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Mellin transforms and asymptotics: Harmonic sums

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Dedicated to Don Knuth and N.G. de Bruijn
for their pioneering works on Mellin transforms and combinatorics

Abstract

This survey presents a unified and essentially self-contained approach to the asymptotic analysis of a large class of sums that arise in combinatorial mathematics, discrete probabilistic models, and the average-case analysis of algorithms. It relies on the Mellin transform, a close relative of the integral transforms of Laplace and Fourier. The method applies to harmonic sums that are superpositions of rather arbitrary "harmonics" of a common base function. Its principle is a precise correspondence between individual terms in the asymptotic expansion of an original function and singularities of the transformed function. The main applications are in the area of digital data structures, probabilistic algorithms, and communication theory.

Die Theorie der reziproken Funktionen und Integrale
ist ein centrales Gebiet, welches manche anderen Gebiete
der Analysis miteinander verbindet.

Hjalmar Mellin

Introduction

Hjalmar Mellin (1854–1933, see [59] for a summary of his works) gave his name to the *Mellin transform* that associates to a function $f(x)$ defined over the positive reals the complex function $f^*(s)$ where

$$f^*(s) = \int_0^{\infty} f(x) x^{s-1}\, dx. \tag{1}$$

The change of variables $x = e^{-u}$ shows that the Mellin transform is closely related to the Laplace transform and the Fourier transform. However, despite this connection,

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there are numerous applications where it proves convenient to operate directly with the Mellin form (1) rather than the Laplace–Fourier version. This is often the case in complex function theory (asymptotics of Gamma-related functions [64, 65]), in number theory (coefficients of Dirichlet series, after Riemann), in applied mathematics (asymptotic estimation of integral forms), and in the analysis of algorithms (harmonic sums introduced below). Thus, throughout this paper, we operate directly with the Mellin transform.

The major use of the Mellin transform examined here is for the *asymptotic analysis* of sums obeying the general pattern

$$G(x) = \sum_k \lambda_k g(\mu_k x), \tag{2}$$

either as $x \to 0$ or as $x \to \infty$. Following a proposal of [31], sums of this type are called *harmonic sums* as they represent a linear superposition of "harmonics" of a single base function $g(x)$.

Harmonic sums surface at many places in combinatorial mathematics as well as in the analysis of algorithms and data structures. De Bruijn and Knuth are responsible in an essential way for introducing the Mellin transform in this range of problems, as attested by Knuth's account in [56, p. 131] and the classic paper [16] which have been the basis of many later combinatorial applications. For instance, the analyses of the radix-exchange sort algorithm [56] and of the expected height of plane (Catalan) trees [16] involve the quantities

$$S_n = \sum_{k=0}^{\infty} \left[1 - \left(1 - \frac{1}{2^k}\right)^n\right] \quad \text{and} \quad T_n = \sum_{k=0}^{n} d(k) \frac{\binom{2n}{n-k}}{\binom{2n}{n}}, \tag{3}$$

where $d(k)$ is the number of divisors of k and $\binom{a}{b}$ is the binomial coefficient. These discrete quantities have continuous analogues that are harmonic sums

$$G(x) = \sum_{k=0}^{\infty} [1 - \mathrm{e}^{-x/2^k}] \quad \text{and} \quad H(x) = \sum_{k=0}^{\infty} d(k) \mathrm{e}^{-k^2 x^2}, \tag{4}$$

and elementary arguments establish that asymptotically $S_n \sim G(n)$ and $T_n \sim H(n)$.

There are inherent difficulties in the asymptotic analysis of the discrete sums S_n, T_n or their continuous analogues $G(x)$, $H(x)$. The divisor function in T_n and $H(x)$ fluctuates heavily in a rather irregular manner (for instance it equals 2 if and only if k is prime). The quantities S_n and $G(x)$ look more innocuous; however, a Mellin based analysis to be shown later reveals that they involve subtle fluctuations of a tiny amplitude, less than 10^{-5}. Such behaviors preclude the use of elementary real asymptotic techniques in most sums of this type.

Over the past 20 years, perhaps some 50 odd analyses of algorithms or related evaluations of parameters of combinatorial structures have involved in a crucial way a Mellin treatment of harmonic sums of sorts. We shall organize and detail in later sections several of these examples that can be broadly categorized, in terms of their range of applications, as follows.

- Digital searching methods: radix-exchange sort, digital trees, digital search trees [56] and their generalizations like "bucket" trees [32], Patricia trees [56, 80], suffix trees [48].
- Digital trees, variance and biased-bit models [54, 79].
- Tree manipulation algorithms: height and stack depth [16], register allocation [50].
- String searching and the occurrence of patterns in strings [68] (and references therein), with applications to carry propagation in binary adders [57] and data compression [48, 47].
- Multidimensional searching problems [29] and Euclidean matching [74].
- Extendible hashing and grid file methods [72, 73].
- Parallel and distributed algorithms like the leader election technique of [70], communication protocols based on the Capetanakis–Tsybakov scheme [20, 39, 46, 63], parallel sorting networks based on Batcher's odd–even scheme [75].
- Divide-and-conquer algorithms: mergesort [25] and geometric maxima finding [24].
- Randomized data structures, like skip lists [52].
- Probabilistic estimation algorithms: probabilistic counting [27], adaptive sampling [23], or approximate counting [22, 71].

In many cases, notably digital search trees, harmonic sums are an alternative to the method of "Rice integrals" that is discussed in detail elsewhere [34].

