Generating Functions for Laguerre Polynomials: New Identities for Lacunary Series

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Abstract

We present a number of identities involving standard and associated Laguerre polynomials. They include double-, and triple-lacunary, ordinary and exponential generating functions of certain classes of Laguerre polynomials.

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The purpose of this note is to list a number of identities satisfied by standard Laguerre polynomials $L_n(x)$ and their associated counterparts $L_n^{(\alpha)}(x)$, with $L_n^{(0)}(x) = L_n(x)$. For $L_n^{(\alpha)}(x)$ as well as for the Hermite polynomials $H_n(x)$ we employ the definitions given in [1]. We present these identities below:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_{2n}(x) = \mathrm{e}^t \sum_{r=0}^{\infty} \frac{(ix\sqrt{t})^r}{(r!)^2} H_r(i\sqrt{t}); \tag{1}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_{2n}^{(1)}(x) = \mathrm{e}^t \sum_{r=0}^{\infty} \frac{p_2(r;x,t)}{r!(r+3)!} (ix\sqrt{t})^r H_r(i\sqrt{t}), \tag{2}$$

$$p_2(r;x,t) = (1+2t)r^2 + (5-4xt+10t)r + (6+12t-12xt+2tx^2); \tag{3}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_{2n}^{(2)}(x) = \mathrm{e}^t \sum_{r=0}^{\infty} \frac{p_4(r;x,t)}{r!(r+6)!} (ix\sqrt{t})^r H_r(i\sqrt{t}), \tag{4}$$

$$\begin{aligned} p_4(r;x,t) &= (2+10t+4t^2)r^4 + [36+(180-20x)t+(72-16x)t^2]r^3 \\ &+ [238+(1190-300x+10x^2)t+(476-240x+24x^2)t^2]r^2 \\ &+ [684+(3420-1480x+110x^2)t+(1368-1184x+264x^2-16x^3)t^2]r \\ &+ 720+(3600-2400x+300x^2)t \\ &+ (1440-1920x+720x^2-96x^3+4x^4)t^2; \end{aligned} \tag{5}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_{2n+k}(x) = \mathrm{e}^t k! \sum_{r=0}^{\infty} \frac{(ix\sqrt{t})^r}{r!(r+k)!} L_k^{(r)}(x) H_r(i\sqrt{t}); \tag{6}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_{3n+k}(x) = \mathrm{e}^t k! \sum_{r=0}^{\infty} \frac{L_k^{(r)}(x)}{(r+k)!} \left[\sum_{s=0}^{[r/3]} \frac{(-tx^3)^s (ix\sqrt{3t})^{r-3s}}{s!(r-3s)!} H_{r-3s}\left(i\frac{\sqrt{3t}}{2} \right) \right]; \tag{7}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_{3n}^{(1)}(x) = \mathrm{e}^t \sum_{r=0}^{\infty} \frac{q_3(r;x,t)}{(r+4)!} \left[\sum_{s=0}^{[r/3]} \frac{(-tx^3)^s (ix\sqrt{3t})^{r-3s}}{s!(r-3s)!} H_{r-3s}\left(i\frac{\sqrt{3t}}{2} \right) \right]; \tag{8}$$

$$\begin{aligned} q_3(r;x,t) &= (1+3t)r^3 + (9+27t-9tx)r^2 + (26+78t-63tx+9tx^2)r \\ &+ (24+72t-108tx+36tx^2-3tx^3); \end{aligned} \tag{9}$$

In Eqs. (7) and (8) $[n]$ is the floor function.

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n t^n}{\left(1+\frac{\alpha}{2}\right)_n} L_{2n}^{(\alpha)}(x) = (1-t)^{-\frac{1+\alpha}{2}} \sum_{r=0}^{\infty} \frac{L_r^{(r+\alpha)}\left(\frac{x}{2}\right)}{\left(1+\frac{\alpha}{2}\right)_r} \left[-\frac{tx}{2(1-t)}\right]^r; \tag{10}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{(1+m)_n} t^n L_{2n}^{(2m)}(x) = \frac{1}{\sqrt{1-t}} \left(\frac{x\sqrt{t}}{2}\right)^{-m} \exp\left(-\frac{tx}{1-t}\right) I_m\left(\frac{x\sqrt{t}}{1-t}\right), \tag{11}$$

where $m = 0, 1, 2, \ldots$ and $I_m(z)$ is the modified Bessel function;

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n t^n L_{3n}^{(\alpha)}(x)}{\left(1+\frac{\alpha}{3}\right)_n \left(\frac{2}{3}+\frac{\alpha}{3}\right)_n} &= (1-t)^{-\frac{1+\alpha}{3}} \sum_{r=0}^{\infty} \frac{\Gamma(3r+\alpha+1)}{\left(1+\frac{\alpha}{3}\right)_r \left(\frac{2}{3}+\frac{\alpha}{3}\right)_r} \left[-\frac{tx}{9(1-t)}\right]^r \\ &\times \left[\sum_{s=0}^{r} \frac{(-x)^s L_s^{(s+\alpha+r)}\left(\frac{x}{3}\right)}{(r-s)!\,\Gamma(2s+\alpha+r+1)}\right];\end{aligned} \tag{12}$$

$$\sum_{n=0}^{\infty} t^n L_{2n}(x) = \frac{1}{1-t} \sum_{r=0}^{\infty} \frac{L_r^{(r)}\left(\frac{x}{2}\right)}{\left(\frac{1}{2}\right)_r} \left[-\frac{tx}{2(1-t)}\right]^r; \tag{13}$$

$$\sum_{n=0}^{\infty} t^n L_{3n}(x) = \frac{1}{1-t} \sum_{r=0}^{\infty} \left(-\frac{3tx}{1-t}\right)^r \left[\sum_{s=0}^{r} \frac{r!(-x)^s}{(r-s)!(r+2s)!} L_s^{(s+r)}\left(\frac{x}{3}\right)\right]; \tag{14}$$

$$\sum_{n=0}^{\infty} t^n L_{2n}^{(\alpha-2n)}(x) = (1-t)^{\frac{\alpha}{2}} \cosh\left[\sqrt{t}x - i\alpha \arcsin\left(\frac{\sqrt{t}}{\sqrt{t-1}}\right)\right]. \tag{15}$$

To the best of our knowledge all the above formulas are new. The Eq. (11) is the corrected version of Eq. (5.11.2.10), p. 704 of [1]. We shall present the detailed derivation of identities Eqs. (1)-(15) in the forthcoming publication [2].

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[1] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series, vol. 2: Special Functions* (Gordon and Breach, New York, 1992).

[2] D. Babusci, G. Dattoli, K. Górska, and K. A. Penson, unpublished.