Identities and generating functions on Chebyshev polynomials

Clemente Cesarano

Faculty of Engineering, International Telematic University UNINETTUNO
Corso Vittorio Emanuele II, 39 – 00186 – Roma, Italy
email: c.cesarano@uninettunouniversity.net

ABSTRACT – In this article we present the classical theory of Chebyshev polynomials starting from the definition of a family of complex polynomials, including both the first and second kind classical Chebyshev ones, which are related to its real and imaginary part. This point of view permits to derive a lot of generating functions and relations between the two kinds Chebyshev families, which are essentially new, as exponential generating functions, bilinear and bilinear exponential generating functions. We also deduce relevant relations of products of Chebyshev polynomials and the related generating functions.

Keywords: Chebyshev polynomials, Mixed type generating functions.

2000 Mathematics Subject Classification: 33C45.

1 Introduction

The Chebyshev polynomials and their nice properties permitting applications in Approximation theory, Quadrature rules, etc. are well known [1], [2], [3]. In this article the introduction of the twin families of the first and second kind Chebyshev polynomials is performed starting by their complex representation. This original point of view was first applied by G. Dattoli and his group in [4] and [5].

This allows to derive, in a constructive way, many identities involving unusual generating functions of exponential, bilinear, and mixed type [6].

There are a number of distinct families of polynomials that go by the name of Chebyshev Polynomials. The Chebyshev Polynomials *par excellence* can be defined by:

Definition 1 – Let x a real variable, we say Chebyshev Polynomials of first kind, the polynomials defined by the following relation:

$$T_n(x) = \cos n(\arccos(x)). \tag{1.1}$$

In the same way we can also introduce the second kind Chebyshev polynomials, by using again the link with the circular functions.

Definition 2 – Let x a real variable, we say Chebyshev Polynomials of second kind, the polynomials

$$U_n(x) = \frac{\sin\left[(n+1)\arccos(x)\right]}{\sqrt{1-x^2}}. \tag{1.2}$$

The study of the properties of the Chebyshev polynomials can be simplified by introducing the following complex quantity:

$$\mathbf{T}_n(x) = \exp\left[in(\arccos(x)\right] \tag{1.3}$$

so that:

$$\begin{aligned} \mathrm{Re}\left[\mathbf{T}_n(x)\right] &= \cos n(\arccos(x)) \\ \mathrm{Im}\left[\mathbf{T}_n(x)\right] &= \sin n(\arccos(x)). \end{aligned} \tag{1.4}$$

The above relations can be recast directly in terms of the Chebyshev Polynomials of the first and second kind. In fact, by noting that the second kind Chebyshev polynomials of degree $n-1$ reads:

$$U_{n-1}(x) = \frac{\sin\left[n\arccos(x)\right]}{\sqrt{1-x^2}} \tag{1.5}$$

we can immediately conclude that:

$$\begin{aligned} T_n(x) &= \mathrm{Re}\left[\mathbf{T}_n(x)\right] \\ U_{n-1}(x) &= \frac{\mathrm{Im}\left[\mathbf{T}_n(x)\right]}{\sqrt{1-x^2}} \end{aligned} \tag{1.6}$$

2 Generating functions

To derive the related generating functions of the Chebyshev polynomials of the first and second kind [1], [6], we can consider the generating functions of the complex quantity, introduced in the (1.3); let, in fact, the real number ξ, such that $|\xi| < 1$, we can immediately write:

$$\sum_{n=0}^{+\infty} \xi^n \mathbf{T}_n(x) = \sum_{n=0}^{+\infty} \left(\xi e^{i\arccos(x)}\right)^n = \frac{1}{1-\xi e^{i\arccos(x)}}. \tag{2.1}$$

Proposition 1 – Let $\xi \in \mathbf{R}$, such that $|\xi| < 1$; the generating function of the first kind Chebyshev polynomials reads:

$$\sum_{n=0}^{+\infty} \xi^n T_n(x) = \frac{1-\xi x}{1-2\xi x+\xi^2}. \tag{2.2}$$

Proof – By using the link stated in equation (1.6) and by the (1.7), for a real number ξ, such that $|\xi| < 1$, we can write:

$$\sum_{n=0}^{+\infty} \xi^n T_n(x) = \sum_{n=0}^{+\infty} \xi^n \mathrm{Re}\left[\mathbf{T}_n(x)\right] = \mathrm{Re}\left[\frac{1}{1-\xi e^{i\arccos(x)}}\right]. \tag{2.3}$$

By manipulating the r.h.s. of the previous relation, we find:

$$\mathrm{Re}\left[\frac{1}{1-\xi e^{i\arccos(x)}}\right] = \mathrm{Re}\left\{\frac{[1-\xi\cos(\arccos(x))]+i\xi\sin(\arccos(x))}{[1-\xi\cos(\arccos(x))]^2+\xi^2\sin^2(\arccos(x))}\right\} \tag{2.4}$$

that is:

$$\mathrm{Re}\left[\frac{1}{1-\xi e^{i\arccos(x)}}\right] = \frac{1-\xi\cos(\arccos(x))}{1-2\xi\cos(\arccos(x))+\xi^2} \tag{2.5}$$

and then, we immediately obtain the (1.8).