Some brief accounts of the method are given in the books by Hofri [44, p. 48ff], Kemp [51, p. 141], Mahmoud [62], and in the handbook chapter [84]. We follow here the architecture of the informal survey [31].

General properties of the Mellin transform are usually treated in detail in books on integral transforms, like those of Doetsch [18], Widder [86], or Titchmarsh [82]. Asymptotic methods in connection with Mellin transforms are discussed within the context of applied mathematics in treatises by Davies [13], Dingle [17], and Wong [88]. In particular, our work is close in spirit to Wong's who discusses extensively an analogue of harmonic sums in the form of "harmonic integrals"

$$G(x) = \int_0^\infty \lambda(\kappa) g(\mu(\kappa)x)\, \mathrm{d}\kappa. \tag{5}$$

(In this continuous case, by a change of variables, one may always assume that $\mu(\kappa) \equiv \kappa$.) Some of the many uses in number theory are treated for instance in [4, 7, 42].

Principles. A major use of Mellin transform in asymptotic analysis is for estimating asymptotically harmonic sums (2). The λ_k are the "amplitudes", the μ_k are the "frequencies", and $g(x)$ is the "base function". Harmonic sums reduce to usual Fourier series when the base function $g(x)$ is taken to be a complex exponential, $g(x) = \mathrm{e}^{\pm \mathrm{i}x}$, and the frequencies are the integers, $\mu_k \equiv k$.

1. *Mellin transforms and the "separation" property.* The Mellin transform as defined in (1) converges in a strip of the complex plane called the fundamental strip. By

a direct change of variables, the Mellin transform of $g(\mu x)$ is μ^{-s} times the transform $g^*(s)$ of $g(x)$. Thus, by linearity, the Mellin transform of a general harmonic sum is (conditionally)

$$G^*(s) = \Lambda(s) \cdot g^*(s), \tag{6}$$

where

$$\Lambda(s) = \sum_k \lambda_k \mu_k^{-s}, \qquad g^*(s) = \int_0^\infty g(x) x^{s-1}\, \mathrm{d}x.$$

It therefore factors as the product of the transform of the base function and of a generalized Dirichlet series: *Mellin transforms applied to harmonic sums "separate" the amplitude–frequency pair from the base function.*

The *inversion theorem* for Mellin transforms is analogous to Fourier inversion,

$$f(x) = \frac{1}{2\mathrm{i}\pi} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} f^*(s) x^{-s}\, \mathrm{d}s. \tag{7}$$

There, the integration line $\Re(s) = c$ should be taken in the fundamental strip of the Mellin transform.

Basic functional properties of the Mellin transform are recalled in Section 1 (Theorem 1) and the separation property is expressed by Lemma 2 of Section 3.

2. *Poles of transforms and asymptotics of originals.* There is a fundamental correspondence between the asymptotic expansion of a function (either at 0 or ∞) and singularities of the transformed function.

First and foremost, Mellin asymptotics crucially relies on *analytic continuation* of transforms. Consider the problem of asymptotically expanding $f(x)$ as $x \to 0$ when $f^*(s)$ is known to be meromorphically continuable in $\mathbb{C}$. The starting point is the inversion formula (7). The line of integration is then shifted to the *left* while taking residues into account. For instance when $c > 0$ and $f^*(s)$ has simple poles at the nonpositive integers, each simple pole at $s = -m$ contributes a term proportional to x^m since

$$\operatorname{Res}(f^*(s) x^{-s})_{s=-m} = \operatorname{Res}(f^*(s))_{s=-m} \cdot x^m.$$

(There $\operatorname{Res}(g(s))_{s=s_0}$ denotes the residues of $g(s)$ at $s = s_0$.) Thus, globally integrating along an infinite rectangle with sides $\Re(s) = c$ and $\Re(s) = -M - 1/2$ (M an integer) gives by the residue theorem

$$f(x) = \sum_{m=0}^{M} (\operatorname{Res} f^*(s))_{s=-m}\, x^m + \mathrm{O}(x^{M+1/2}). \tag{8}$$

Some additional decay condition on $f^*(s)$ is evidently necessary in order to justify (8).

The computation outlined above reflects a general phenomenon: *Poles of a Mellin transform are in direct correspondence with terms in the asymptotic expansion of the original function at either* 0 *or* $+\infty$.

For the asymptotic evaluation of a harmonic sum $G(x)$ this principle applies to $G^*(s)$ provided the Dirichlet series $\Lambda(s)$ and the transform $g^*(s)$ are each analytically continuable and of controlled growth. This assumption is realistic for many naturally occurring base functions and for amplitude and frequencies given by "laws" that are regular enough.

The fundamental correspondence is explored in Section 2 (Theorems 3 and 4); its application to harmonic sums is spelled out in Section 3.

Example. The mode of operation of Mellin asymptotics is well illustrated by the harmonic sum

$$G(x) = \sum_{k=1}^{\infty} (-1)^k (\log k) \mathrm{e}^{-k^2 x}. \tag{9}$$

In this case, the classical equations

$$\Gamma(s) = \int_0^{\infty} \mathrm{e}^{-x} x^{s-1} \,\mathrm{d}x \quad \text{and} \quad \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

define the Gamma function of Euler and the Riemann zeta function [85]. Thus, $\Gamma(s)$ is the Mellin transform of the base function $g(x) = \mathrm{e}^{-x}$ and a simple computation yields

$$\Lambda\left(\frac{s}{2}\right) = \frac{\mathrm{d}}{\mathrm{d}s}(1 - 2^{1-s})\zeta(s).$$

In this way, the transform of $G(x)$ is found to be

$$G^*(s) = [2^{1-2s}(\log 2)\,\zeta(2s) + (1 - 2^{1-2s})\zeta'(2s)] \cdot \Gamma(s). \tag{10}$$

Analytically, the representation (10) is easily justified when $\Re(s) > 1$, a condition that ensures simultaneously absolute convergence of the Dirichlet series and of the Mellin integral.

