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# Riordan arrays and harmonic number identities

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ABSTRACT

Let the numbers $P(r, n, k)$ be defined by

$$P(r, n, k) := P_r\left(H_n^{(1)} - H_k^{(1)}, \dots, H_n^{(r)} - H_k^{(r)}\right),$$

where $P_r(x_1, \dots, x_r) = (-1)^r Y_r(-0!x_1, -1!x_2, \dots, -(r-1)!x_r)$ and $Y_r$ are the exponential complete Bell polynomials. By observing that the numbers $P(r, n, k)$ generate two Riordan arrays, we establish several general summation formulas, from which series of harmonic number identities are obtained. In particular, some of these harmonic number identities also involve other special combinatorial sequences, such as the Stirling numbers of both kinds, the Lah numbers, the Bernoulli numbers and polynomials and the Cauchy numbers of both kinds.



## 1. Introduction

The *generalized harmonic numbers* are defined to be partial sums of the Riemann–Zeta series:

$$H_0^{(r)} = 0 \quad \text{and} \quad H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r} \quad \text{for } n, r = 1, 2, \dots.$$

When $r = 1$, they reduce to the classical harmonic numbers, denoted as $H_n = H_n^{(1)}$. Harmonic numbers are important in various branches of combinatorics and number theory, and they also frequently appear in the analysis of algorithms and expressions for special functions.

Recently, many works have been devoted to the study of harmonic number identities by various methods. In [1–3], Chu established some harmonic number identities by decomposing rational functions into partial fractions. In [4–6], Chu, De Donno and Fu derived series of identities by classical hypergeometric summation theorems. Sprugnoli [7] presented some harmonic number identities by means of the Riordan array method, and Sofo [8–10] obtained many from the integral representations of some special binomial sums. Moreover, a variety of harmonic number identities are derived by symbolic computation; readers may consult the papers by Driver et al. [11,12], Lyons et al. [13], Osburn and Schneider [14] and Paule and Schneider [15].

The concept of Riordan arrays was introduced by Shapiro et al. [16]. In the present paper, we make use of the method of Riordan arrays in a constructive way to establish some general summation formulas, from which series of harmonic number identities can be obtained. In particular, besides the harmonic numbers, some identities also involve the Stirling numbers of both kinds, the Lah numbers, the Bernoulli numbers and polynomials and the Cauchy numbers of both kinds. It can be found that no harmonic number identities presented in [1–15] referred to above have other special combinatorial sequences, and actually, there are not many identities involving both harmonic numbers and other combinatorial numbers in the literature. From this point of view, our results extend the range of harmonic number identities.

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The study of this paper follows Zave and Spieß's results [17,18]. Let the polynomials $P_r(x_1, \ldots, x_r)$ be defined by $P_0 = 1$ and

$$P_r(x_1, \ldots, x_r) = (-1)^r Y_r(-0!x_1, -1!x_2, \ldots, -(r-1)!x_r),$$

where $Y_r$ are the *exponential complete Bell polynomials* (see [19, Section 3.3]). Some of the polynomials are

$$P_1(x_1) = x_1,$$
$$P_2(x_1, x_2) = x_1^2 - x_2,$$
$$P_3(x_1, x_2, x_3) = x_1^3 - 3x_1x_2 + 2x_3,$$
$$P_4(x_1, x_2, x_3, x_4) = x_1^4 - 6x_1^2x_2 + 8x_1x_3 + 3x_2^2 - 6x_4.$$

In [18], Zave established the following series expansion:

$$\sum_{n=k}^{\infty} \binom{n}{k} P_r\left(H_n^{(1)} - H_k^{(1)}, \ldots, H_n^{(r)} - H_k^{(r)}\right) t^{n-k} = \frac{(-\log(1-t))^r}{(1-t)^{k+1}}. \tag{1.1}$$

Obviously, this expansion contains as special cases two well-known generating functions related to the harmonic numbers:

$$\sum_{n=0}^{\infty} \binom{n+k}{k} (H_{n+k} - H_k) t^n = \frac{-\log(1-t)}{(1-t)^{k+1}}, \tag{1.2}$$

$$\sum_{n=0}^{\infty} H_n t^n = \frac{-\log(1-t)}{1-t}. \tag{1.3}$$

Based on Zave's result (1.1), Spieß [17] introduced the symbol

$$P(r, n, k) = P_r\left(H_n^{(1)} - H_k^{(1)}, \ldots, H_n^{(r)} - H_k^{(r)}\right) \tag{1.4}$$

and derived some general summation formulas by standard combinatorial techniques. He further obtained such nice harmonic number identities as

$$\sum_{k=0}^{n} \binom{s+k}{p} \binom{n-k}{q} H_{s+k} = \binom{n+s+1}{p+q+1} (H_{n+s+1} - H_{p+q+1} + H_p),$$
$$\sum_{k=0}^{n} (-1)^k \binom{p}{k} \binom{n-k}{m} (H_{n-k} - H_m) = \binom{n-p}{m-p} (H_{n-p} - H_{m-p}),$$

which generalize the identities in [20, Eqs. (7.62)–(7.64)], though this fact was not emphasized in [17].

By means of (1.4) and the fact that $P(r, n, k) = 0$ for $n < r + k$, Zave's result can be restated as

$$\sum_{n=r+k}^{\infty} \binom{n}{k} P(r, n, k) t^{n-k} = \frac{(-\log(1-t))^r}{(1-t)^{k+1}}, \tag{1.5}$$

which is equivalent to

$$\sum_{n=r}^{\infty} \binom{n+k}{k} P(r, n+k, k) t^n = \frac{(-\log(1-t))^r}{(1-t)^{k+1}}. \tag{1.6}$$

From these two generating functions, we observe that the numbers $P(r, n, k)$ generate two Riordan arrays. Thus, by the Riordan array method, we can establish many general summation formulas associated with $P(r, n, k)$, which lead us to series of identities involving harmonic numbers and some other special combinatorial sequences.

This paper is organized as follows. In Section 2, we briefly introduce the theory of Riordan arrays and present the arrays generated by $P(r, n, k)$. Section 3 gives two general summation formulas concerning $P(r, n, k)$. From these two formulas, we obtain many harmonic number identities, some of which involve the Stirling numbers. Next, in Section 4, we further study the relations between $P(r, n, k)$ and the Stirling numbers of both kinds. Finally, in Section 5, we establish some formulas relating $P(r, n, k)$ to the Bernoulli numbers (polynomials) and the Cauchy numbers of both kinds.

There are too many special cases for the general summation formulas obtained in this paper and it is impossible to list them exhaustively, so we present only the simplest ones. From the special cases, readers may see that the Riordan array method is powerful for dealing with harmonic number identities.

## 2. Riordan arrays

Let $f(t)$ be a formal power series in the indeterminate $t$; then $f(t)$ has the form

$$f(t)=\sum_{k=0}^{\infty} f_k t^k.$$

As usual, the coefficient of $t^n$ in $f(t)$ may be denoted by $[t^n]f(t)$.

A *Riordan array* is a pair $(g(t), f(t))$ of formal power series with $f_0=f(0)=0$. It defines an infinite, lower triangular array $(d_{n,k})_{n,k\in\mathbb{N}}$ according to the rule

$$d_{n,k}=[t^n]g(t)(f(t))^k. \tag{2.1}$$

Hence we write $\mathscr{R}(d_{n,k})=(g(t), f(t))$. The product of two Riordan arrays is still a Riordan array, i.e.,

$$(g(t), f(t)) * (h(t), l(t)) = (g(t)h(f(t)), l(f(t))).$$

Moreover, if $g_0=g(0)\neq 0$ and $f_1=[t]f(t)\neq 0$, then the Riordan array $(g(t), f(t))$ has inverse $(1/g(\bar{f}(t)), \bar{f}(t))$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$. The summation property for a Riordan array $\mathscr{R}(d_{n,k})$ is

$$\sum_{k=0}^{n} d_{n,k}h_k=[t^n]g(t)h(f(t)), \tag{2.2}$$

where $h(t)$ is the generating function of the sequence $(h_k)_{k\in\mathbb{N}}$, i.e., $h(t)=\sum_{k=0}^{\infty} h_k t^k$ or $h(t)=\mathscr{G}(h_k)$. This property has been extensively used in, for example, [7,21,22], and has been proved a constructive and efficient way to deal with combinatorial sums.