By following the same procedure, we can also derive the related generating function for the Chebyshev polynomials $U_n(x)$.

It is easy in fact to note, from the second of (1.6) and (1.7) that:

$$\sum_{n=0}^{+\infty} \xi^n U_{n-1}(x) = \sum_{n=0}^{+\infty} \xi^n \frac{\mathrm{Im}\left[\mathbf{T}_n(x)\right]}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}\mathrm{Im}\left[\frac{1}{1-\xi e^{i\arccos(x)}}\right]. \tag{2.6}$$

By using the same manipulation exploited in the previous proposition, we end up with:

$$\sum_{n=0}^{+\infty} \xi^n U_{n-1}(x) = \frac{\xi}{1-2\xi x+\xi^2} \tag{2.7}$$

which is the generating function of the Chebyshev polynomials of second kind of degree $n-1$, with again $|\xi|<1$.

It is also possible to derive different generating functions for these families of Chebyshev polynomials, by using the complex quantity in (1.7). In fact by noting that:

$$\sum_{n=0}^{+\infty} \frac{\xi^n}{n!}\mathbf{T}_n(x) = \sum_{n=0}^{+\infty} \frac{1}{n!}\left(\xi e^{i\arccos(x)}\right)^n = \exp\left[\xi e^{i\arccos(x)}\right]$$

we have:

Proposition 2 – For the first and second kind Chebyshev polynomials, the following results hold:

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{\xi^n}{n!} T_n(x) &= e^{\xi x}\cos\left(\xi\sqrt{1-x^2}\right) \\ \sum_{n=0}^{+\infty} \frac{\xi^n}{n!} U_{n-1}(x) &= e^{\xi x}\frac{\sin\left(\xi\sqrt{1-x^2}\right)}{\sqrt{1-x^2}} \end{aligned} \tag{2.8}$$

where $|\xi|<1$.

Proof – From the identity:

$$\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} T_n(x) = \mathrm{Re}\left\{\exp\left[\xi e^{i\arccos(x)}\right]\right\} \tag{2.9}$$

after setting:

$$\psi = \arccos(x)$$

we can rearranging the r.h.s. in the following from:

$$\mathrm{Re}\left\{\exp\left[\xi\left(\cos(\psi)+i\sin(\psi)\right)\right]\right\} = \exp\left(\xi\cos(\psi)\right)\mathrm{Re}\left[\exp\left(i\xi\sin(\psi)\right)\right]. \tag{2.10}$$

By noting that:

$$\mathrm{Re}\left[\exp\left(i\xi\sin(\psi)\right)\right] = \mathrm{Re}\left[\cos\left(\xi\sin(\psi)\right) + i\sin\left(\xi\sin(\psi)\right)\right] = \cos\left(\xi\sin(\psi)\right) \tag{2.11}$$

we immediately obtain the first of the (1.14).

For the second kind Chebyshev polynomials, by using the complex quantity $\mathbf{T}_n(x)$, we write:

$$\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} U_{n-1}(x) = \frac{1}{\sqrt{1-x^2}} \mathrm{Im}\left\{\exp\left[\xi e^{i\arccos(x)}\right]\right\}. \tag{2.12}$$

By using the same position $\psi = \arccos(x)$, we can write the r.h.s. of the above identity in the form:

$$\frac{1}{\sqrt{1-x^2}} \mathrm{Im}\left\{\exp\left[\xi\left(\cos(\psi) + i\sin(\psi)\right)\right]\right\} = \frac{1}{\sqrt{1-x^2}} \exp\left(\xi\cos(\psi)\right)\sin\left(\xi\sin(\psi)\right) \tag{2.13}$$

and then the second of the (1.14) immediately follows.

The use of the complex representation of Chebyshev polynomials can be also exploited to derive less trivial relations involving first and second kind Chebyshev polynomials. From definition of the first kind Chebyshev polynomials, given in (1.1), we can generalize it, by putting:

$$T_{n+l}(x) = \left[\cos(n+l)\arccos(x)\right] \tag{2.14}$$

and, from the (1.3), we can immediately write:

$$\mathbf{T}_{n+l}(x) = \exp\left[i(n+l)\arccos(x)\right] \tag{2.15}$$

then:

$$\begin{aligned} \mathrm{Re}\left[\mathbf{T}_{n+l}(x)\right] &= T_{n+l}(x) \\ \mathrm{Im}\left[\mathbf{T}_{n+l}(x)\right] &= \frac{U_{n-1+l}(x)}{\sqrt{1-x^2}}. \end{aligned} \tag{2.16}$$

By using the same procedure exploited in the propositions 1 and 2 to derive the generating functions of the polynomials $T_n(x)$ and $U_n(x)$, help us to state the following results [4], [6].