It is known that the Gamma function and the zeta function are both continuable to the complex plane: $\Gamma(s)$ has simple poles at the nonpositive integers while $\zeta(s)$ has only a simple pole at $s = 1$. Also, $\Gamma(s)$ decreases fast along vertical lines while $\zeta(s)$ is only of moderate growth. Thus, globally, $G^*(s)$ gets small for $s \to \pm \mathrm{i}\infty$ and the integration contour can legitimately be shifted to the left. For instance, by sweeping the integration contour till the vertical line $\Re(s) = -5/2$, one gets from the poles at $s = 0, -1, -2$

$$G(x) = \log \sqrt{\frac{\pi}{2}} + c_1 x + c_2 x^2 + \mathrm{O}(x^{5/2}),$$

with $c_1 = 7\zeta'(-2)$ and $c_2 = -\frac{31}{2}\zeta'(-4)$. The first few terms are then determined in a matter of seconds with the help of a computer algebra system like Maple that "knows" the expansions of $\zeta(s)$ and $\Gamma(s)$ at points of interest, like $\zeta(0) = -\frac{1}{2}$, $\zeta'(0) = -\log\sqrt{2\pi}$.

The computation is easily carried out to any order, and one finds

$$G(x) \sim \log\sqrt{\frac{\pi}{2}} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!}(2^{1+2k}-1)\zeta'(-2k)x^k.$$

Such an expansion might be obtained otherwise via a reduction to Euler–Maclaurin summation but application of this formula would be hampered by the alternation of sign and by the presence of the logarithmic factor. Mellin asymptotics is the approach of choice here.

The method is susceptible to a large number of variations. It is applicable each time the Dirichlet series of an amplitude–frequency pair is reducible to a standard function, like for instance

$$\sum_{k=0}^{\infty} \frac{1}{(2^k)^s} = \frac{1}{1-2^{-s}}, \qquad \sum_{k=1}^{\infty} \frac{d(k)}{k^s} = \zeta^2(s).$$

In this way sums involving geometrically increasing frequencies or highly fluctuating amplitudes of an arithmetic nature can be analyzed effortlessly, see Sections 4–7. Another feature is that the correspondence between poles of transforms, and asymptotic terms of originals fares both ways; this fact allows for base functions that admit of no explicit transform, like

$$g(x) = e^{-x^2}\sqrt{1+x},$$

as well as exotic Dirichlet series like

$$\Lambda(s) = \sum_{k=1}^{\infty} \frac{\log k}{\sqrt{1+k^2}} \frac{(-1)^k}{k^s}.$$

This flexibility explains the power of the Mellin method in asymptotic analysis, which would apply for instance to a harmonic sum like

$$G(x) = \sum_{k=1}^{\infty} (-1)^k (\log k) e^{-k^2x^2} \sqrt{\frac{1+kx}{1+k^2}}.$$

The general methodology for analyzing such "implicit" sums is discussed in Section 8.

Plan of the paper: Part I is devoted to the general functional and asymptotic properties of Mellin transforms. Our presentation assumes only rudiments of complex analysis (contour integrals, residue theorem) as found for instance in [81, 85]. The general framework is conveniently built on the Lebesgue integral [81, Ch. X] allowing dominated convergence properties, rather than on the Riemann integral. This distinction is however somewhat immaterial as specific examples can be dealt with using Riemann integrals only. A number of examples related to combinatorial analysis and the analysis of algorithms are then presented in Part II. Finally, methods for dealing with wider classes of sums form the subject of Part III.

PART I. MELLIN TRANSFORMS AND ASYMPTOTICS

In this part, we lay down the basic framework of Mellin asymptotics. Section 1 gives the basic functional properties of Mellin transforms and Section 2 discusses the fundamental correspondence between asymptotic properties of an original function and singularities of its transform. Section 3 develops the basic treatment of harmonic sums that form the subject of this paper. Section 4 briefly examines consequences for general summatory formulae.

1. Basic properties

We start by recalling the salient properties of the Mellin transform. The definition domain of a Mellin transform turns out to be a strip. We thus introduce the notation $\langle \alpha, \beta \rangle$ for the *open strip* of complex numbers $s = \sigma + it$ such that $\alpha < \sigma < \beta$.

Definition 1. Let $f(x)$ be a locally Lebesgue integrable over $(0, +\infty)$. The *Mellin transform* of $f(x)$ is defined by

$$\mathcal{M}[f(x); s] = f^*(s) = \int_0^{+\infty} f(x) x^{s-1} \, \mathrm{d}x. \tag{11}$$

The largest open strip $\langle \alpha, \beta \rangle$ in which the integral converges is called the *fundamental strip.*

From the decomposition $\int_0^\infty = \int_0^1 + \int_1^\infty$, we see that α is the infimum of all A such that $f(x)x^{A-1}$ is integrable over $(0, 1]$ and β is the supremum of all B such that $f(x)x^{B-1}$ is integrable over $[1, +\infty)$. Most functions have an order at 0 and ∞ so that an existence strip for $f^*(s)$ can be guaranteed.