An important example of a Riordan array is the Pascal triangle, defined by

$$\mathscr{R}\left(\binom{n}{k}\right)=\left(\frac{1}{1-t}, \frac{t}{1-t}\right),$$

which has the inverse

$$\mathscr{R}\left((-1)^{n-k}\binom{n}{k}\right)=\left(\frac{1}{1+t}, \frac{t}{1+t}\right).$$

Another two interesting Riordan arrays, associated with the Stirling numbers of both kinds, are defined by

$$\mathscr{R}\left(\frac{k!}{n!}\begin{bmatrix} n \\ k \end{bmatrix}\right)=(1, -\log(1-t)), \tag{2.3}$$

$$\mathscr{R}\left((-1)^{n-k}\frac{k!}{n!}\begin{Bmatrix} n \\ k \end{Bmatrix}\right)=(1, 1-\mathrm{e}^{-t}), \tag{2.4}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ are the (unsigned) Stirling numbers of the first kind and $\begin{Bmatrix} n \\ k \end{Bmatrix}$ are the Stirling numbers of the second kind. From (2.3), it can be verified that

$$\sum_{n=k}^{\infty}\frac{k!}{n!}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}t^n=\frac{(-\log(1-t))^k}{1-t}. \tag{2.5}$$

Thus, comparing with (1.6), we obtain the following connection between the numbers $P(r, n, k)$ and the Stirling numbers of the first kind:

$$P(k, n, 0)=\frac{k!}{n!}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}. \tag{2.6}$$

Note that the connection (2.6) coincides with [19, p. 217, Theorem B]. By appealing to this connection and the values of $P_r(x_1, \ldots, x_r)$, we readily see that

$$\begin{bmatrix} n+1 \\ 1 \end{bmatrix}=n!,$$

$$\begin{bmatrix} n+1 \\ 2 \end{bmatrix}=n!H_n,$$

$$\begin{bmatrix} n+1 \\ 3 \end{bmatrix}=\frac{n!}{2}\left(H_n^2-H_n^{(2)}\right),$$

$$\begin{bmatrix} n+1 \\ 4 \end{bmatrix}=\frac{n!}{6}\left(H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}\right).$$

These can also be found in [19, p. 217] and will be frequently made use of later.

Finally, based on the generating functions (1.5) and (1.6), we obtain the next two Riordan arrays, to which we pay particular attention in the present paper:

$$\mathscr{R}\left(\binom{n}{k}P(r,n,k)\right)=\left(\frac{(-\log(1-t))^r}{1-t},\frac{t}{1-t}\right),\tag{2.7}$$

$$\mathscr{R}\left(\binom{n+r}{r}P(k,n+r,r)\right)=\left(\frac{1}{(1-t)^{r+1}},-\log(1-t)\right).\tag{2.8}$$

Note that besides Sprugnoli [7], Cheon and his collaborators also used the theory of Riordan arrays to study some generalizations of harmonic numbers as well as some polynomials associated with harmonic numbers; readers may consult [23,24] for their results.

## 3. General summation formulas

In this section, we establish two general formulas related to the numbers $P(r,n,k)$ (see Theorems 3.1 and 3.5). From these two general formulas, various identities involving harmonic numbers and Stirling numbers are obtained.

**Theorem 3.1.** *The following identity*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}P(r,n+i,k+i)P(s,k+i,i)=(-1)^s\frac{(r+s)!}{n!}\left[{n \atop r+s}\right]\binom{n+i}{i}^{-1}\tag{3.1}$$

*relates the Stirling numbers of the first kind and the numbers $P(r,n,k)$.*

**Proof.** From (1.5), it can be seen that

$$\sum_{k=0}^{\infty}(-1)^{k-i}\binom{k}{i}P(s,k,i)t^k=\frac{t^i(-\log(1+t))^s}{(1+t)^{i+1}}.\tag{3.2}$$

Then applying the summation property (2.2) to the Riordan array (2.7) and the generating function (3.2) yields

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}P(r,n,k)(-1)^{k-i}\binom{k}{i}P(s,k,i)\\&\quad=[t^n]\frac{(-\log(1-t))^r}{1-t}\left[\frac{y^i(-\log(1+y))^s}{(1+y)^{i+1}}\middle|\,y=\frac{t}{1-t}\right]=[t^n]t^i(-1)^r(\log(1-t))^{r+s}\\&\quad=[t^{n-i}](-1)^s(-\log(1-t))^{r+s}=(-1)^s\frac{(r+s)!}{(n-i)!}\left[{n-i \atop r+s}\right],\end{aligned}$$

which indicates that

$$\sum_{k=i}^{n}(-1)^{k-i}\binom{n-i}{k-i}P(r,n,k)P(s,k,i)=(-1)^s\frac{(r+s)!}{(n-i)!}\left[{n-i \atop r+s}\right]\binom{n}{i}^{-1}.$$

Replacing $k$ by $k+i$ and $n$ by $n+i$, we can obtain the desired result. □

**Corollary 3.2.** *The following relations hold:*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}P(r,n+i,k+i)=\frac{r!}{n!}\left[{n \atop r}\right]\binom{n+i}{i}^{-1},\tag{3.3}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}P(r,k+i,i)=(-1)^r\frac{r!}{n!}\left[{n \atop r}\right]\binom{n+i}{i}^{-1}.\tag{3.4}$$

**Proof.** Recall that $P(0,n,k)=1$. Thus, setting $s=0$ in Theorem 3.1 gives (3.3); setting $r=0$ in Theorem 3.1 and then replacing $s$ by $r$ gives (3.4). □

**Example 3.1.** The substitution $r=1$ in (3.3) or (3.4) yields (3.5), which can also be found in [11, p. 142] and [17, p. 847]; the substitution $r=2$ in (3.3) and (3.4) yields (3.6) and (3.7), respectively.

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_{k+i}=-\frac{1}{n}\binom{n+i}{i}^{-1},\tag{3.5}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(H_{k+i}^2+H_{k+i}^{(2)}\right)=\frac{2}{n}(H_{n-1}-H_{n+i})\binom{n+i}{i}^{-1},\tag{3.6}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(H_{k+i}^2-H_{k+i}^{(2)}\right)=\frac{2}{n}(H_{n-1}-H_{i})\binom{n+i}{i}^{-1}.\tag{3.7}$$

From (3.5), we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_k=-\frac{1}{n}.$$

Summing $n$ from 1 to $m$, interchanging the summation order, applying the binomial relation

$$\sum_{n=k}^{m}\binom{n}{k}=\binom{m+1}{k+1}$$

and then replacing $m$ by $n$, we obtain (3.8). In a similar way, (3.9) can be derived from (3.6). Moreover, combining (3.6) and (3.7), we obtain the last two identities (3.10) and (3.11).

$$\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}H_k=-H_n,\tag{3.8}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}\left(H_k^2+H_k^{(2)}\right)=-2H_n^{(2)},\tag{3.9}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_{k+i}^2=\frac{1}{n}(2H_{n-1}-H_{n+i}-H_i)\binom{n+i}{i}^{-1},\tag{3.10}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_{k+i}^{(2)}=\frac{1}{n}(H_i-H_{n+i})\binom{n+i}{i}^{-1}.\tag{3.11}$$

**Corollary 3.3.** *The following relation holds:*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\begin{bmatrix}k+1\\ s+1\end{bmatrix}\frac{s!}{k!}P(r,n,k)=(-1)^s\frac{(r+s)!}{n!}\begin{bmatrix}n\\ r+s\end{bmatrix}.\tag{3.12}$$

**Proof.** To obtain the result, set $i=0$ in Theorem 3.1 and make use of the relation (2.6). □

**Example 3.2.** The substitutions $s=1$ and $s=2$ in (3.12) give us (3.13) and (3.14), respectively.

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_kP(r,n,k)=-\frac{(r+1)!}{n!}\begin{bmatrix}n\\ r+1\end{bmatrix},\tag{3.13}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(H_k^2-H_k^{(2)}\right)P(r,n,k)=\frac{(r+2)!}{n!}\begin{bmatrix}n\\ r+2\end{bmatrix}.\tag{3.14}$$

Setting $r=2$ in (3.13) and making use of (3.5) and (3.10) yields (3.15). Setting $r=1$ in (3.14) and making use of (3.7) yields (3.16). Next, combining (3.15) with (3.16) yields (3.17) and (3.18).