Proposition 3 – Let $\xi \in \mathbf{R}$, such that $|\xi| < 1$; the following identities hold:

$$\sum_{n=0}^{+\infty} \xi^n T_{n+l}(x) = \frac{(1-\xi x)\, T_l(x) - \xi\left(1-x^2\right) U_{l-1}}{1-2\xi x+\xi^2} \tag{2.17}$$

and:

$$\sum_{n=0}^{+\infty} \xi^n U_{n-1+l}(x) = \frac{\xi T_l(x) + (1-\xi x)\, U_{l-1}}{1-2\xi x+\xi^2}. \tag{2.18}$$

Proof – From the previous results, it is easy to note that:

$$\sum_{n=0}^{+\infty} \xi^n \mathbf{T}_{n+l}(x) = \sum_{n=0}^{+\infty} \xi^n e^{in\arccos(x)} e^{il\arccos(x)} = e^{il\arccos(x)} \frac{1}{1-\xi e^{i\arccos(x)}}. \tag{2.19}$$

Otherwise:

$$e^{il\arccos(x)} = \cos(l\arccos(x)) + i\sin(\arccos(x)) \tag{2.20}$$

and so:

$$\sum_{n=0}^{+\infty} \xi^n T_{n+l}(x) = \sum_{n=0}^{+\infty} \xi^n \mathrm{Re}\left[\mathbf{T}_{n+l}(x)\right] = \mathrm{Re}\left[\frac{\cos(l\arccos(x)) + i\sin(\arccos(x))}{1-\xi e^{i\arccos(x)}}\right]. \tag{2.21}$$

The r.h.s. can be rearranged in the more convenient form:

$$\mathrm{Re}\left[\frac{\cos(l\arccos(x)) + i\sin(\arccos(x))}{1-\xi e^{i\arccos(x)}}\right] = \tag{2.22}$$

$$= \mathrm{Re}\left\{\frac{\left[\cos(l\arccos(x)) + i\sin(\arccos(x))\right]\left[1-\xi x + i\xi\sin(\arccos(x)\right]}{1-2\xi x+\xi^2}\right\}$$

to give:

$$\mathrm{Re}\left[\frac{\cos(l\arccos(x)) + i\sin(\arccos(x))}{1-\xi e^{i\arccos(x)}}\right] = \frac{(1-\xi x)\, T_l(x) - \xi\left(1-x^2\right) U_{l-1}}{1-2\xi x+\xi^2} \tag{2.23}$$

which prove the first statement.

In an analogous way, we note that:

$$\sum_{n=0}^{+\infty} \xi^n U_{n-1+l}(x) = \frac{1}{\sqrt{1-x^2}} \sum_{n=0}^{+\infty} \xi^n \mathrm{Im}\left[\mathbf{T}_{n+l}(x)\right] = \tag{2.24}$$

$$= \mathrm{Im}\left[\frac{\cos(l\arccos(x)) + i\sin(\arccos(x))}{1-\xi e^{i\arccos(x)}}\right]$$

and by following the same procedure, we immediately obtain the (1.24).

The corresponding generating functions stated in the *Proposition 2*, for the Chebyshev polynomials are also easily obtained.

Proposition 4 – For a real ξ, $|\xi| < 1$, the polynomials $T_n(x)$ and $U_n(x)$ satisfy the following relations:

$$\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} T_{n+l}(x) = e^{\xi x} \left[\cos\left(\xi\sqrt{1-x^2}\right) T_l(x) - \sqrt{1-x^2} \sin\left(\xi\sqrt{1-x^2}\right) U_{l-1}(x) \right] \tag{2.25}$$

and:

$$\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} U_{n-1+l}(x) = e^{\xi x} \left[\sqrt{1-x^2} \cos\left(\xi\sqrt{1-x^2}\right) U_{l-1}(x) + \sin\left(\xi\sqrt{1-x^2}\right) T_l(x) \right]. \tag{2.26}$$

Proof – From the (1.15) it follows that:

$$\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} T_{n+l}(x) = \mathrm{Re}\left[e^{il \arccos(x)} e^{\xi e^{i \arccos(x)}} \right] \tag{2.27}$$

or in a more convenient form, by setting $\psi = \arccos(x)$:

$$\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} T_{n+l}(x) = \mathrm{Re}\left\{ \left[\cos(l\psi) + i \sin(l\psi) \right] \left[e^{\xi \cos\psi} e^{\xi i \sin\psi} \right] \right\}. \tag{2.28}$$

