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# Identities Involving Lucas or Fibonacci and Lucas Numbers as Binomial Sums

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**Abstract**

As in [1, 2], for rapid numerical calculations of identities pertaining to Lucas or both Fibonacci and Lucas numbers we present each identity as a binomial sum.

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## 1. Preliminaries

The two most well-known linear homogeneous recurrence relations of order two with constant coefficients are those that define *Fibonacci* and *Lucas numbers* (or Fibonacci and Lucas sequences). They are defined recursively as

$$F_{n+2} = F_n + F_{n+1}, \quad \text{where} \quad F_0 = 0,\ F_1 = 1,\ n \geq 0,$$

and

$$L_{n+2} = L_n + L_{n+1}, \quad \text{where} \quad L_1 = 1,\ L_2 = 3,\ n \geq 1.$$

We note that aside from the boundary conditions, Fibonacci and Lucas numbers are represented by the same recurrence relation. This is the reason that Fibonacci and Lucas numbers have so many common or very similar properties. For example, the ratio of two consecutive Fibonacci numbers as well as the ratio of two consecutive Lucas numbers both converge to the famous golden ratio. We observe that for $n \geq 0$,

$$L_{n+1} = F_n + F_{n+2} \text{ and } L_{n+1} + L_{n+3} = 5F_{n+2}.$$

Hundreds of Fibonacci and Lucas identities, as well as identities involving both Fibonacci and Lucas numbers, appeared in various journals and books over the years. While *The Fibonacci Quarterly* is the main source for most original identities, one can find hundreds of known identities in numerous books such as Thomas Koshy's book [25] and Marjorie Bicknel and Verner E. Hoggatt's book [7]. We acknowledge that the following individuals authored at least one of the identities that we have presented in this paper: W. C. Barely [5], M. Bicknel and V. E. Hoggatt Jr. [7], R. Blazej [8, 9], L. Carlitz [11-13], W. Chevez [14], H. H. Ferns [15-17], V. E. Hoggatt Jr.[19-24], D. Jarden [25], T. Koshy [26-28], and H. L. Umansky [30-32]. Our goal in this paper is to present some known identities concerning Lucas, or both Fibonacci and Lucas numbers, as binomial sums for quick numerical calculations.

Throughout the paper we use $F_i$ $(i \geq 0)$ and $L_j$ $(j \geq 1)$ to represent a Fibonacci or a Lucas number, respectively. To proceed, first we recall the following theorem from [2].

**Theorem 1.1** [2]. If $F_n$ is any Fibonacci number, then

$$\begin{aligned} F_{n+1} &= \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + ... + \binom{n - \lfloor \frac{n}{2} \rfloor + 1}{\lfloor \frac{n}{2} \rfloor - 1} + \binom{n - \lfloor \frac{n}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor} \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}, \quad n \geq 0. \end{aligned}$$

## 2. Lucas Identities

**Theorem 2.1**.

(*i*) $$\frac{1}{5}\left(2L_{2n+2} - L_{n+1}^2\right) = \frac{1}{5}\left[L_{2n+2} + 2(-1)^n\right] = \left[\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}\right]^2$$

(*ii*) $$\frac{1}{5}\left[L_{n+1}L_{n+3} + (-1)^n\right] = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-i}{i}$$

(*iii*) $$\frac{1}{5}\left[L_{n+5} - L_{n+1}\right] = \sum_{i=0}^{\lfloor \frac{n+2}{2} \rfloor} \binom{n+2-i}{i}$$

**Theorem 2.2**.

$(i)$ $\frac{1}{5}\left[L_{n+1}^2 + L_{n+2}^2\right] = \sum_{i=0}^{n+1}\binom{2n+2-i}{i}$

$(ii)$ $L_{2n+2}\sum_{i=1}^{n+1}L_{2n+4i} = \sum_{i=0}^{\lfloor\frac{8n+7}{2}\rfloor}\binom{8n+7-i}{i}$

**Theorem 2.3**.

$(i)$ $\prod_{i=0}^{n}L_{2^i} = \sum_{i=0}^{\lfloor\frac{2^{n+1}-1}{2}\rfloor}\binom{2^{n+1}-1-i}{i}$

$(ii)$ $\prod_{i=0}^{n}(L_{2.3^i}-1) = \sum_{i=0}^{\lfloor\frac{3^{n+1}-1}{2}\rfloor}\binom{3^{n+1}-1-i}{i}$

## 3. Identities involving Fibonacci and Lucas Numbers

**Theorem 3.1**.

$(i)$ $3 - F_{n+3} + \sum_{i=1}^{n}L_i = \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-i}{i}$

$(ii)$ $L_{n+1}^2 - 4F_nF_{n+2} = 2^{-2n}\left[1 + \sum_{i=1}^{n}2^{2i-2}L_iF_{i+3}\right] = \left[\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-i}{i}\right]^2$

$(iii)$ $\frac{1}{5}\left(4F_{3n+3} - 3F_{n+1}L_{n+1}^2\right) = 7F_{n+2}^3 - (F_n^3 + 3F_{n+2}F_{n+3}L_{n+1})$

$= L_{n+1}^3 - 2F_n(3F_{n+2}^2 + F_n^2) = \left[\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-i}{i}\right]^3$

**Theorem 3.2**.

$(i)$ $2^{-2n}\left[1 + 5\sum_{i=1}^{n}2^{2i-2}F_iL_{i+3}\right] - F_n(F_n + 2F_{n+2})$

$= L_{n+1}L_{n+2} - F_n(F_n + 2F_{n+2}) - \left[2 + \sum_{i=1}^{n}L_i^2\right] = \left[\sum_{i=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{n+1-i}{i}\right]^2$