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_k\left(H_k^2+H_k^{(2)}\right)=\frac{2}{n}\left(H_{n-1}^{(2)}+\frac{1}{n}H_{n-1}-\frac{1}{n^2}\right),\tag{3.15}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_k\left(H_k^2-H_k^{(2)}\right)=\frac{1}{n}\left(\frac{2}{n}H_{n-1}-H_{n-1}^2+3H_{n-1}^{(2)}\right),\tag{3.16}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_k^3=\frac{1}{2n}\left(5H_{n-1}^{(2)}+\frac{4}{n}H_{n-1}-H_{n-1}^2-\frac{2}{n^2}\right),\tag{3.17}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_kH_k^{(2)}=\frac{1}{2n}\left(H_{n-1}^2-H_{n-1}^{(2)}-\frac{2}{n^2}\right).\tag{3.18}$$

Finally, we may make the substitutions $r = 0, 1, 2$ in (3.12) to obtain (3.19)–(3.21).

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\begin{bmatrix}k+1\\s+1\end{bmatrix}\frac{s!}{k!}=(-1)^s\frac{s!}{n!}\begin{bmatrix}n\\s\end{bmatrix},\tag{3.19}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\begin{bmatrix}k+1\\s+1\end{bmatrix}\frac{s!}{k!}H_k=(-1)^s\frac{s!}{n!}\left(H_n\begin{bmatrix}n\\s\end{bmatrix}-(s+1)\begin{bmatrix}n\\s+1\end{bmatrix}\right),\tag{3.20}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\begin{bmatrix}k+1\\s+1\end{bmatrix}\frac{s!}{k!}\left(H_k^2+H_k^{(2)}\right)$$
$$=(-1)^s\frac{s!}{n!}\left(\left(H_n^2+H_n^{(2)}\right)\begin{bmatrix}n\\s\end{bmatrix}-2(s+1)H_n\begin{bmatrix}n\\s+1\end{bmatrix}+(s+2)(s+1)\begin{bmatrix}n\\s+2\end{bmatrix}\right).\tag{3.21}$$

It should be noticed that more harmonic number identities can be established from those in Examples 3.1 and 3.2 by the binomial inverse relation; readers may list them.

**Corollary 3.4.** *The numbers $P(r, n, k)$ have the following expressions:*

$$\binom{n+i}{i}P(r,n+i,i)=\sum_{k=0}^{n}\binom{n-k+i}{i}\begin{bmatrix}k\\r\end{bmatrix}\frac{r!}{k!}=\sum_{k=0}^{n}(-1)^{k-r}\binom{n+i}{k+i}\begin{bmatrix}k\\r\end{bmatrix}\frac{r!}{k!},\tag{3.22}$$

$$\binom{n}{i}P(r,n,i)=\frac{i!}{n!}\sum_{k=0}^{n}\begin{bmatrix}n\\r+k\end{bmatrix}\begin{Bmatrix}k+1\\i+1\end{Bmatrix}\frac{(r+k)!}{k!}.\tag{3.23}$$

**Proof.** Eqs. (3.22) and (3.23) are obtained by applying the summation property (2.2) to the next three Riordan array and generating function pairs:

$$\mathscr{R}\left(\binom{n-k+i}{i}\right)=\left(\frac{1}{(1-t)^{i+1}},t\right),\qquad \mathscr{G}\left(\frac{r!}{k!}\begin{bmatrix}k\\r\end{bmatrix}\right)=(-\log(1-t))^r,$$
$$\mathscr{R}\left(\binom{n+i}{k+i}\right)=\left(\frac{1}{(1-t)^{i+1}},\frac{t}{1-t}\right),\qquad \mathscr{G}\left((-1)^{k-r}\frac{r!}{k!}\begin{bmatrix}k\\r\end{bmatrix}\right)=(\log(1+t))^r,$$

and

$$\mathscr{R}\left(\begin{bmatrix}n\\r+k\end{bmatrix}\frac{(r+k)!}{n!}\right)=((-\log(1-t))^r,-\log(1-t)),$$
$$\mathscr{G}\left(\frac{i!}{k!}\begin{Bmatrix}k+1\\i+1\end{Bmatrix}\right)=e^t(e^t-1)^i.\tag{3.24}$$

Note that the three identities given by (3.22) and (3.23) are actually the inverses of (3.3), (3.4) and (3.12), respectively. □

**Example 3.3.** The substitution $i = 0$ in (3.22) gives (3.25). The substitutions $r = 1$ and $r = 2$ in (3.22) give (3.26) and (3.27), which have (3.28) and (3.29) as further special cases.

$$P(r,n,0)=\frac{r!}{n!}\begin{bmatrix}n+1\\r+1\end{bmatrix}=\sum_{k=0}^{n}\begin{bmatrix}k\\r\end{bmatrix}\frac{r!}{k!}=\sum_{k=0}^{n}(-1)^{k-r}\binom{n}{k}\begin{bmatrix}k\\r\end{bmatrix}\frac{r!}{k!},\tag{3.25}$$

$$\sum_{k=1}^{n}\binom{n-k+i}{i}\frac{1}{k}=\sum_{k=1}^{n}(-1)^{k-1}\binom{n+i}{k+i}\frac{1}{k}=\binom{n+i}{i}(H_{n+i}-H_i),\tag{3.26}$$

$$\sum_{k=1}^{n}\binom{n-k+i}{i}\frac{H_{k-1}}{k}=\sum_{k=1}^{n}(-1)^k\binom{n+i}{k+i}\frac{H_{k-1}}{k}$$
$$=\frac{1}{2}\binom{n+i}{i}\left((H_{n+i}-H_i)^2-\left(H_{n+i}^{(2)}-H_i^{(2)}\right)\right),\tag{3.27}$$

$$\sum_{k=1}^{n}\frac{1}{k}=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{k}=H_n,\tag{3.28}$$

$$\sum_{k=1}^{n}\frac{H_{k-1}}{k}=\sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{H_{k-1}}{k}=\frac{1}{2}\left(H_n^2-H_n^{(2)}\right).\tag{3.29}$$

Note that (3.28) is indicated by [20, Eq. (6.72)]. Moreover, from (3.29), we have

$$\sum_{k=1}^{n} \frac{H_k}{k} = \frac{1}{2}\left(H_n^2 + H_n^{(2)}\right),$$

which is just [20, Eq. (6.71)]. Next, setting $r = 0, 1$ in (3.23) gives (3.30) and (3.31).

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k+1 \\ i+1 \end{Bmatrix} = \binom{n}{i} \frac{n!}{i!}, \tag{3.30}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k+1 \end{bmatrix} \begin{Bmatrix} k+1 \\ i+1 \end{Bmatrix} (k+1) = \binom{n}{i} \frac{n!}{i!} (H_n - H_i). \tag{3.31}$$

More results will be deduced from these two identities as well as (3.23) by replacing $i$ by special values. However, it may be found that they are also special cases of (4.12), (4.13) and (4.15), so we choose not to list them here.