Lemma 1. *The conditions*

$$f(x) \underset{x \to 0^+}{=} \mathrm{O}(x^u), \qquad f(x) \underset{x \to +\infty}{=} \mathrm{O}(x^v).$$

when $u > v$, guarantee that $f^*(s)$ exists in the strip $\langle -u, -v \rangle$.

Thus, existence of a Mellin transform is granted for locally integrable functions[1] whose exponent in the order at 0 is strictly larger than the exponent of the order at infinity. *The asymptotic form of $f(x)$ at 0 constrains the leftmost boundary of the fundamental strip of $f^*(s)$; the asymptotic form at $+\infty$ constrains the rightmost*

[1] In the sequel, all functions considered are tacitly assumed to be locally integrable.

boundary. Monomials x^c, including constants, thus do not have transforms under Definition 1.

As the integral defining $f^*(s)$ depends analytically on the complex parameter s, a Mellin transform is in addition analytic in its fundamental strip.

For instance, the function $f(x) = (1 + x)^{-1}$ is $O(x^0)$ at 0 and $O(x^{-1})$ at infinity, hence a guaranteed existence strip for $f^*(s)$ that is $\langle 0, 1 \rangle$, which here coincides with the fundamental strip. In this case, the Mellin transform may be found from the classical Beta integral [85, p. 254] to be

$$f^*(s) = \frac{\pi}{\sin \pi s},$$

which is analytic in $\langle 0, 1 \rangle$, as predicted.

The function $g(x) = e^{-x}$ satisfies

$$e^{-x} \underset{x \to 0^+}{\sim} 1, \qquad e^{-x} \underset{x \to +\infty}{=} O(x^{-b}) \quad \text{for any } b > 0$$

so that its transform, known as the Gamma function [85, Ch. XII]

$$\Gamma(s) = \int_0^{+\infty} e^{-x} x^{s-1} \, dx, \tag{12}$$

is a priori defined in $\langle 0, +\infty \rangle$ and analytic there.

Let $H(x)$ be the (Heaviside-like) step function defined by

$$H(x) = 1 \quad \text{if } 0 \leqslant x < 1, \qquad H(x) = 0 \quad \text{if } 1 < x. \tag{13}$$

Then,

$$H^*(s) = \frac{1}{s}, \quad s \in \langle 0, +\infty \rangle.$$

The complementary function $\bar{H}(x) = 1 - H(x)$ satisfies $\bar{H}^*(s) = -s^{-1}$ for $s \in \langle -\infty, 0 \rangle$, where the fundamental strips corresponding to H and $\bar{H}$ are disjoint.

Functional properties. Simple changes of variables in the definition of Mellin transforms yield a basic set of transformation rules summarized in Fig. 1 and Theorem 1 below.

Theorem 1 (Functional properties). *Let $f(x)$ be a function whose transform admits the fundamental strip $\langle \alpha, \beta \rangle$. Let ρ be a nonzero real number, and μ, ν be positive reals. Then the following relations hold*:

$$\mathcal{M}[f(\mu x); s] = \mu^{-s} f^*(s), \quad s \in \langle \alpha, \beta \rangle.$$

$$\mathcal{M}[\textstyle\sum_k \lambda_k f(\mu_k x); s] = (\sum_{k \in \mathcal{K}} \lambda_k \mu_k^{-s}) \cdot f^*(s), \quad \mathcal{K} \textit{ finite}, \lambda_k > 0.$$

$$\mathcal{M}[x^\nu f(x); s] = f^*(s + \nu), \quad s \in \langle \alpha, \beta \rangle.$$

$$\mathcal{M}[f(x^\rho); s] = \frac{1}{\rho} f^*\left(\frac{s}{\rho}\right), \quad s \in \langle \rho\alpha, \rho\beta \rangle.$$

	$f(x)$	$f^*(s)$	$\langle\alpha, \beta\rangle$	
F_1	$x^{\nu} f(x)$	$f^*(s+\nu)$	$\langle\alpha-\nu, \beta-\nu\rangle$	
F_2	$f(x^{\rho})$	$\frac{1}{\rho} f^*\left(\frac{s}{\rho}\right)$	$\langle\rho\alpha, \rho\beta\rangle$	$\rho > 0$
	$f(1/x)$	$-f^*(-s)$	$\langle -\beta, -\alpha\rangle$	
F_3	$f(\mu x)$	$\frac{1}{\mu^s} f^*(s)$	$\langle\alpha, \beta\rangle$	$\mu > 0$
	$\sum_k \lambda_k f(\mu_k x)$	$(\sum_k \lambda_k \mu_k^{-s}) \cdot f^*(s)$		Theorems 1, 5, Lemma 1
F_4	$f(x) \log x$	$\frac{\mathrm{d}}{\mathrm{d}s} f^*(s)$	$\langle\alpha, \beta\rangle$	
F_5	$\Theta f(x)$	$-s f^*(s)$	$\langle\alpha', \beta'\rangle$	$\Theta = x \frac{\mathrm{d}}{\mathrm{d}x}$
	$\frac{\mathrm{d}}{\mathrm{d}x} f(x)$	$-(s-1) f^*(s-1)$	$\langle\alpha'-1, \beta'-1\rangle$	
	$\int_0^x f(t)\,\mathrm{d}t$	$-\frac{1}{s} f^*(s+1)$		

Fig. 1. Basic functional properties of Mellin transforms. The table lists the original function, its Mellin transform, and the validity strips.

