Int. Journal of Math. Analysis, Vol. 8, 2014, no. 10, 489 - 494
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijma.2014.4249

On Some Identities of k-Jacobsthal-Lucas Numbers

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Abstract

In this paper we present the sequence of the k-Jacobsthal-Lucas numbers that generalizes the Jacobsthal-Lucas sequence introduced by Horadam in 1988. For this new sequence we establish an explicit formula for the term of order n, the well-known Binet's formula, Catalan's and d'Ocagne's Identities and a generating function.

Mathematics Subject Classification 2010: 11B37, 11B83

Keywords: Jacobsthal numbers; Jacobsthal-Lucas numbers; Binet's formula

1 Introduction

Several recurrence sequences of positive integers have been object of study for many researchers. Examples of these are the Fibonacci, Lucas, Pell, Pell-Lucas, Modified Pell, Jacobhstal, Jacobsthal-Lucas sequences among others(see [8], [10], [12], [13]). About them there is a vast literature studying several properties, ones involving the well-known Binet's formula, Catalan's, Cassini's and d'Ocagne's identities and there is also a vast literature dedicated to the study of other properties involving each sequence (see [7] and [14]).

More recently, some of these sequences were generalized for any positive real number k: the study of the k-Fibonacci sequence, the k-Lucas sequence, the k-Pell sequence, the k-Pell-Lucas sequence,the Modified k-Pell sequence and the k-Jacobhstal sequence appeared (see [1], [11], [2], [4], [5], [6] and [3]).

In this paper we generalize the sequence of Jacobsthal-Lucas numbers and study by introducing the sequence of the k-Jacobsthal-Lucas numbers. We give an explicit formula for the term of order n of this sequence, the well-know Binet's formula, Catalan's and d'Ocagne's Identities and a generating function for this recurrence sequence.

2 Identities

Let us define the sequence of the k-Jacobsthal-Lucas numbers $\{j_{k,n}\}_{n\in\mathbb{N}}$ as follows:

$$j_{k,n+1} = kj_{k,n} + 2j_{k,n-1} \tag{1}$$

where the initial conditions are:

$$\begin{cases} j_{k,0} &= 2 \\ j_{k,1} &= k \end{cases} \tag{2}$$

for any positive real number k. If $k = 1$ we get the sequence of Jacobsthal-Lucas numbers defined by Horadam in [9]. The characteristic equation associated to the recurrence relation (1) is

$$x^2 = kx + 2 \tag{3}$$

with roots r_1 and r_2 given by $r_1 = \frac{k+\sqrt{k^2+8}}{2}$ and $r_2 = \frac{k-\sqrt{k^2+8}}{2}$.

Note that $r_1r_2 = -2$; $r_1 + r_2 = k$ and $r_1 - r_2 = \sqrt{k^2+8}$. Associated to (1) the term of order n of the k-Jacobsthal-Lucas sequence, can be written by the following identity $j_{k,n} = c_1{r_1}^n + c_2{r_2}^n$ for some constants c_1, c_2.

Solving the system of two linear equations corresponding to the initial conditions (2),

$$\begin{cases} 2 &= c_1 + c_2 \\ k &= c_1r_1 + c_2r_2, \end{cases} \tag{4}$$

we obtain $c_1 = c_2 = 1$. So, we get the next Proposition:

Proposition 2.1 *(Binet's Formula): The nth k-Jacobsthal-Lucas number $j_{k,n}$ is given by*

$$j_{k,n} = r_1^n + r_2^n, \tag{5}$$

where r_1 and r_2 are the roots of the characteristic equation (3) and $r_1 > r_2$.