**Theorem 3.5.** *The following relations hold:*

$$\sum_{k=0}^{n} \binom{r+n-k-1}{n-k} \binom{k+s}{s} P(i, k+s, s) = \binom{n+r+s}{r+s} P(i, n+r+s, r+s), \tag{3.32}$$

$$\binom{n+i}{i} P(r, n+i, i) = \sum_{k=0}^{n} \binom{i+n-k-1}{n-k} \begin{bmatrix} k+1 \\ r+1 \end{bmatrix} \frac{r!}{k!}. \tag{3.33}$$

**Proof.** To obtain (3.32), apply the summation property (2.2) to the Riordan array

$$\mathscr{R}\left(\binom{r+n-k-1}{n-k}\right) = \left(\frac{1}{(1-t)^r}, t\right)$$

and the generating function (1.6). Setting $s = 0$ in (3.32) and interchanging $r$ and $i$ gives the expression (3.33), which can be compared with Corollary 3.4. □

**Example 3.4.** Setting $i = 0, 1, 2$ in (3.32) gives (3.34)–(3.36), respectively. It is easy to verify that (3.34) can be obtained by the Vandermonde convolution formula.

$$\sum_{k=0}^{n} \binom{r+n-k-1}{n-k} \binom{k+s}{s} = \binom{n+r+s}{r+s}, \tag{3.34}$$

$$\sum_{k=0}^{n} \binom{r+n-k-1}{n-k} \binom{k+s}{s} H_{k+s} = \binom{n+r+s}{r+s} (H_{n+r+s} - H_{r+s} + H_s), \tag{3.35}$$

$$\sum_{k=0}^{n} \binom{r+n-k-1}{n-k} \binom{k+s}{s} \left(H_{k+s}^2 - H_{k+s}^{(2)}\right)$$
$$= \binom{n+r+s}{r+s} \left((H_{n+r+s} - H_{r+s})(H_{n+r+s} - H_{r+s} + 2H_s) - H_{n+r+s}^{(2)} + H_{r+s}^{(2)} - H_s^{(2)} + H_s^2\right). \tag{3.36}$$

More special cases can be obtained from these three identities. For example, when $s = 0$, (3.35) reduces to (3.37). Setting further $r = 1, 2$ in (3.37) gives us (3.38) and (3.39), which are two known identities (see [20, Eqs. (6.67) and (6.68)]).

$$\sum_{k=0}^{n} \binom{r+n-k-1}{n-k} H_k = \binom{n+r}{r} (H_{n+r} - H_r), \tag{3.37}$$

$$\sum_{k=0}^{n} H_k = (n+1)(H_{n+1} - 1), \tag{3.38}$$

$$\sum_{k=0}^{n} kH_k = \frac{(n+1)n}{2} H_{n+1} - \frac{(n+1)n}{4}. \tag{3.39}$$

**Corollary 3.6.** *The following identities relate the Stirling numbers of both kinds and the numbers $P(r, n, k)$:*

$$\sum_{k=0}^{n}(-1)^{k-i}\frac{i!}{k!}\left\{ {k+1 \atop i+1} \right\}P(k,n+r,r)=\binom{r+n-i-1}{n-i}\binom{n+r}{r}^{-1}, \tag{3.40}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{r}{n-k}\binom{k+r}{r}P(i,k+r,r)=\frac{i!}{n!}\left[ {n+1 \atop i+1} \right]. \tag{3.41}$$

**Proof.** From the Riordan array (2.8) and the generating function (3.24), we have

$$\sum_{k=0}^{n}\binom{n+r}{r}P(k,n+r,r)(-1)^{k-i}\frac{i!}{k!}\left\{ {k+1 \atop i+1} \right\}$$
$$=[t^n]\frac{1}{(1-t)^{r+1}}\left[e^{-y}(1-e^{-y})^i|y=-\log(1-t)\right]=[t^n]\frac{t^i}{(1-t)^r}=\binom{r+n-i-1}{n-i},$$

which gives (3.40). Next, from the Riordan array

$$\mathscr{R}\left((-1)^{n-k}\binom{r}{n-k}\right)=((1-t)^r,t)$$

and the generating function (1.6), and taking into account (2.5), we obtain (3.41). Note that identities (3.40) and (3.41) can be viewed as the inverses of (3.33). □

**Example 3.5.** The substitutions $i=0,1,2$ in (3.40) give us the next three identities, where the first is also a special case of the Vandermonde convolution formula.

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{r}{n-k}\binom{k+r}{r}=1, \tag{3.42}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{r}{n-k}\binom{k+r}{r}H_{k+r}=H_n+H_r, \tag{3.43}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{r}{n-k}\binom{k+r}{r}\left(H_{k+r}^2-H_{k+r}^{(2)}\right)=(H_n+H_r)^2-\left(H_n^{(2)}+H_r^{(2)}\right). \tag{3.44}$$

## 4. Connections with Stirling numbers

In Section 3, many identities involve both harmonic numbers and Stirling numbers. In this section, we will provide more results of this type, some of which also involve Lah numbers.

**Theorem 4.1.** *Let $\langle x\rangle_n$ be the rising factorial polynomials defined by $\langle x\rangle_0=1$ and $\langle x\rangle_n=x(x+1)\cdots(x+n-1)$ for $n\ge 1$. Then the following identities relate the Stirling numbers of both kinds and the numbers $P(r,n,k)$:*

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{ {n \atop k} \right\}\binom{k+i}{i}k!P(r,k+i,i)=r!\binom{n}{r}(i+1)^{n-r}, \tag{4.1}$$

$$r!\sum_{k=0}^{n}\left[ {n \atop k} \right]\binom{k}{r}(i+1)^{k-r}=n!\binom{n+i}{i}P(r,n+i,i)=\left.\frac{d^r}{dx^r}\langle x\rangle_n\right|_{x=i+1}, \tag{4.2}$$

$$\sum_{k=r}^{n}P(k,n+i,i)\frac{(-i-1)^{k-r}}{(k-r)!}=\frac{r!}{n!}\left[ {n \atop r} \right]\binom{n+i}{i}^{-1}. \tag{4.3}$$

**Proof.** It is easy to verify that

$$\sum_{k=0}^{\infty}\frac{r!}{k!}\binom{k}{r}x^{k-r}t^k=\sum_{k=r}^{\infty}\frac{x^{k-r}}{(k-r)!}t^k=t^re^{xt}. \tag{4.4}$$

Then, by applying (2.2) to the Riordan array (2.4) and the generating function (1.6), we have

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n \atop k}\right\}\binom{k+i}{i}k!P(r,k+i,i)=n!\sum_{k=0}^{n}(-1)^{n-k}\frac{k!}{n!}\left\{{n \atop k}\right\}\binom{k+i}{i}P(r,k+i,i)$$
$$=n![t^n]\left[\frac{(-\log(1-y))^r}{(1-y)^{i+1}}\middle| y=1-\mathrm{e}^{-t}\right]=n![t^n]t^r\mathrm{e}^{(i+1)t}=r!\binom{n}{r}(i+1)^{n-r},$$

which gives (4.1). In a similar way, from the Riordan array (2.3) and the generating function (4.4), we find

$$r!\sum_{k=0}^{n}\left[{n \atop k}\right]\binom{k}{r}(i+1)^{k-r}=n!\sum_{k=0}^{n}\frac{k!}{n!}\left[{n \atop k}\right]\frac{r!}{k!}\binom{k}{r}(i+1)^{k-r}$$
$$=n![t^n]\left[y^r\mathrm{e}^{(i+1)y}|y=-\log(1-t)\right]=n![t^n]\frac{(-\log(1-t))^r}{(1-t)^{i+1}}=n!\binom{n+i}{i}P(r,n+i,i),$$

which is the first equation of (4.2) and can also be obtained directly by the inverse relation of Stirling numbers of both kinds. To obtain the second equation of (4.2), we make use of (see [19, p. 213, Eq. (5f)])

$$\sum_{k=0}^{n}\left[{n \atop k}\right]x^k=\langle x\rangle_n;$$

then

$$\sum_{k=0}^{n}\left[{n \atop k}\right](k)_r(i+1)^{k-r}=\frac{\mathrm{d}^r}{\mathrm{d}x^r}\sum_{k=0}^{n}\left[{n \atop k}\right]x^k\Bigg|_{x=i+1}=\frac{\mathrm{d}^r}{\mathrm{d}x^r}\langle x\rangle_n\Bigg|_{x=i+1}.$$

Finally, combining the Riordan array (2.8) with the generating function (4.4) gives

$$\sum_{k=0}^{n}\binom{n+i}{i}P(k,n+i,i)\binom{k}{r}\frac{r!}{k!}(-i-1)^{k-r}$$
$$=[t^n]\frac{1}{(1-t)^{i+1}}\left[y^r\mathrm{e}^{-(i+1)y}|\,y=-\log(1-t)\right]=[t^n](-\log(1-t))^r=\frac{r!}{n!}\left[{n \atop r}\right],$$

from which we can establish (4.3). Then the proof is complete. □

**Example 4.1.** Setting $r=0,1,2$ in Eq. (4.1) gives (4.5)–(4.7), respectively. Setting $i=0$ in Eq. (4.1) yields (4.8). Eqs. (4.9)–(4.11) are $i=0$ cases of (4.5)–(4.7), respectively; they are also special cases of (4.8).