The most important rule is the *rescaling rule* that gives the transform of $f(\mu x)$ as $\mu^{-s} f^*(s)$ via the change of variables $x \mapsto \mu x$, provided that $\mu > 0$. By linearity of the transform, one also has

$$\mathscr{M}\left[\sum_k \lambda_k f(\mu_k x); s\right] = \left(\sum_k \frac{\lambda_k}{\mu_k^s}\right) \cdot f^*(s),$$

whenever k ranges over a finite set of indices. This formula can usually be extended to infinite sums that define the harmonic sums already mentioned in the introduction, see Lemma 1 and Theorem 5 below. For instance from

$$f(x) = \frac{\mathrm{e}^{-x}}{1-\mathrm{e}^{-x}} = \mathrm{e}^{-x} + \mathrm{e}^{-2x} + \mathrm{e}^{-3x} + \cdots,$$

one finds

$$f^*(s) = \Gamma(s) \cdot \left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\right) = \Gamma(s)\zeta(s),$$

as long as $\Re(s) > 1$, a condition that simultaneously entails absolute convergence of the Mellin integral of $f(x)$ and of the sum defining the zeta function.

A particular case of the rule for powers is

$$\mathcal{M}\left[f\left(\frac{1}{x}\right); s\right] = -f^*(-s).$$

Since the function $1/x$ exchanges 0 and ∞, this permits to limit consideration of asymptotic properties of a function to one of the two places $0, +\infty$. A useful application of the rule is for the transform of the Gaussian function,

$$\mathcal{M}[\mathrm{e}^{-x^2}; s] = \frac{1}{2}\Gamma\left(\frac{s}{2}\right),$$

which arises in estimates of sums involving binomial coefficients.

The formal rule

$$\frac{\mathrm{d}}{\mathrm{d}s}f^*(s) = \int_0^\infty f(x)(\log x)\, x^{s-1}\,\mathrm{d}x$$

is readily justified analytically by "differentiation under the integral sign". (As complex-differentiable implies analytic, this also supports our earlier claim that Mellin transforms are analytic in the fundamental strip.) Thus

$$\mathcal{M}[f(x)(\log x); s] = \frac{\mathrm{d}}{\mathrm{d}s}f^*(s).$$

For instance, the transform of $H(x)\log x$ is $-1/s^2$, the transform of $\mathrm{e}^{-x}\log x$ is $\Gamma'(s)$.

Conversely, the rule for transforming the derivative of an original function is best enunciated in terms of the operator

$$\Theta = x\frac{\mathrm{d}}{\mathrm{d}x}.$$

Frequently the fundamental strips of $f(x)$ and $\Theta f(x)$ have a nonempty intersection $\langle\alpha', \beta'\rangle$. This is ensured in the common case where $f(x) = \mathrm{O}(x^{-\alpha})$ as $x \to 0$ and $f(x) = \mathrm{O}(x^{-\beta})$ as $x \to \infty$ whenever $\Theta f(x)$ satisfies the same estimates (i.e., the asymptotic form of the derivative of f coincides with the derivative of the asymptotic form of f). Under this sufficient condition, with $f(x)$ continuous and piecewise differentiable, integration by parts yields

$$\int_0^{+\infty} \Theta f(x)\, x^{s-1}\,\mathrm{d}x = [f(x)x^s]_0^{+\infty} - s\int_0^{+\infty} f(x)\, x^{s-1}\,\mathrm{d}x, \quad \alpha < \Re(s) < \beta,$$

and the term $[f(x)x^{s-1}]_0^\infty$ equals 0 for $s \in \langle\alpha, \beta\rangle$. Thus

$$\mathcal{M}[\Theta f(x); s] = -sf^*(s).$$

This permits to deduce Mellin transforms of primitive functions by identification. For instance, from the transform of $(1+x)^{-1}$, we find

$$\mathcal{M}[\log(1+x), s] = \frac{\pi}{s\sin\pi s}, \quad -1 < \Re(s) < 0.$$

e^{-x}	$\Gamma(s)$	$\langle 0, +\infty \rangle$
$e^{-x} - 1$	$\Gamma(s)$	$\langle -1, 0 \rangle$
$e^{-x} - 1 + x$	$\Gamma(s)$	$\langle -2, -1 \rangle$
e^{-x^2}	$\frac{1}{2}\Gamma(\frac{1}{2}s)$	$\langle 0, +\infty \rangle$
$\dfrac{1}{(1+x)}$	$\dfrac{\pi}{\sin \pi s}$	$\langle 0, 1 \rangle$
$\log(1+x)$	$\dfrac{\pi}{s \sin \pi s}$	$\langle -1, 0 \rangle$
$H(x) \equiv 1_{0<x<1}$	$\dfrac{1}{s}$	$\langle 0, +\infty \rangle$
$x^{\alpha}(\log x)^k H(x)$	$\dfrac{(-1)^k k!}{(s+\alpha)^{k+1}}$	$\langle -\alpha, +\infty \rangle$ $k \in \mathbb{N}$

Fig. 2. A microdictionary of Mellin transforms.

In the same vein,

$$\mathscr{M}[e^{-x} - 1, s] = \frac{\Gamma(s+1)}{s} = \Gamma(s), \quad -1 < \Re(s) < 0,$$

and so on, see Fig. 2.