Proof. We use induction on n. Taking into account the initial conditions (2), we note that the equation (5) is valid for $n = 0$ and $n = 1$. Now assume that (5) is true for $0 \leq s \leq n$, that is, $j_{k,s} = r_1^s + r_2^s$, for every $s \in \{0, \ldots, n\}$. Using (1) and taking in account that $r_1 r_2 = -2$ we have

$$\begin{aligned} j_{k,n+1} &= k j_{k,n} + 2 j_{k,n-1} \\ &= k\left(r_1^n + r_2^n\right) + 2\left(r_1^{n-1} + r_2^{n-1}\right) \\ &= r_1^{n-1}\left(k r_1 + 2\right) + r_2^{n-1}\left(k r_2 + 2\right) \\ &= r_1^{n-1}\left(\left(r_1 + r_2\right) r_1 + 2\right) + r_2^{n-1}\left(\left(r_1 + r_2\right) r_2 + 2\right) \\ &= r_1^{n-1}\left(r_1^2 + r_1 r_2 + 2\right) + r_2^{n-1}\left(r_1 r_2 + r_2^2 + 2\right) \\ &= r_1^{n+1} + r_2^{n+1}. \end{aligned}$$

Consequently, the Binet's Formula is true for any positive integer n. □

The use of the Binet's Formula (5) and the fact that $r_1 r_2 = -2$ allows us to obtain Catalan's Identity.

Proposition 2.2 *(Catalan's Identity):*

$$j_{k,n-r} j_{k,n+r} - j_{k,n}^2 = (-2)^{n-r}(j_{k,r}^2 - (-2)^{r+2}). \tag{6}$$

Proof. We have

$$\begin{aligned} j_{k,n-r} j_{k,n+r} - j_{k,n}^2 &= \left(r_1^{n-r} + r_2^{n-r}\right)\left(r_1^{n+r} + r_2^{n+r}\right) - \left(r_1^n + r_2^n\right)^2 \\ &= (-2)^n \left(\frac{r_2}{r_1}\right)^r + (-2)^n \left(\frac{r_1}{r_2}\right)^r - 2(-2)^n \\ &= (-2)^n \left(\frac{r_2^r}{r_1^r} + \frac{r_1^r}{r_2^r} - 2\right) \\ &= (-2)^n \left[\frac{r_2^{2r} + r_1^{2r} - 2(r_1 r_2)^r}{(r_1 r_2)^r}\right] \\ &= (-2)^n \left[\frac{r_2^{2r} + r_1^{2r} - 2(r_1 r_2)^r}{(-2)^r}\right] \\ &= (-2)^{n-r}\left(r_2^{2r} + r_1^{2r} - 2(r_1 r_2)^r\right) \\ &= (-2)^{n-r}\left(\left(r_1^r + r_2^r\right)^2 - 4(r_1 r_2)^r\right) \\ &= (-2)^{n-r}\left(j_{k,r}^2 - 4(-2)^r\right), \end{aligned}$$

as required. □

Substituting $r = 1$ in Catalan's Identity (6), yields

$$j_{k,n-1} j_{k,n+1} - j_{k,n}^2 = (-2)^{n-1}\left(j_{k,1}^2 - 4(-2)\right)$$

and using the initial condition $j_{k,1} = k$, we obtain the Cassini's identity for k-Jacobsthal-Lucas sequence.

Proposition 2.3 *(Cassini's Identity):*

$$j_{k,n-1}j_{k,n+1}-j_{k,n}^2=(-2)^{n-1}\left(k^2+8\right). \tag{7}$$

The d'Ocagne's identity can also be obtained from the Binet's Formula (5) and the fact that $r_1r_2=-2$ and $m>n$.

Proposition 2.4 *(d'Ocagne's Identity): For $m>n$,*

$$j_{k,m}j_{k,n+1}-j_{k,m+1}j_{k,n}=(-2)^n\sqrt{k^2+8}\left(j_{k,m-n}-2^{n-m+1}\left(k+\sqrt{k^2+8}\right)^{m-n}\right).$$