By exploiting the r.h.s., we obtain:

$$\begin{aligned} &\mathrm{Re}\left\{ \left[\cos(l\psi) + i \sin(l\psi) \right] \left[e^{\xi \cos\psi} e^{\xi i \sin\psi} \right] \right\} = \\ &= \mathrm{Re}\Big\{ \cos(l\psi)\, e^{\xi \cos(\psi)} \left[\cos(\xi \sin(\psi)) + i \sin(\xi \sin(\psi)) \right] + \\ &+ \sin(l\psi)\, e^{\xi \cos(\psi)} \left[\cos(\xi \sin(\psi)) + i \sin(\xi \sin(\psi)) \right] \Big\} \end{aligned} \tag{2.29}$$

and then, after substituting the previous position of ψ:

$$\begin{aligned} &\mathrm{Re}\left\{ \left[\cos(l \arccos(x)) + i \sin(l \arccos(x)) \right] \left[e^{\xi x} e^{\xi i \sin(\arccos(x))} \right] \right\} = \\ &= e^{\xi x} \left[\cos\left(\xi\sqrt{1-x^2}\right) T_l(x) - \sqrt{1-x^2} \sin\left(\xi\sqrt{1-x^2}\right) U_{l-1}(x) \right] \end{aligned} \tag{2.30}$$

that is equation (1.31).

Regarding the second statement, we have:

$$\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} U_{n-1+l}(x) = \frac{1}{\sqrt{1-x^2}} \mathrm{Im}\left[e^{il\arccos(x)} e^{\xi e^{i\arccos(x)}} \right] \tag{2.31}$$

and it is easy, by following the same procedure previous outlined, to state the second identity (1.32).

In the next sections it will be shown that the simple method we have proposed in these introductory remarks offers a fairly important tool of analysis for wide classes of properties of the Chebyshev polynomials.

3 Products of Chebyshev polynomials.

In this section we will show some important identities related to the generating functions of products of Chebyshev polynomials. We permit the following results [1], [5].

Proposition 5 – For the polynomials $T_n(x)$ and $U_n(x)$ and for their complex representation $\mathbf{T}_n(x)$, the following identities are true:

$$\begin{aligned} |\mathbf{T}_n(x)|^2 &= [T_n(x)]^2 + \left(1-x^2\right)[U_{n-1}(x)]^2 = 1, \\ \mathrm{Re}\,[\mathbf{T}_n(x)]^2 &= [T_n(x)]^2 - \left(1-x^2\right)[U_{n-1}(x)]^2, \\ \mathrm{Im}\,[\mathbf{T}_n(x)]^2 &= 2\sqrt{1-x^2}\,T_n(x)U_{n-1}(x). \end{aligned} \tag{3.1}$$

Proof – By noting that:

$$|\mathbf{T}_n(x)|^2 = \mathrm{Re}\mathbf{T}_n(x)^2 + \mathrm{Im}\mathbf{T}_n(x)^2 \tag{3.2}$$

that is:

$$|\mathbf{T}_n(x)|^2 = [T_n(x)]^2 + \left(1-x^2\right)[U_{n-1}(x)]^2 . \tag{3.3}$$

After substituting the explicit forms of the polynomials $T_n(x)$ and $U_n(x)$, we obtain the first of the (2.1).

We can also note that:

$$[\mathbf{T}_n(x)]^2 = \left[T_n(x) + i\sqrt{1-x^2}U_{n-1}(x) \right]^2 \tag{3.4}$$

and by exploiting the r.h.s.:

$$[\mathbf{T}_n(x)]^2 = [T_n(x)]^2 + i2\sqrt{1-x^2}T_n(x)U_{n-1}(x) - \left(1-x^2\right)[U_{n-1}(x)]^2 \quad (3.5)$$

which once explicated as real and imaginary part allows us to recognize the second and the third identities of the statement.

From the (2.1) it also immediately follows that:

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{\xi^n}{n!} |\mathbf{T}_n(x)|^2 &= \exp(\xi) \quad (3.6) \\ \sum_{n=0}^{+\infty} \frac{\xi^n}{n!} |\mathbf{T}_n(x)|^2 &= \exp[\xi \exp(2i \arccos(x))] . \end{aligned}$$

The (2.1) and the (2.6) can be used to state further relations linking the Chebyshev polynomials of the first and second kind [1], [2], [6]. We have in fact:

Proposition 6 – The polynomials $T_n(x)$ and $U_n(x)$ satisfy the following identities:

$$\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} |T_n(x)|^2 = \frac{1}{2}\left[e^{\xi} + e^{\xi\left(2x^2-1\right)} \cos\left(2\xi x\sqrt{1-x^2}\right)\right] \quad (3.7)$$

and:

$$\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} |U_{n-1}(x)|^2 = \frac{1}{2\left(1-x^2\right)}\left[e^{\xi} - e^{\xi\left(2x^2-1\right)} \cos\left(2\xi x\sqrt{1-x^2}\right)\right] . \quad (3.8)$$