Note on transforms of analytic functions: Although we do not make use of it here, there is a fruitful approach to the determination of transforms of analytic functions that is based on a classical loop integral representation due to Hankel [83, 85]. A *Hankel contour* $\mathscr{H}$ is a simple loop that starts in the upper half-plane near $+\infty$, circles around the origin counterclockwise and returns to $+\infty$ in the lower half-plane. If $f(x)$ is analytic in some open set that contains $[0, +\infty)$ and satisfies reasonable growth conditions, then *Hankel's formula* holds,

$$\mathscr{M}[f(x), s] = \frac{-1}{2i \sin \pi s} \int_{\mathscr{H}} f(w)(-w)^{s-1} \, dw, \quad 0 < \sigma < \beta. \tag{14}$$

The proof is based on shrinking the contour towards the real axis and using the fact that in the infinitesimal limit the integrand is $f(x)\, x^{s-1}$ multiplied by $(e^{-i\pi s} - e^{i\pi s}) = -2i \sin \pi s$. Hankel's formula permits to determine all transforms of rational functions as well as many transforms of meromorphic functions. See [77, 83, 85].

Inversion. With again $s = \sigma + it$ and the change of variables $x = e^{-y}$, the Mellin transform becomes a Fourier transform

$$f^*(s) = \int_0^{\infty} f(x) x^{s-1} \, dx = -\int_{-\infty}^{+\infty} f(e^{-y}) e^{-\sigma y} e^{-ity} \, dy.$$

This links the two transforms; hence it provides directly inversion theorems for Mellin transforms. We cite here two main versions, one related to Lebesgue integration, the other to Riemann integration.

Theorem 2 (Inversion). (i) *Let $f(x)$ be integrable with fundamental strip $\langle \alpha, \beta \rangle$. If c is such that $\alpha < c < \beta$ and $f^*(c+it)$ is integrable, then the equality*

$$\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} f^*(s)\,x^{-s}\,ds = f(x)$$

holds almost everywhere. Moreover, if $f(x)$ is continuous, then equality holds everywhere on $(0, +\infty)$.

(ii) *Let $f(x)$ be locally integrable with fundamental strip $\langle \alpha, \beta \rangle$ and be of bounded variation in a neighborhood of x_0. Then, for any c in the interval* (α, β),

$$\lim_{T\to\infty} \frac{1}{2i\pi}\int_{c-iT}^{c+iT} f^*(s)\,x^{-s}\,ds = \frac{f(x_0^+)+f(x_0^-)}{2}.$$

Proof. See [18, 82, 86]. □

Fig. 2 presents a few classical original-transform pairs (f, f^*). More can be found in standard tables of integral transforms [19, 66, 67] and in [61, 77].

2. The fundamental correspondence

There is a very precise correspondence between the asymptotic expansion of a function at 0 (and ∞), and poles of its Mellin transform in a left (resp. right) half-plane. Each individual term in such an asymptotic expansion of $f(x)$ having the form $x^c(\log x)^k$ is associated to a pole of $f^*(s)$ at $s = -c$ with multiplicity $k+1$. The correspondence fares both ways. It is thus the basis of an asymptotic process: *For estimating asymptotically a function $F(x)$ – typically a harmonic sum –, determine its Mellin transform and translate back its singularities into asymptotic terms in the expansion of $F(x)$.*

Let $\phi(s)$ be meromorphic at $s = s_0$: it admits near s_0 a *Laurent expansion*

$$\phi(s) = \sum_{k \geqslant -r} c_k (s-s_0)^k. \tag{15}$$

The function $\phi(s)$ has a pole of order r if $r > 0$ and $c_{-r} \neq 0$, it is analytic (regular, holomorphic) at s_0 if $r = 0$. A *singular element* of $\phi(s)$ at s_0 is an initial sum of the Laurent expansion (15) truncated at terms of order $O(1)$ or smaller.

Definition 2 (*Singular expansion*). Let $\phi(s)$ be meromorphic in Ω with $\mathcal{S}$ including all the poles of $\phi(s)$ in Ω. A *singular expansion* of $\phi(s)$ in Ω is a formal sum of singular elements of $\phi(s)$ at all points of $\mathcal{S}$.

When E is a singular expansion of $\phi(s)$, in Ω, we write

$$\phi(s) \asymp E \quad (s \in \Omega).$$

For instance, one has

$$\frac{1}{s^2(s+1)} \asymp \left[\frac{1}{s+1} + 2 + 3(s+1)\right]_{s=-1} + \left[\frac{1}{s^2} - \frac{1}{s}\right]_{s=0} + \left[\frac{1}{2}\right]_{s=1} \quad (s \in \langle -2, +2 \rangle), \tag{16}$$

where the point of expansion may be indicated whenever needed as a subscript to the corresponding singular element. The expansion (16) is a concise way of combining information contained in the Laurent expansions of the function $\phi(s) \equiv s^{-2}(s+1)^{-1}$ at the three points of $\mathscr{S} = \{-1, 0, 1\}$:

$$\phi(s) \underset{s \to -1}{=} (s+1)^{-1} + 2 + 3(s+1) + \cdots, \qquad \phi(s) \underset{s \to 0}{=} s^{-2} - s^{-1} + 1 + \cdots,$$

$$\phi(s) \underset{s \to 1}{=} \tfrac{1}{2} - \tfrac{5}{4}(s-1) + \tfrac{17}{8}(s-1)^2 + \cdots.$$

Example 1. *The Gamma function.* The Mellin transform of the function e^{-x} defines the Gamma function,

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx,$$

for $\Re(s) > 0$. Integration by part permits to verify the well-known functional equation

$$\Gamma(s+1) = s\Gamma(s),$$

which allow to extend $\Gamma(s)$ to a meromorphic function in the whole of $\mathbb{C}$. The function $\Gamma(s)$ satisfies $\Gamma(1) = 1$, and from the functional equation one has

$$\Gamma(s) = \frac{\Gamma(s+m+1)}{s(s+1)(s+2)\cdots(s+m)}.$$

Thus, $\Gamma(s)$ has poles at the points $s = -m$ with $m \in \mathbb{N}$, near which

$$\Gamma(s) \sim \frac{(-1)^m}{m!} \frac{1}{s+m},$$

so that the Gamma function admits

$$\Gamma(s) \asymp \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{s+k} \quad (s \in \mathbb{C}) \tag{17}$$

as singular expansion in the whole of $\mathbb{C}$.