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n \atop k}\right\}\binom{k+i}{i}k!=(i+1)^n, \tag{4.5}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n \atop k}\right\}\binom{k+i}{i}k!H_{k+i}=n(i+1)^{n-1}+H_i(i+1)^n, \tag{4.6}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n \atop k}\right\}\binom{k+i}{i}k!\left(H_{k+i}^2-H_{k+i}^{(2)}\right)=n(n-1)(i+1)^{n-2}+2nH_i(i+1)^{n-1}+\left(H_i^2-H_i^{(2)}\right)(i+1)^n, \tag{4.7}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n \atop k}\right\}\left[{k+1 \atop r+1}\right]=\binom{n}{r}, \tag{4.8}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n \atop k}\right\}k!=1, \tag{4.9}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n \atop k}\right\}k!H_k=n, \tag{4.10}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n \atop k}\right\}k!\left(H_k^2-H_k^{(2)}\right)=n(n-1). \tag{4.11}$$

Similarly, setting $r=0,1,2$ in Eq. (4.2) gives (4.12)–(4.14), respectively, and setting $i=0$ in Eq. (4.2) yields (4.15). Eqs. (4.16)–(4.18) are $i=0$ cases of (4.12)–(4.14), respectively. It can be found that (4.12), (4.15)–(4.18) can be obtained from (4.5), (4.8)–(4.11) by the inverse relation associated with the Stirling numbers of both kinds.

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (i+1)^k = n! \binom{n+i}{i}, \tag{4.12}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} k(i+1)^{k-1} = n! \binom{n+i}{i} (H_{n+i} - H_i), \tag{4.13}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} k(k-1)(i+1)^{k-2} = n! \binom{n+i}{i} \left( (H_{n+i} - H_i)^2 - \left( H_{n+i}^{(2)} - H_i^{(2)} \right) \right), \tag{4.14}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \binom{k}{r} = \begin{bmatrix} n+1 \\ r+1 \end{bmatrix}, \tag{4.15}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = n!, \tag{4.16}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} k = n! H_n, \tag{4.17}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} k(k-1) = n! \left( H_n^2 - H_n^{(2)} \right). \tag{4.18}$$

Finally, the substitution $i = 0$ in (4.3) gives (4.19). Setting further $r = 0, 1, 2$ in (4.19) yields (4.20)–(4.22), where $n \geq 1$ is a positive integer.

$$\sum_{k=r}^{n} (-1)^{k-r} \binom{k}{r} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = \begin{bmatrix} n \\ r \end{bmatrix}, \tag{4.19}$$

$$\sum_{k=0}^{n} (-1)^{k} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = 0 \quad (n \geq 1), \tag{4.20}$$

$$\sum_{k=0}^{n} (-1)^{k} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} k = -(n-1)!, \tag{4.21}$$

$$\sum_{k=0}^{n} (-1)^{k} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} k(k-1) = 2(n-1)! H_{n-1}. \tag{4.22}$$

Recall that the Lah numbers are defined by $\binom{n-1}{k-1} \frac{n!}{k!}$ (see [19, p. 135] and [25, p. 43]). These numbers occur in the following theorem.

**Theorem 4.2.** *For $n \geq i \geq 1$, the following identities relate the Stirling numbers of both kinds, the Lah numbers and the numbers $P(r, n, k)$:*

$$\sum_{k=i}^{n} (-1)^{n-k} \begin{Bmatrix} n \\ k \end{Bmatrix} \binom{k-1}{i-1} \frac{k!}{i!} P(r, k-1, i-1) = r! \binom{n}{r} \begin{Bmatrix} n-r \\ i \end{Bmatrix}, \tag{4.23}$$

$$r! \sum_{k=i}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \binom{k}{r} \begin{Bmatrix} k-r \\ i \end{Bmatrix} = \binom{n-1}{i-1} \frac{n!}{i!} P(r, n-1, i-1). \tag{4.24}$$

**Proof.** First of all we observe that

$$\sum_{k=i+r}^{\infty} \binom{k-1}{i-1} P(r, k-1, i-1) t^k = \frac{t^i}{(1-t)^i} (-\log(1-t))^r, \tag{4.25}$$

$$\sum_{k=i+r}^{\infty} \frac{i!}{(k-r)!} \begin{Bmatrix} k-r \\ i \end{Bmatrix} t^k = t^r (e^t - 1)^i. \tag{4.26}$$

Then the summation property (2.2) gives (4.23):

$$\frac{n!}{i!}\sum_{k=i}^{n}(-1)^{n-k}\frac{k!}{n!}\left\{{n \atop k}\right\}\binom{k-1}{i-1}P(r,k-1,i-1)$$
$$=\frac{n!}{i!}[t^n]\left[\frac{y^i}{(1-y)^i}(-\log(1-y))^r\Big|y=1-\mathrm{e}^{-t}\right]=\frac{n!}{i!}[t^n]t^r(\mathrm{e}^t-1)^i=r!\binom{n}{r}\left\{{n-r \atop i}\right\}.$$

Analogously, using (4.25) and (4.26), we have

$$\frac{n!}{i!}\sum_{k=i}^{n}\frac{k!}{n!}\left[{n \atop k}\right]\frac{i!}{(k-r)!}\left\{{k-r \atop i}\right\}=\frac{n!}{i!}[t^n][y^r(\mathrm{e}^y-1)^i|y=-\log(1-t)]$$
$$=\frac{n!}{i!}[t^n]\left(\frac{t}{1-t}\right)^i(-\log(1-t))^r=\binom{n-1}{i-1}\frac{n!}{i!}P(r,n-1,i-1),$$

which is (4.24). Obviously, identity (4.24) can be obtained from (4.23) by the inverse relation of Stirling numbers of both kinds. Note also that, when $n<r+i$, both sides of (4.23) and (4.24) will vanish. □

**Example 4.2.** Setting $r=0,1,2$ in (4.23) gives (4.27)–(4.29); setting $r=0,1,2$ in (4.24) gives (4.30)–(4.32). In particular, (4.30) is the inverse of (4.27) and is a well-known relation between Lah numbers and Stirling numbers (for example, see [19, p. 156, Exercise 2 (4)] and [25, p. 43, Exercise 16 (d)]).

$$\sum_{k=i}^{n}(-1)^{n-k}\left\{{n \atop k}\right\}\binom{k-1}{i-1}\frac{k!}{i!}=\left\{{n \atop i}\right\}, \tag{4.27}$$

$$\sum_{k=i}^{n}(-1)^{n-k}\left\{{n \atop k}\right\}\binom{k-1}{i-1}\frac{k!}{i!}H_{k-1}=n\left\{{n-1 \atop i}\right\}+H_{i-1}\left\{{n \atop i}\right\}, \tag{4.28}$$

$$\sum_{k=i}^{n}(-1)^{n-k}\left\{{n \atop k}\right\}\binom{k-1}{i-1}\frac{k!}{i!}\left(H_{k-1}^2-H_{k-1}^{(2)}\right)$$
$$=n(n-1)\left\{{n-2 \atop i}\right\}+2nH_{i-1}\left\{{n-1 \atop i}\right\}+\left(H_{i-1}^2-H_{i-1}^{(2)}\right)\left\{{n \atop i}\right\}, \tag{4.29}$$

$$\sum_{k=i}^{n}\left[{n \atop k}\right]\left\{{k \atop i}\right\}=\binom{n-1}{i-1}\frac{n!}{i!}, \tag{4.30}$$

$$\sum_{k=i}^{n}\left[{n \atop k}\right]\left\{{k-1 \atop i}\right\}k=\binom{n-1}{i-1}\frac{n!}{i!}(H_{n-1}-H_{i-1}), \tag{4.31}$$

$$\sum_{k=i}^{n}\left[{n \atop k}\right]\left\{{k-2 \atop i}\right\}k(k-1)=\binom{n-1}{i-1}\frac{n!}{i!}\left((H_{n-1}-H_{i-1})^2-\left(H_{n-1}^{(2)}-H_{i-1}^{(2)}\right)\right). \tag{4.32}$$