$(ii)$ $\frac{1}{3}\left[L_{n+3}^2 + L_{n+1}^2 + 10(-1)^n\right] - F_{n+1}(F_{n+1} + 2F_{n+3}) = \left[\sum_{i=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{n+2-i}{i}\right]^2$

**Theorem 3.3.**

$$(i)\ \ 2-F_{2n+3}+\sum_{i=1}^{n+1}L_{2i-1}=L_{n+1}^2+L_{n+2}^2-(L_{2n+4}+F_{2n+3})$$

$$=F_{n+2}L_{n+3}-L_{n+1}L_{n+2}=\sum_{i=0}^{n}\tbinom{2n-i}{i}$$

$$(ii)\ \ \frac{1}{5}(2+L_{4n+2})=\left[\sum_{i=0}^{n}\binom{2n-i}{i}\right]^2$$

$$(iii)\ \ (-1)^n+L_{n+1}L_{n+2}-F_{2n+4}=1-F_{2n}+\sum_{i=1}^{n}L_{2i}=\sum_{i=0}^{\lfloor\frac{2n+1}{2}\rfloor}\binom{2n+1-i}{i}$$

$$(iv)\ \ \frac{1}{5}(L_{4n+4}+L_{4n+6})-F_{2n+3}^2=\left[\sum_{i=0}^{n}\binom{2n+1-i}{i}\right]^2$$

$$(v)\ \ F_{n+2}L_{n+3}-F_{n+3}L_{n+1}=\sum_{i=0}^{n+1}\binom{2n+2-i}{i}$$

**Theorem 3.4.**

$$(i)\ \ L_{n+1}\left[L_{2n+2}+(-1)^n\right]-F_{3n+4}=\sum_{i=0}^{\lfloor\frac{3n+1}{2}\rfloor}\binom{3n+1-i}{i}$$

$$(ii)\ \ F_{2n+2}L_{n+1}+(-1)^nF_{n+1}=F_{n+1}[L_{2n+2}-(-1)^n]=\sum_{i=0}^{\lfloor\frac{3n+2}{2}\rfloor}\binom{3n+2-i}{i}$$

$$(iii)\ \ \frac{1}{4}L_{n+1}(15F_{n+1}^2+L_{n+1}^2)-F_{3n+2}=\sum_{i=0}^{\lfloor\frac{3n+3}{2}\rfloor}\binom{3n+3-i}{i}$$

$$(iv)\ \ \frac{1}{5}(L_{n+3}^3+L_{n+2}^3-L_{n+1}^3)-F_{3n+7}=\sum_{i=0}^{\lfloor\frac{3n+4}{2}\rfloor}\binom{3n+4-i}{i}$$

**Theorem 3.5.**

$$(i)\ \ 1+L_{2n+1}F_{2n+2}=\sum_{i=0}^{2n+1}\binom{4n+2-i}{i}$$

$$(ii)\ \ L_{n+1}\left[L_{2n+2}^2+(-1)^nL_{2n+2}-1\right]-F_{5n+6}=\sum_{i=0}^{\lfloor\frac{5n+3}{2}\rfloor}\binom{5n+3-i}{i}$$

$$(iii)\ \ L_{n+1}^2\left[L_{n+1}^4+6(-1)^nL_{n+1}^2+9\right]+2(-1)^n-F_{6n+7}=\sum_{i=0}^{3n+2}\binom{6n+4-i}{i}$$

$$(iv)\ \ L_{n+1}\left[L_{n+1}^6+7(-1)^nL_{n+1}^4+14L_{n+1}^2+7(-1)^n\right]-F_{7n+8}=\sum_{i=0}^{\lfloor\frac{7n+5}{2}\rfloor}\binom{7n+5-i}{i}$$

$$(v) \quad L_{2n+2}[(L_{4n+4}-3)^2+25F_{2n+2}^2]-F_{10n+9}=\sum_{i=0}^{5n+5}\binom{10n+10-i}{i}$$

**Note 3.6**. For a general theorem that encompasses Theorem 3.5 (iii) and (iv), the reader is referred to [19].

## 4. Proofs and Remarks

To prove Theorems 2.1-2.3 and 3.1-3.5, we can simply use Theorem 1.1, and the fact that each identity on the left-hand side of each part of each theorem can be written as a (power of a ) single Fibonacci number. Or, we could use the principle of mathematical induction, combinatorial arguments, or just simple algebra. Also, in one form or another these identities can be found in the references. However, in order for some identities to fit a specified format they have been slightly modified and they may look different than in the literature. Although we verified the validity of most of these identities, we acknowledge that we did not think it was necessary to verify independently the validity of all.

**Remark 4.1.** We stated in [2] that Fibonacci numbers are the sum of the numbers along the rising diagonals of *Pascal's* (*Khayyām-Pascal's*) triangle. Similarly, Lucas numbers are the sum of the numbers along the rising diagonals of an arithmetic triangle, called the (1, 2)-Pascal triangle or Lucas triangle (for example, see [4] or [25]).

**Remark 4.2.** According to our definition in this article, $L_{n+2} = L_n + L_{n+1}$, where $L_1 = 1$, $L_2 = 3$, $n \geq 1$, and hence, the Lucas numbers are $1, 3, 4, 7, 11, 18, 29, 47, ....$ However, if we start with $L_0 = 2$, then the Lucas numbers will be $2, 1, 3, 4, 7, 11, 18, 29, 47, ....$ For a wealth of properties and references for Lucas numbers the reader is referred to [4] and [28].

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