$f(x)$	$f^*(s)$
Order at 0: $O(x^a)$	Leftmost boundary of f.s. at $\Re(s) = -a$
Order at $+\infty$: $O(x^b)$	Rightmost boundary of f.s. at at $\Re(s) = -b$
Expansion till $O(x^\gamma)$ at 0	Meromorphic continuation till $\Re(s) \geqslant -\gamma$
Expansion till $O(x^\delta)$ at ∞	Meromorphic continuation till $\Re(s) \leqslant -\delta$
Term $x^a(\log x)^k$ at 0	Pole with singular element $\dfrac{(-1)^k k!}{(s+a)^{k+1}}$
Term $x^a(\log x)^k$ at ∞	Pole with singular element $-\dfrac{(-1)^k k!}{(s+a)^{k+1}}$

Fig. 3. The fundamental correspondence: aspects of the direct mapping (Theorem 3).

Direct mapping. The function e^{-x} has a Taylor expansion at $x = 0$:

$$\mathrm{e}^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k. \tag{18}$$

There is a striking coincidence of coefficients in the Taylor expansion (18) of the original function e^{-x} and in the singular expansion (17) of the transform $\Gamma(s)$ expressed by the rule

$$x^k \mapsto \frac{1}{s+k}.$$

This is in fact a completely general phenomenon.

Theorem 3 (Direct mapping; see Fig. 3). *Let $f(x)$ have a transform $f^*(s)$ with nonempty fundamental strip $\langle \alpha, \beta \rangle$.*

(i) *Assume that $f(x)$ admits as $x \to 0^+$ a finite asymptotic expansion of the form*

$$f(x) = \sum_{(\xi,k) \in A} c_{\xi,k} x^{\xi} (\log x)^k + O(x^\gamma) \tag{19}$$

where the ξ satisfy $-\gamma < -\xi \leqslant \alpha$ and the k are nonnegative. Then $f^*(s)$ is continuable to a meromorphic function in the strip $\langle -\gamma, \beta \rangle$ where it admits the singular expansion

$$f^*(s) \asymp \sum_{(\xi,k) \in A} c_{\xi,k} \frac{(-1)^k k!}{(s+\xi)^{k+1}} \quad (s \in \langle -\gamma, \beta \rangle).$$

(ii) *Similarly, assume that $f(x)$ admits as $x \to +\infty$ a finite asymptotic expansion of the form* (19) *where now $\beta \leqslant -\xi < -\gamma$. Then $f^*(s)$ is continuable to a meromorphic* *function in the strip* $\langle \alpha, -\gamma \rangle$ *where*

$$f^*(s) \asymp -\sum_{(\xi,k)\in A} c_{\xi,k} \frac{(-1)^k k!}{(s+\xi)^{k+1}} \quad (s \in \langle \alpha, -\gamma \rangle).$$

Thus terms in the asymptotic expansion of $f(x)$ at 0 induce poles of $f^*(s)$ in a strip to the *left* of the fundamental strip; terms in the expansion at $+\infty$ induce poles in a strip to the *right*.

Proof (see [18]). Since $\mathcal{M}(f(1/x), s) = -\mathcal{M}(f(x), -s)$, it suffices to treat the case (i) corresponding to $x \to 0^+$. By assumption, the function $g(x)$,

$$g(x) = f(x) - \sum_{(\xi,k)\in A} c_{\xi,k} x^{\xi} (\log x)^k,$$

satisfies $g(x) = O(x^{\gamma})$.

For s in the fundamental strip, a split of the definition domains yields

$$\begin{aligned} f^*(s) &= \int_0^1 f(x) x^{s-1} \mathrm{d}x + \int_1^{\infty} f(x) x^{s-1} s\mathrm{d}x \\ &= \int_0^1 g(x) x^{s-1} \mathrm{d}x + \int_0^1 \left(\sum_{(\xi,k)\in A} c_{\xi,k} x^{\xi} (\log x)^{x} \right) x^{s-1} \mathrm{d}x \\ &\quad + \int_1^{+\infty} f(x) x^{s-1} \mathrm{d}x. \end{aligned} \tag{20}$$

In the last line of (20), the first integral defines an analytic function of s in the strip $\langle -\gamma, +\infty \rangle$ since $g(x) = O(x^{\gamma})$ as $x \to 0$; the third integral is analytic in $\langle -\infty, \beta \rangle$, so that the sum of these two is analytic in $\langle -\gamma, \beta \rangle$. Finally, straight integration expresses the middle integral as

$$\sum_{(\xi,k)\in A} c_{\xi,k} \frac{(-1)^k k!}{(s+\xi)^{k+1}},$$

which is meromorphic in all $\mathbb{C}$ and provides the singular expansion of $f^*(s)$ in the extended strip. $\square$

In the case where there exists a complete expansion of $f(x)$ at 0 (or ∞), the transform $f^*(s)$ becomes meromorphic in a complete left (or right) half-plane. This situation always occurs for functions that are analytic at 0 (or $+\infty$).