Proof – By summing term to term the first two identities of the (2.1), we have:

$$2T_n^2(x) = |\mathbf{T}_n(x)|^2 + \mathrm{Re}\mathbf{T}_n^2(x). \quad (3.9)$$

By multiplying both sides of the previous relation by $\frac{\xi^n}{n!}$ and then summing up, we find:

$$\sum_{n=0}^{+\infty} 2\frac{\xi^n}{n!} T_n^2(x) = \sum_{n=0}^{+\infty} \frac{\xi^n}{n!} |\mathbf{T}_n(x)|^2 + \sum_{n=0}^{+\infty} \frac{\xi^n}{n!} \mathrm{Re}\mathbf{T}_n^2(x) \quad (3.10)$$

and by using the (2.6), we can write:

$$\sum_{n=0}^{+\infty} 2\frac{\xi^n}{n!} T_n^2(x) = \exp(\xi) + \operatorname{Re}\exp\left[\xi \exp\left(2i \arccos(x)\right)\right]. \tag{3.11}$$

By expanding the r.h.s. of the above identity, we obtain:

$$\sum_{n=0}^{+\infty} 2\frac{\xi^n}{n!} T_n^2(x) = \exp(\xi) + \exp\left[\xi\left(2x^2-1\right)\right] \operatorname{Re}\left[\exp\left(i2\xi x\sqrt{1-x^2}\right)\right] \tag{3.12}$$

that is:

$$\sum_{n=0}^{+\infty} 2\frac{\xi^n}{n!} T_n^2(x) = \tag{3.13}$$

$$= \exp(\xi) + \exp\left[\xi\left(2x^2-1\right)\right] \operatorname{Re}\left[\cos\left(2\xi x\sqrt{1-x^2}\right) + i\sin\left(2\xi x\sqrt{1-x^2}\right)\right],$$

which proves equation (2.7).

The second identity of the statement can be derived in the same way; in fact it is enough to note, that by subtracting term to term the first two relations of the (2.1), we find:

$$2\left(1-x^2\right) U_{n-1}^2(x) = |\mathbf{T}_n(x)|^2 - \operatorname{Re}\mathbf{T}_n^2(x) \tag{3.14}$$

and by following the same procedure we obtain the (2.8).

The last equation of the (2.1) allows us to state the further identity:

$$\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} T_n(x) U_{n-1}(x) = \frac{\exp\left[\xi\left(2x^2-1\right)\right]}{2\sqrt{1-x^2}} \sin\left(2\xi x\sqrt{1-x^2}\right). \tag{3.15}$$

In the previous section we have derived different generating functions for the Chebyshev polynomials $T_n(x)$ and $U_{n-1}(x)$; we can generalize those results for their products. We firstly note that, from (2.1) and from the choice of ξ, $|\xi| < 1$, that:

$$\xi |\mathbf{T}_n(x)|^2 < 1$$

which implies:

$$\sum_{n=0}^{+\infty} \xi^n |\mathbf{T}_n(x)|^2 = \frac{1}{1-\xi}. \tag{3.16}$$

Otherwise, can also be noted that:

$$\mathbf{T}_n^2(x) = \left[\exp\left(i \arccos(x)\right)^n\right]^2 \leq 1$$

and since $|\xi| < 1$, it follows that:

$$\sum_{n=0}^{+\infty} \xi^n \mathbf{T}_n^2(x) = \frac{1}{1 - \xi \exp\left(2i \arccos(x)\right)}. \tag{3.17}$$

Proposition 7 – Let $\xi \in \mathbf{R}$, $|\xi| < 1$; the following identities hold:

$$\sum_{n=0}^{+\infty} \xi^n T_n^2(x) = \frac{1}{2} \frac{1}{1-\xi} \left[1 + \frac{(1-\xi)\left(1 - \xi\left(2x^2 - 1\right)\right)}{1 - 2\xi\left(2x^2 - 1\right) + \xi^2}\right] \tag{3.18}$$

and:

$$\sum_{n=0}^{+\infty} \xi^n U_{n-1}^2(x) = \frac{1}{2\left(1 - x^2\right)} \frac{1}{1-\xi} \left[1 - \frac{(1-\xi)\left(1 - \xi\left(2x^2 - 1\right)\right)}{1 - 2\xi\left(2x^2 - 1\right) + \xi^2}\right]. \tag{3.19}$$

Proof – By multiplying both sides of the (2.9) by ξ^n and then summing up, we find:

$$2\sum_{n=0}^{+\infty} \xi^n T_n^2(x) = \sum_{n=0}^{+\infty} \xi^n |\mathbf{T}_n(x)|^2 + \sum_{n=0}^{+\infty} \xi^n \mathrm{Re}\mathbf{T}_n^2(x) \tag{3.20}$$

and from the (2.16) and (2.17), we can write:

$$2\sum_{n=0}^{+\infty} \xi^n T_n^2(x) = \frac{1}{1-\xi} + \mathrm{Re}\left[\frac{1}{1 - \xi \exp\left(2i \arccos(x)\right)}\right]. \tag{3.21}$$