The $i=1$ cases of (4.23), (4.24) and (4.27)–(4.32) are essentially equivalent to (4.8)–(4.11) and (4.15)–(4.18). As an instance, according to the recurrence relation of the Stirling numbers of the first kind, for $n\geq r+1$ we have

$$\sum_{k=r}^{n}(-1)^{n-k}\left\{{n \atop k}\right\}\left[{k+1 \atop r+1}\right]=\sum_{k=r}^{n}(-1)^{n-k}\left\{{n \atop k}\right\}\left(\left[{k \atop r}\right]+k\left[{k \atop r+1}\right]\right)$$
$$=\sum_{k=r+1}^{n}(-1)^{n-k}\left\{{n \atop k}\right\}\left[{k \atop r+1}\right]k.$$

Then (4.8) indicates that

$$\sum_{k=r+1}^{n}(-1)^{n-k}\left\{{n \atop k}\right\}\left[{k \atop r+1}\right]k=\binom{n}{r}, \tag{4.33}$$

where $n\geq r+1$. Identity (4.33) coincides with the $i=1$ case of (4.23). Setting further $r=2$ in it gives

$$\sum_{k=3}^{n}(-1)^{n-k}\left\{{n \atop k}\right\}k!\left(H_{k-1}^2-H_{k-1}^{(2)}\right)=n(n-1), \tag{4.34}$$

which is equivalent to (4.11).

The results given below have only slight differences from those presented above.

**Theorem 4.3.** *The following identities relate the Stirling numbers of both kinds and the numbers $P(r,n,k)$:*

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n+1 \atop k+1}\right\}\binom{k+i}{i}k!P(r,k+i,i)=r!\binom{n}{r}i^{n-r}, \tag{4.35}$$

$$r!\sum_{k=0}^{n}\left[{n+1 \atop k+1}\right]\binom{k}{r}i^{k-r}=n!\binom{n+i}{i}P(r,n+i,i), \tag{4.36}$$

$$\sum_{k=r}^{n}P(k,n+i,i)\frac{(-i)^{k-r}}{(k-r)!}=\frac{r!}{n!}\left[{n+1 \atop r+1}\right]\binom{n+i}{i}^{-1}, \tag{4.37}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n+1 \atop k+1}\right\}\binom{k}{i}k!P(r,k,i)=i!r!\binom{n}{r}\left\{{n-r \atop i}\right\}, \tag{4.38}$$

$$r!\sum_{k=i}^{n}\left[{n+1 \atop k+1}\right]\binom{k}{r}\left\{{k-r \atop i}\right\}=\binom{n}{i}\frac{n!}{i!}P(r,n,i), \tag{4.39}$$

$$\sum_{k=0}^{n}(-1)^{k-i}\frac{i!}{k!}\left\{{k \atop i}\right\}P(k,n+r,r)=\binom{r+n-i}{n-i}\binom{n+r}{r}^{-1}, \tag{4.40}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{r+1}{n-k}\binom{k+r}{r}P(i,k+r,r)=\frac{i!}{n!}\left[{n \atop i}\right]. \tag{4.41}$$

Eqs. (4.35)–(4.37) may be compared with Eqs. (4.1)–(4.3), Eqs. (4.38)–(4.39) may be compared with Eqs. (4.23)–(4.24), and Eqs. (4.40)–(4.41) may be compared with Eqs. (3.40)–(3.41). All of these seven identities can be established by the Riordan array method, and we omit the proofs.

**Remark 4.4.** Actually, Eq. (4.35) can be obtained from (4.1) by the recurrence

$$\left\{{n+1 \atop k+1}\right\}=\left\{{n \atop k}\right\}+(k+1)\left\{{n \atop k+1}\right\}$$

of the Stirling numbers of the second kind and the recurrence

$$\binom{k+i}{i}P(r,k+i,i)=\binom{k+i-1}{i-1}P(r,k+i-1,i-1)+\binom{k+i-1}{i}P(r,k+i-1,i)$$

of the numbers $P(r,n,k)$. Similarly, (4.38) can be derived from (4.23). Note also that (4.41) was first given in [17, Theorem 10]. For the identities of Theorem 4.3, we do not discuss further the relations between them and those presented before, but consider special harmonic number identities obtained from them.

**Example 4.3.** Setting $r=0,1,2$ in (4.35) gives (4.42)–(4.44). Setting further $i=0$ in (4.43) and (4.44) yields (4.45) and (4.46), which are in fact special cases of the orthogonal relation

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n+1 \atop k+1}\right\}\left[{k+1 \atop r+1}\right]=\delta_{n,r}.$$

Readers may compare these five identities with (4.5)–(4.7) and (4.10)–(4.11).

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n+1 \atop k+1}\right\}\binom{k+i}{i}k!=i^n, \tag{4.42}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n+1 \atop k+1}\right\}\binom{k+i}{i}k!H_{k+i}=n\cdot i^{n-1}+H_i\cdot i^n, \tag{4.43}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n+1 \atop k+1}\right\}\binom{k+i}{i}k!\left(H_{k+i}^2-H_{k+i}^{(2)}\right)=n(n-1)\cdot i^{n-2}+2nH_i\cdot i^{n-1}+\left(H_i^2-H_i^{(2)}\right)\cdot i^n, \tag{4.44}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n+1 \atop k+1}\right\}k!H_k=\delta_{n,1}, \tag{4.45}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n+1 \atop k+1}\right\}k!\left(H_k^2-H_k^{(2)}\right)=2\delta_{n,2}. \tag{4.46}$$

Setting $r=0,1,2$ in (4.38) gives (4.47)–(4.49), which may be compared with (4.27)–(4.29). Setting further $i=0$ in (4.48) and (4.49) yields (4.45) and (4.46) again; while setting further $i=1$ in (4.47)–(4.49) yields (4.50)–(4.52).

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n+1 \atop k+1}\right\}\binom{k}{i}k!=i!\left\{{n \atop i}\right\}, \tag{4.47}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n+1 \atop k+1}\right\}\binom{k}{i}k!H_k=i!\left(n\left\{{n-1 \atop i}\right\}+H_i\left\{{n \atop i}\right\}\right), \tag{4.48}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n+1 \atop k+1}\right\}\binom{k}{i}k!\left(H_k^2-H_k^{(2)}\right)$$
$$=i!\left(n(n-1)\left\{{n-2 \atop i}\right\}+2nH_i\left\{{n-1 \atop i}\right\}+\left(H_i^2-H_i^{(2)}\right)\left\{{n \atop i}\right\}\right), \tag{4.49}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n+1 \atop k+1}\right\}k\cdot k!=1\quad(n\geq 1), \tag{4.50}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n+1 \atop k+1}\right\}k\cdot k!H_k=n+1\quad(n\geq 2), \tag{4.51}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\left\{{n+1 \atop k+1}\right\}k\cdot k!\left(H_k^2-H_k^{(2)}\right)=n(n+1)\quad(n\geq 3). \tag{4.52}$$

The substitution $r=0$ in (4.39) and (4.40) leads us to (4.53) and (4.54), respectively, which may be compared with (3.30), (4.8), (4.30) and the orthogonal relation between the Stirling numbers of both kinds. Next, the substitutions $r=1,2$ in (4.39) yield (4.55)–(4.56), which may be compared with (4.31)–(4.32).

$$\sum_{k=i}^{n}\left[{n+1 \atop k+1}\right]\left\{{k \atop i}\right\}=\binom{n}{i}\frac{n!}{i!}, \tag{4.53}$$

$$\sum_{k=i}^{n}(-1)^{k-i}\left[{n+1 \atop k+1}\right]\left\{{k \atop i}\right\}=\frac{n!}{i!}, \tag{4.54}$$

$$\sum_{k=i}^{n}\left[{n+1 \atop k+1}\right]\left\{{k-1 \atop i}\right\}k=\binom{n}{i}\frac{n!}{i!}(H_n-H_i), \tag{4.55}$$

$$\sum_{k=i}^{n}\left[{n+1 \atop k+1}\right]\left\{{k-2 \atop i}\right\}k(k-1)=\binom{n}{i}\frac{n!}{i!}\left((H_n-H_i)^2-\left(H_n^{(2)}-H_i^{(2)}\right)\right). \tag{4.56}$$

Finally, to compare with (3.42)–(3.44), we list identities (4.57)–(4.59), though they have been given in [17, p. 846]. These three identities can be obtained from (4.41) by the substitutions $i=0,1,2$.

