathbb{Z}_p^*$ yields a sum of Bernoulli polynomials. Hence, we have the p-adic interpolation of Kummer's congruences on Bernoulli polynomials. We'll discuss this in another paper.

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