Let set $\psi = \arccos(x)$, the r.h.s. of the above relation can be recast in the form:

$$\frac{1}{1-\xi} + \mathrm{Re}\left[\frac{1}{1 - \xi \exp\left(2i \arccos(x)\right)}\right] = \frac{1}{1-\xi} + \mathrm{Re}\left[\frac{1 - \xi e^{-i\psi}}{\left(1 - \xi e^{i\psi}\right)\left(1 - \xi e^{-i\psi}\right)}\right]. \tag{3.22}$$

After exploiting the r.h.s., rewriting in terms of x, we obtain:

$$\frac{1}{1-\xi} + \mathrm{Re}\left[\frac{1}{1 - \xi \exp\left(2i \arccos(x)\right)}\right] = \frac{1}{1-\xi} + \frac{1 - \xi\left(2x^2 - 1\right)}{1 - 2\xi\left[\cos\left(2\arccos(x)\right)\right] + \xi^2} \tag{3.23}$$

which gives the (2.18).

From the equation written in the (1.14) and by using again the (2.16) and (2.17), we have:

$$2\left(1-x^2\right)\sum_{n=0}^{+\infty}\xi^n U_{n-1}^2(x)=\frac{1}{1-\xi}-\mathrm{Re}\left[\frac{1}{1-\xi\exp\left(2i\arccos(x)\right)}\right] \tag{3.24}$$

which once exploited gives us the (2.19).

4 Further generalizations

In this section we show further identities and generating functions involving products of first and second kind Chebyshev polynomials [1], [4].

By using the equation (2.15), we can state the further identity:

$$\sum_{n=0}^{+\infty}\xi^n T_n(x)U_{n-1}(x)=\frac{x\xi}{\left[1-2\xi\left(2x^2-1\right)+\xi^2\right]}. \tag{4.1}$$

In fact, by multiplying both sides of the third equation of the (2.1) by ξ^n and then summing up, we obtain:

$$2\sqrt{1-x^2}\sum_{n=0}^{+\infty}\xi^n T_n(x)U_{n-1}(x)=\mathrm{Im}\left[\frac{1}{1-\xi\exp\left(2i\arccos(x)\right)}\right] \tag{4.2}$$

which, by using the same procedure exploited in the above proposition, gives the (2.25).

In the first section (see (1.3)) we have introduced the complex quantity $\mathbf{T}_n(x)$ to better derive the properties of the Chebyshev polynomials $T_n(x)$ and $U_{n-1}(x)$. To deduce further properties involving generating functions of Chebyshev polynomials, we will indicate with $\overline{\mathbf{T}}_n(x)$ the complex conjugation of the Chebyshev representation $\mathbf{T}_n(x)$.

By using the identities stated in (2.1), we can immediately obtain:

$$\begin{aligned}\mathrm{Re}\left[\mathbf{T}_n(x)\overline{\mathbf{T}}_n(y)\right] &= T_n(x)T_n(y)+\sqrt{\left(1-x^2\right)\left(1-y^2\right)}U_{n-1}(x)U_{n-1}(y)\\ \mathrm{Im}\left[\mathbf{T}_n(x)\overline{\mathbf{T}}_n(y)\right] &= \sqrt{1-x^2}U_{n-1}(x)T_n(y)-\sqrt{1-y^2}U_{n-1}(y)T_n(x)\end{aligned}$$

and:

$$
\begin{aligned}
\mathrm{Re}\left[\mathbf{T}_n(x)\mathbf{T}_n(y)\right] &= T_n(x)T_n(y) - \sqrt{(1-x^2)(1-y^2)}U_{n-1}(x)U_{n-1}(y) \\
\mathrm{Im}\left[\mathbf{T}_n(x)\mathbf{T}_n(y)\right] &= \sqrt{1-x^2}U_{n-1}(x)T_n(y) - \sqrt{1-y^2}U_{n-1}(y)T_n(x).
\end{aligned}
$$

Proposition 8 – Let $\xi \in \mathbf{R}$, $|\xi| < 1$, the polynomials $T_n(x)$ and $U_n(x)$ satisfy the following identities:

$$
\begin{aligned}
\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} T_n(x)T_n(y) &= \frac{1}{2}\left[e^{\xi F_+}\cos(\xi G_-) + e^{\xi F_-}\cos(\xi G_+)\right] \qquad (4.3) \\
\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} U_{n-1}(x)U_{n-1}(y) &= -\frac{1}{2}\frac{\left[e^{\xi F_-}\cos(\xi G_+) + e^{\xi F_+}\cos(\xi G_-)\right]}{\sqrt{1-x^2}\,(1-y^2)}
\end{aligned}
$$

where:

$$
\begin{aligned}
F_\pm &= xy \pm \sqrt{(1-x^2)(1-y^2)}, \qquad (4.4) \\
G_\pm &= y\sqrt{1-x^2} \pm \sqrt{1-y^2}.
\end{aligned}
$$