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{r+1}{n-k}\binom{k+r}{r}=\delta_{n,0}, \tag{4.57}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{r+1}{n-k}\binom{k+r}{r}H_{k+r}=\frac{1}{n}, \tag{4.58}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{r+1}{n-k}\binom{k+r}{r}\left(H_{k+r}^2-H_{k+r}^{(2)}\right)=\frac{2}{n}(H_{n-1}+H_r). \tag{4.59}$$

## 5. Connections with Bernoulli numbers and Cauchy numbers

The higher-order Bernoulli numbers (or Nörlund polynomials) $B_n^{(z)}$ and the Bernoulli polynomials $B_n(x)$ are defined by

$$\sum_{n=0}^{\infty} B_n^{(z)} \frac{t^n}{n!} = \left(\frac{t}{e^t - 1}\right)^z \quad \text{and} \quad \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{xt}. \tag{5.1}$$

They are generalizations of the Bernoulli numbers $B_n$, i.e., $B_n = B_n^{(1)} = B_n(0)$. These numbers and polynomials are among the most interesting and important sequences in mathematics and they have numerous applications in combinatorics, number theory, numerical analysis, and other fields. In this section, we first present some results concerning these numbers and polynomials.

**Theorem 5.1.** *The following identities relate the Bernoulli numbers (polynomials) and the numbers $P(r, n, k)$:*

$$\sum_{k=0}^{n} (-1)^k P(k, n + r, r) \frac{B_k^{(i)}}{k!} = \binom{n + i + r}{r} \binom{n + r}{r}^{-1} P(i, n + i + r, r), \tag{5.2}$$

$$\sum_{k=0}^{n} P(k, n + r, r) \frac{B_k(i + 1)}{k!} = \binom{n + r + i + 1}{r + i} \binom{n + r}{r}^{-1} (H_{n+r+i+1} - H_{r+i}). \tag{5.3}$$

**Proof.** Applying the summation property (2.2) to the Riordan array (2.8) and the generating function of $B_k^{(i)}$, we have

$$\sum_{k=0}^{n} \binom{n + r}{r} P(k, n + r, r) \frac{(-1)^k B_k^{(i)}}{k!} = [t^n] \frac{1}{(1 - t)^{r+1}} \left[ \left(\frac{y}{1 - e^{-y}}\right)^i \Bigg| y = -\log(1 - t) \right]$$
$$= [t^{n+i}] \frac{(-\log(1 - t))^i}{(1 - t)^{r+1}} = \binom{n + i + r}{r} P(i, n + i + r, r),$$

which gives (5.2). Analogously, applying (2.2) to (2.8) and the generating function of $B_k(i + 1)$, we have

$$\sum_{k=0}^{n} \binom{n + r}{r} P(k, n + r, r) \frac{B_k(i + 1)}{k!} = [t^n] \frac{1}{(1 - t)^{r+1}} \left[ \frac{y}{e^y - 1} e^{(i+1)y} \Bigg| y = -\log(1 - t) \right]$$
$$= [t^{n+1}] \frac{-\log(1 - t)}{(1 - t)^{r+i+1}} = \binom{n + r + i + 1}{r + i} P(1, n + r + i + 1, r + i),$$

which yields (5.3). □

**Example 5.1.** The substitutions $i = 1$ and $r = 0$ in (5.2) give us (5.4) and (5.5), respectively, which have (5.6) as common special case. Setting $i = 0$ in (5.3) and using $B_k(1) = (-1)^k B_k$ gives (5.4) again; while setting $r = 0$ in (5.3) yields (5.7).

$$\sum_{k=0}^{n} (-1)^k P(k, n + r, r) \frac{B_k}{k!} = \frac{n + r + 1}{n + 1} (H_{n+r+1} - H_r), \tag{5.4}$$

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n + 1 \\ k + 1 \end{bmatrix} B_k^{(i)} = \begin{bmatrix} n + i + 1 \\ i + 1 \end{bmatrix} \binom{n + i}{i}^{-1}, \tag{5.5}$$

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n + 1 \\ k + 1 \end{bmatrix} B_k = n! H_{n+1}, \tag{5.6}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n + 1 \\ k + 1 \end{bmatrix} B_k(i + 1) = n! \binom{n + i + 1}{i} (H_{n+i+1} - H_i). \tag{5.7}$$

Taking into account the inverse relation associated with the Stirling numbers of both kinds, we can obtain from (5.5)–(5.7) the next three identities. Note that (5.9) can be found in Cheon and El-Mikkawy's paper [23, p. 424], and readers may compare it with (4.45).

$$\sum_{k=0}^{n} (-1)^k \begin{Bmatrix} n + 1 \\ k + 1 \end{Bmatrix} \begin{bmatrix} k + i + 1 \\ i + 1 \end{bmatrix} \binom{k + i}{i}^{-1} = B_n^{(i)}, \tag{5.8}$$

$$\sum_{k=0}^{n} (-1)^k \begin{Bmatrix} n + 1 \\ k + 1 \end{Bmatrix} k! H_{k+1} = B_n, \tag{5.9}$$

$$\sum_{k=0}^{n} (-1)^k \begin{Bmatrix} n + 1 \\ k + 1 \end{Bmatrix} \binom{k + i + 1}{i} k! (H_{k+i+1} - H_i) = (-1)^n B_n(i + 1). \tag{5.10}$$

Other identities concerning the Bernoulli numbers and the numbers $P(r, n, k)$ can be obtained by the Riordan array method. For example, the following holds:

$$\sum_{k=0}^{n} P(k, n+r+i, r+i)\frac{B_k^{(i)}}{k!} = \binom{r+i}{i}\binom{n+i}{i}^{-1} P(i, n+r+i, r).$$

However, from it we cannot obtain nice special cases.

By the method of coefficients, particularly, the Riordan array method, many properties of the Cauchy numbers, the Cauchy polynomials and the sums of products of Cauchy numbers are established in [21,24,26], including some identities which relate the Cauchy numbers (polynomials) to the Stirling, Bernoulli and harmonic numbers. Following the notation of [21], we define

$$\sum_{k=0}^{\infty} \mathscr{C}_k \frac{t^k}{k!} = \frac{t}{\log(1+t)}, \tag{5.11}$$

$$\sum_{k=0}^{\infty} \hat{\mathscr{C}}_k \frac{t^k}{k!} = \frac{t}{(1+t)\log(1+t)}, \tag{5.12}$$

where $\mathscr{C}_k$ and $\hat{\mathscr{C}}_k$ are the Cauchy numbers of the first and second kinds. Note that, in general, the Cauchy numbers of the second kind are defined by $(-1)^k\hat{\mathscr{C}}_k$ (see [19, p. 293, Exercise 13]). Moreover, from the generating functions given above, it may be found that $\mathscr{C}_k/k!$ are actually the Bernoulli numbers of the second kind and $\hat{\mathscr{C}}_k$ are actually the Nörlund numbers $B_k^{(k)}$ (for example, see [27,28]). Based on the theory of Riordan arrays, the next theorem can be established.