Proof. For $m>n$, we have

$$\begin{aligned}
j_{k,m}j_{k,n+1}-j_{k,m+1}j_{k,n} &= \left(r_1^m+r_2^m\right)\left(r_1^{n+1}+r_2^{n+1}\right)-\left(r_1^{m+1}+r_2^{m+1}\right)\left(r_1^n+r_2^n\right)\\
&= (-2)^n\left(r_1^{m-n}r_2+r_1r_2^{m-n}-r_1^{m-n}r_1-r_2^{m-n}r_2\right)\\
&= (-2)^n\left(r_1^{m-n}(r_2-r_1)+r_2^{m-n}(r_1-r_2)\right)\\
&= (-2)^n\left(r_1-r_2\right)\left(r_2^{m-n}-r_1^{m-n}\right)\\
&= (-2)^n\sqrt{k^2+8}\left(r_1^{m-n}+r_2^{m-n}-2r_1^{m-n}\right)\\
&= (-2)^n\sqrt{k^2+8}\left(j_{k,m-n}-2r_1^{m-n}\right)\\
&= (-2)^n\sqrt{k^2+8}\left(j_{k,m-n}-2\frac{\left(k+\sqrt{k^2+8}\right)^{m-n}}{2^{m-n}}\right)\\
&= (-2)^n\sqrt{k^2+8}\left(j_{k,m-n}-2^{n-m+1}\left(k+\sqrt{k^2+8}\right)^{m-n}\right).
\end{aligned}$$

as required. □

The limit property stated in the following Proposition is also deduced using Binet's Formula (5).

Proposition 2.5 *For $m>n$,*

$$lim_{n\to\infty}\frac{j_{k,n}}{j_{k,n-1}}=r_1. \tag{8}$$

Proof. We have

$$lim_{n\to\infty}\frac{j_{k,n}}{j_{k,n-1}}=lim_{n\to\infty}\frac{r_1^n+r_2^n}{r_1^{n-1}+r_2^{n-1}}.$$

Since $\left|\frac{r_2}{r_1}\right|<1$, then $lim_{n\to\infty}\left(\frac{r_2}{r_1}\right)^n=0$ and therefore

$$\begin{aligned}
lim_{n\to\infty}\frac{j_{k,n}}{j_{k,n-1}} &= lim_{n\to\infty}\frac{1+\left(\frac{r_2}{r_1}\right)^n}{\frac{1}{r_1}+\left(\frac{r_2}{r_1}\right)^n\frac{1}{r_2}}\\
&= \frac{1}{\frac{1}{r_1}},
\end{aligned}$$

and the result follows. □

3 Generating Function

In the next Proposition we present a generating function for the sequence of the k-Jacobsthal-Lucas numbers.

Proposition 3.1 *(Generating function of the k-Jacobsthal-Lucas numbers)*

$$j_k(x) = \frac{2 - kx}{1 - kx - 2x^2}$$

Proof. Let us suppose that the k-Jacobsthal-Lucas numbers are the coefficients of a power series centered at the origin, that is convergent in $\left]-\frac{1}{r_1}, \frac{1}{r_1}\right[$, taking in account the Proposition (2.5) . To the sum of this power series, $j_k(x)$, we call generating function of the k-Jacobsthal-Lucas numbers. So we have

$$j_k(x) = j_{k,0} + j_{k,1}x + j_{k,2}x^2 + \cdots + j_{k,n}x^n + \cdots$$

and then,

$$\begin{aligned} kxj_k(x) &= kj_{k,0}x + kj_{k,1}x^2 + kj_{k,2}x^3 + \cdots + kj_{k,n}x^{n+1} + \cdots \\ 2x^2 j_k(x) &= 2j_{k,0}x^2 + 2j_{k,1}x^3 + 2j_{k,2}x^4 + \cdots + 2j_{k,n}x^{n+2} + \cdots . \end{aligned}$$

Since (1) e (2) we obtain

$$j_k(x) - kxj_k(x) - 2x^2 j_k(x) = 2 - kx$$

and then we conclude that

$$j_k(x) = \frac{2 - kx}{1 - kx - 2x^2}$$

□

Acknowledgements. This work has been supported by the Portuguese Government through the FCT - Fundação para a Ciência e a Tecnologia - under the project PEst-OE/MAT/ UI4080/2014.

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Received: February 11, 2014