Proof – From the relations in the (2.27) and (2.28), we find:

$$
2T_n(x)T_n(y) = \mathrm{Re}\left[\mathbf{T}_n(x)\overline{\mathbf{T}}_n(y)\right] + \mathrm{Re}\left[\mathbf{T}_n(x)\mathbf{T}_n(y)\right]. \qquad (4.5)
$$

By multiplying both sides by $\frac{\xi^n}{n!}$ and summing up, after setting $\psi = \arccos(x)$, $\phi = \arccos(y)$, it follows that:

$$
2\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} T_n(x)T_n(y) = \mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{-i\phi}\right)\right)\right] + \mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{i\phi}\right)\right)\right]. \qquad (4.6)
$$

By exploiting the r.h.s of the above equation we obtain:

$$
\begin{aligned}
&\mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{-i\phi}\right)\right)\right] + \mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{i\phi}\right)\right)\right] = \\
&= \mathrm{Re}\left\{\exp\left[\xi\left(\cos\psi + i\sin\psi\right)\left(\cos\phi - i\sin\phi\right)\right]\right\} + \qquad (4.7) \\
&+ \mathrm{Re}\left\{\exp\left[\xi\left(\cos\psi + i\sin\psi\right)\left(\cos\phi + i\sin\phi\right)\right]\right\}
\end{aligned}
$$

which gives, after substituting the values of x and y:

$$
\begin{aligned}
&\mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{-i\phi}\right)\right)\right] + \mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{i\phi}\right)\right)\right] \\
&= \mathrm{Re}\left\{\exp\left[\xi\left(xy - ix\sqrt{1-y^2} + iy\sqrt{1-x^2} + \sqrt{1-x^2}\sqrt{1-y^2}\right)\right]\right\} + \\
&+ \mathrm{Re}\left\{\exp\left[\xi\left(xy + ix\sqrt{1-y^2} + iy\sqrt{1-x^2} - \sqrt{1-x^2}\sqrt{1-y^2}\right)\right]\right\}. \qquad (4.8)
\end{aligned}
$$

By using the identities in (2.30), the above relation can be recast in the more convenient form:

$$\begin{aligned}
&\mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{-i\phi}\right)\right)\right]+\mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{i\phi}\right)\right)\right]=\\
&=e^{\xi F_+}\mathrm{Re}\left[\cos\left(\xi y\sqrt{1-x^2}\right)\cos\left(\xi x\sqrt{1-y^2}\right)-i\cos\left(\xi y\sqrt{1-x^2}\right)\sin\left(\xi x\sqrt{1-y^2}\right)+\right.\\
&\left.+\,i\cos\left(\xi x\sqrt{1-y^2}\right)\sin\left(\xi y\sqrt{1-x^2}\right)+\sin\left(\xi y\sqrt{1-x^2}\right)\sin\left(\xi x\sqrt{1-y^2}\right)\right]+\\
&+e^{\xi F_-}\mathrm{Re}\left[\cos\left(\xi x\sqrt{1-y^2}\right)\cos\left(\xi y\sqrt{1-x^2}\right)+i\cos\left(\xi x\sqrt{1-y^2}\right)\sin\left(\xi y\sqrt{1-x^2}\right)+\right.\\
&\left.+\,i\cos\left(\xi y\sqrt{1-x^2}\right)\sin\left(\xi x\sqrt{1-y^2}\right)-\sin\left(\xi x\sqrt{1-y^2}\right)\sin\left(\xi y\sqrt{1-x^2}\right)\right].
\end{aligned}$$

By remembering that:

$$\begin{aligned}
\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) &= \cos(\alpha+\beta)\\
\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta) &= \cos(\alpha-\beta)
\end{aligned}$$

we can rearrange the r.h.s. of the above equation in the form:

$$\mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{-i\phi}\right)\right)\right]+\mathrm{Re}\left[\exp\left(\xi\left(e^{i\psi}e^{i\phi}\right)\right)\right]= \tag{4.9}$$

$$=e^{\xi F_+}\cos\left[\xi\left(y\sqrt{1-x^2}-x\sqrt{1-y^2}\right)\right]+e^{\xi F_-}\cos\left[\xi\left(y\sqrt{1-x^2}+x\sqrt{1-y^2}\right)\right]$$

and immediately follows the first one of the (2.29).

By using again the relations (2.27) and (2.28), we can write:

$$2\sqrt{\left(1-x^2\right)\left(1-y^2\right)}U_{n-1}(x)U_{n-1}(y)= \tag{4.10}$$

$$=\mathrm{Re}\left[\mathbf{T}_n(x)\overline{\mathbf{T}}_n(y)\right]-\mathrm{Re}\left[\mathbf{T}_n(x)\mathbf{T}_n(y)\right]$$

which, once following the same procedure previous exploited, gives:

$$2\sqrt{\left(1-x^2\right)\left(1-y^2\right)}\sum_{n=0}^{+\infty}\frac{\xi^n}{n!}U_{n-1}(x)U_{n-1}(y)=\exp\left[\xi\left(e^{i\psi}e^{-i\phi}\right)\right]-\exp\left[\xi\left(e^{i\psi}e^{i\phi}\right)\right] \tag{4.11}$$

and then, the second of the (2.29) can easily be derived.