**Theorem 5.2.** *The following identities relate the Cauchy numbers of both kinds and the numbers $P(r, n, k)$:*

$$\sum_{k=0}^{n} \binom{n}{k} P(r, n, k)\frac{\mathscr{C}_k}{k!} = nP(r-1, n, 1), \tag{5.13}$$

$$\sum_{k=0}^{n} \binom{n}{k} P(r, n, k)\frac{\hat{\mathscr{C}}_k}{k!} = P(r-1, n-1, 0) = \frac{(r-1)!}{(n-1)!}\left[{n \atop r}\right]. \tag{5.14}$$

**Proof.** From the Riordan array (2.7) and the generating function (5.11), we have

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} P(r, n, k)\frac{\mathscr{C}_k}{k!} &= [t^n]\frac{(-\log(1-t))^r}{1-t}\left[\frac{y}{\log(1+y)}\middle| y = \frac{t}{1-t}\right] \\ &= [t^{n-1}]\frac{(-\log(1-t))^{r-1}}{(1-t)^2} = nP(r-1, n, 1),\end{aligned}$$

which is (5.13). Analogously, from (2.7) and (5.12) and taking into account (2.6), we have

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} P(r, n, k)\frac{\hat{\mathscr{C}}_k}{k!} &= [t^n]\frac{(-\log(1-t))^r}{1-t}\left[\frac{y}{(1+y)\log(1+y)}\middle| y = \frac{t}{1-t}\right] \\ &= [t^{n-1}]\frac{(-\log(1-t))^{r-1}}{1-t} = P(r-1, n-1, 0) = \frac{(r-1)!}{(n-1)!}\left[{n \atop r}\right].\end{aligned}$$

This gives (5.14). □

**Example 5.2.** Setting $r = 1, 2$ in (5.13) yields (5.15) and (5.16). Setting $r = 1, 2$ in (5.14) and using the identity

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\hat{\mathscr{C}}_k}{k!} = (-1)^n \frac{\hat{\mathscr{C}}_n}{n!}$$

(see [21, Theorem 2.7]) gives (5.17) and (5.18). The last two identities (5.19) and (5.20) are further special cases.

$$\sum_{k=0}^{n} \binom{n}{k} (H_n - H_k)\frac{\mathscr{C}_k}{k!} = n, \tag{5.15}$$

$$\sum_{k=0}^{n} \binom{n}{k} \left(H_n^2 + H_n^{(2)} - H_k^2 - H_k^{(2)}\right)\frac{\mathscr{C}_k}{k!} = nH_n + n, \tag{5.16}$$

$$\sum_{k=0}^{n}\binom{n}{k}H_k\frac{\hat{\mathscr{C}}_k}{k!}=(-1)^nH_n\frac{\hat{\mathscr{C}}_n}{n!}-1\quad(n\geq 1), \tag{5.17}$$

$$\sum_{k=0}^{n}\binom{n}{k}\left(H_k^2+H_k^{(2)}\right)\frac{\hat{\mathscr{C}}_k}{k!}=(-1)^n\left(H_n^2+H_n^{(2)}\right)\frac{\hat{\mathscr{C}}_n}{n!}-H_n-\frac{1}{n}, \tag{5.18}$$

$$\sum_{k=0}^{n-1}\binom{n}{k}H_k\frac{\hat{\mathscr{C}}_k}{k!}=-1\quad(n\text{ even}), \tag{5.19}$$

$$\sum_{k=0}^{n-1}\binom{n}{k}\left(H_k^2+H_k^{(2)}\right)\frac{\hat{\mathscr{C}}_k}{k!}=-H_n-\frac{1}{n}\quad(n\text{ even}). \tag{5.20}$$

## 6. Conclusion

Series of identities involving the harmonic numbers, the Stirling numbers of both kinds, the Lah numbers, the Bernoulli numbers and polynomials and the Cauchy numbers of both kinds are presented in this paper. Actually, they are special cases of several general summation formulas, which are established by applying the Riordan array method to the numbers $P(r, n, k)$. We believe that more results on harmonic numbers and other special combinatorial sequences will be found by the theory of Riordan arrays.

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## References

[1] W. Chu, A binomial coefficient identity associated with Beukers' conjecture on Apéry numbers, Electron. J. Combin. 11 (1) (2004) Note 15, 3 pp.
[2] W. Chu, Harmonic number identities and Hermite–Padé approximations to the logarithm function, J. Approx. Theory 137 (1) (2005) 42–56.
[3] W. Chu, Partial-fraction decompositions and harmonic number identities, J. Combin. Math. Combin. Comput. 60 (2007) 139–153.
[4] W. Chu, Hypergeometric approach to Weideman's conjecture, Arch. Math. (Basel) 87 (5) (2006) 400–406.
[5] W. Chu, L. De Donno, Hypergeometric series and harmonic number identities, Adv. in Appl. Math. 34 (1) (2005) 123–137.
[6] W. Chu, A.M. Fu, Dougall–Dixon formula and harmonic number identities, Ramanujan J. 18 (1) (2009) 11–31.
[7] R. Sprugnoli, Riordan arrays and the Abel–Gould identity, Discrete Math. 142 (1–3) (1995) 213–233.
[8] A. Sofo, Some more identities involving rational sums, Appl. Anal. Discrete Math. 2 (1) (2008) 56–64.
[9] A. Sofo, Sums of derivatives of binomial coefficients, Adv. in Appl. Math. 42 (1) (2009) 123–134.
[10] A. Sofo, Integral forms of sums associated with harmonic numbers, Appl. Math. Comput. 207 (2) (2009) 365–372.
[11] K. Driver, H. Prodinger, C. Schneider, J.A.C. Weideman, Padé approximations to the logarithm. II. Identities, recurrences, and symbolic computation, Ramanujan J. 11 (2) (2006) 139–158.
[12] K. Driver, H. Prodinger, C. Schneider, J.A.C. Weideman, Padé approximations to the logarithm. III. Alternative methods and additional results, Ramanujan J. 12 (3) (2006) 299–314.
[13] R. Lyons, P. Paule, A. Riese, A computer proof of a series evaluation in terms of harmonic numbers, Appl. Algebra Engrg. Comm. Comput. 13 (4) (2002) 327–333.
[14] R. Osburn, C. Schneider, Gaussian hypergeometric series and supercongruences, Math. Comp. 78 (265) (2009) 275–292.
[15] P. Paule, C. Schneider, Computer proofs of a new family of harmonic number identities, Adv. in Appl. Math. 31 (2) (2003) 359–378.
[16] L.W. Shapiro, S. Getu, W.J. Woan, L.C. Woodson, The Riordan group, Discrete Appl. Math. 34 (1–3) (1991) 229–239.
[17] J. Spieß, Some identities involving harmonic numbers, Math. Comp. 55 (192) (1990) 839–863.
[18] D.A. Zave, A series expansion involving the harmonic numbers, Inform. Process. Lett. 5 (3) (1976) 75–77.
[19] L. Comtet, Advanced Combinatorics, D. Reidel Publishing Co., Dordrecht, 1974.
[20] R.L. Graham, D.E. Knuth, O. Patashnik, Concrete Mathematics, Addison-Wesley Publishing Company, Reading, MA, 1994.
[21] D. Merlini, R. Sprugnoli, M.C. Verri, The Cauchy numbers, Discrete Math. 306 (16) (2006) 1906–1920.
[22] R. Sprugnoli, Riordan arrays and combinatorial sums, Discrete Math. 132 (1–3) (1994) 267–290.
[23] G.-S. Cheon, M.E.A. El-Mikkawy, Generalized harmonic numbers with Riordan arrays, J. Number Theory 128 (2) (2008) 413–425.
[24] G.-S. Cheon, S.-G. Hwang, S.-G. Lee, Several polynomials associated with the harmonic numbers, Discrete Appl. Math. 155 (18) (2007) 2573–2584.
[25] J. Riordan, An Introduction to Combinatorial Analysis, Dover Publications, Inc., Mineola, NY, 2002, Reprint of the 1958 original.
[26] F.-Z. Zhao, Sums of products of Cauchy numbers, Discrete Math. 309 (12) (2009) 3830–3842.
[27] C. Jordan, Calculus of Finite Differences, third ed., Chelsea Publishing Co., New York, 1965.
[28] N.E. Nörlund, Vorlesungen über Differenzenrechnung, Springer, Berlin, 1924.