These results can be used to find similar identities linking products of the polynomials $T_n(x)$ and $U_n(x)$. We note in fact that the relations written in the (2.27) and (2.28), for the imaginary part, can be combined to give:

$$\begin{aligned}
2\sqrt{1-x^2}U_{n-1}(x)T_n(y) &= \mathrm{Im}\left[\mathbf{T}_n(x)\mathbf{T}_n(y)\right]+\mathrm{Im}\left[\mathbf{T}_n(x)\overline{\mathbf{T}}_n(y)\right] \quad (4.12)\\
2\sqrt{1-y^2}U_{n-1}(y)T_n(x) &= \mathrm{Im}\left[\mathbf{T}_n(x)\mathbf{T}_n(y)\right]-\mathrm{Im}\left[\mathbf{T}_n(x)\overline{\mathbf{T}}_n(y)\right].
\end{aligned}$$

By using again the positions in the (2.30) and the above identities we can state the following result:

Proposition 9 – Let $\xi \in \mathbf{R}$, $|\xi| < 1$, the polynomials $T_n(x)$ and $U_n(x)$ satisfy the following identities, involving $T - U$ products:

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{\xi^n}{n!} U_{n-1}(x) T_n(y) &= \frac{1}{2} \frac{e^{\xi F_+} \sin(\xi G_-) + e^{\xi F_-} \sin(\xi G_+)}{\sqrt{1-x^2}} \\ \sum_{n=0}^{+\infty} \frac{\xi^n}{n!} U_{n-1}(y) T_n(x) &= \frac{1}{2} \frac{e^{\xi F_-} \sin(\xi G_+) + e^{\xi F_+} \sin(\xi G_-)}{\sqrt{1-y^2}} \end{aligned} \tag{4.13}$$

Proof – From the (2.38), we get:

$$\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} U_{n-1}(x) T_n(y) = \frac{1}{2\sqrt{1-x^2}} \mathrm{Im}\left[\exp\left(\xi e^{i\psi} e^{-i\phi}\right)\right] + \mathrm{Im}\left[\exp\left(\xi e^{i\psi} e^{i\phi}\right)\right] \tag{4.14}$$

$$\sum_{n=0}^{+\infty} \frac{\xi^n}{n!} U_{n-1}(y) T_n(x) = \frac{1}{2\sqrt{1-y^2}} \mathrm{Im}\left[\exp\left(\xi e^{i\psi} e^{i\phi}\right)\right] - \mathrm{Im}\left[\exp\left(\xi e^{i\psi} e^{-i\phi}\right)\right] \tag{4.15}$$

where is $\psi = \arccos(x)$ and $\phi = \arccos(y)$. By following the same procedure used in the previous proposition we easily obtain the thesis.

The relations stated in the *Proposition 7* can be extended to the two-variable case. By noting in fact that:

$$|\mathbf{T}(x)| = |\exp(i \arccos(x))| = 1$$

and by choosing $|\xi| < 1$, we have:

$$\xi |\mathbf{T}(x)||\mathbf{T}| < 1$$

and finally:

$$\begin{aligned} \sum_{n=0}^{+\infty} \xi^n \mathbf{T}_n(x) \mathbf{T}_n(y) &= \frac{1}{1 - \xi\left(e^{i \arccos(x)}\right)\left(e^{i \arccos(y)}\right)} \\ \sum_{n=0}^{+\infty} \xi^n \mathbf{T}_n(x) \overline{\mathbf{T}}_n(y) &= \frac{1}{1 - \xi\left(e^{i \arccos(x)}\right)\left(e^{i \arccos(y)}\right)}. \end{aligned} \tag{4.16}$$

References

[1] P. Davis, *Interpolation and Approximation*, Dover, New York, 1975.

[2] E.D. Rainville, *Special Functions*, MacMillan, New York, 1960.

[3] L. C. Andrews, *Special Functions for Engineers and Applied Mathematics*, MacMillan, New York, 1958.

[4] G. Dattoli, S. Lorenzutta and A. Torre, *The generating functions method and evaluation of integrals involving combinations of special function*s, Quaderni del Gruppo Naz. Per l'Informatica Matematica, Vol. 8, CNR, Firenze, 1997.

[5] G. Dattoli, C. Cesarano and D. Saccheti, A note on Chebyshev polynomials, *Ann. Univ. Ferrara*, **7** (2001), no. 47, 107–115.

[6] H.W. Srivastava, H.L. Manocha, *A treatise on generating functions*, Wiley, New York, 1984.