Example 2. *The zeta function.* We gave earlier the Mellin pair

$$f(x) = \frac{\mathrm{e}^{-x}}{1-\mathrm{e}^{-x}}, \qquad f^*(s) = \Gamma(s)\zeta(s)$$

with fundamental strip $\langle 1, \infty\rangle$. The function $f(x)$ is exponentially small at infinity and it admits a complete expansion near $x = 0$,

$$\frac{e^{-x}}{1-e^{-x}} = \sum_{k=-1}^{\infty} B_{k+1} \frac{x^k}{(k+1)!},$$

which defines the Bernoulli numbers B_k: $B_0 = 1$, $B_1 = -\frac{1}{2}$, etc. Thus, $f^*(s) = \Gamma(s)\zeta(s)$ is meromorphic in the whole of $\mathbb{C}$ with singular expansion

$$\Gamma(s)\zeta(s) \asymp \sum_{k=-1}^{\infty} \frac{B_{k+1}}{(k+1)!} \frac{1}{s+k}. \tag{21}$$

Therefore, $\zeta(s)$ is meromorphic in the whole of $\mathbb{C}$ (this is often proved by means of its functional equation). In addition comparison of the singular expansions of $\Gamma(s)$ in (17) and $\Gamma(s)\zeta(s)$ in (21) yields

$$\zeta(s) \underset{s\to 1}{\sim} \frac{1}{s-1}, \qquad \zeta(0) = -\tfrac{1}{2}, \qquad \zeta(-m) = -\frac{B_{m+1}}{m+1}.$$

Example 3. *The transform of* $(1+x)^{-1}$. The function $f(x) = (1+x)^{-1}$ has $\langle 0, 1\rangle$ as its fundamental strip. The two expansions,

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad (x \to 0^+) \quad \text{and} \quad \frac{1}{1+x} = \sum_{n=1}^{\infty} (-1)^{n-1} x^{-n} \quad (x \to +\infty),$$

translate into

$$f^*(s) \asymp \sum_{n=0}^{\infty} \frac{(-1)^n}{s+n} \quad (s \in \langle -\infty, 1\rangle) \qquad \text{and}$$

$$f^*(s) \asymp -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{s-n} \quad (s \in \langle 0, +\infty\rangle),$$

which is consistent with the known form,

$$f^*(s) \equiv \frac{\pi}{\sin \pi s} \asymp \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{s+n} \quad (s \in \mathbb{C}).$$

The next example illustrates the fact the Mellin analysis may be applied to functions without explicit transforms.

Example 4. *A nonexplicit transform.* The function $f(x) = (\cosh(x))^{-1/2}$ is exponentially small at ∞ and near 0 it satisfies

$$\frac{1}{\sqrt{\cosh(x)}} = 1 - \frac{1}{4}x^2 + \frac{7}{96}x^4 - \frac{139}{5760}x^6 + \frac{5473}{645120}x^8 + O(x^{10}),$$

so that its transform $f^*(s)$ is meromorphic in $\mathbb{C}$ and

$$f^*(s) \asymp \frac{1}{s} - \frac{1}{4}\frac{1}{s+2} + \frac{7}{96}\frac{1}{s+4} - \frac{139}{5760}\frac{1}{s+6} + \frac{5473}{645120}\frac{1}{s+8} + \cdots.$$

A general principle also derives from the proof of Theorem 3: *subtracting from a function a truncated form of its asymptotic expansion at either* 0 *or* ∞ *does not alter its Mellin transform and only shifts the fundamental strip.* An instance is provided by the equalities

$$\mathscr{M}(\mathrm{e}^{-x}, s) = \Gamma(s), \quad s \in \langle 0, +\infty \rangle, \qquad \mathscr{M}(\mathrm{e}^{-x} - 1, s) = \Gamma(s), \quad s \in \langle -1, 0 \rangle, \tag{22}$$

previously establishing using integration by parts and specific properties of the exponential. The following proof of (22) demonstrates the general technique on this particular example. Take the function

$$F^*(s) = \frac{1}{s} + \int_0^1 (\mathrm{e}^{-x} - 1) x^{s-1}\,\mathrm{d}x + \int_1^\infty \mathrm{e}^{-x} x^{s-1}\,\mathrm{d}x.$$

Consideration of both integrals shows that the function is meromorphic in $\langle -1, +\infty \rangle$. Its restriction to $\langle 0, +\infty \rangle$ is

$$\Gamma(s) = \int_0^\infty \mathrm{e}^{-x} x^{s-1}\,\mathrm{d}x,$$

and its restriction to $\langle -1, 0 \rangle$ is

$$\int_0^\infty (\mathrm{e}^{-x} - 1) x^{s-1}\,\mathrm{d}x.$$

This argument shows that the transforms of e^{-x} and of $\mathrm{e}^{-x} - 1$ are elements of the same meromorphic function in different strips.

Converse mapping. Under a set of mild conditions, a converse to the Direct Mapping theorem also holds: The singularities of a Mellin transform which is small enough towards $\pm \mathrm{i}\infty$ encode the asymptotic properties of the original function.

Theorem 4 (Converse mapping; see Fig. 4). *Let $f(x)$ be continuous in $]0, +\infty[$ with Mellin transform $f^*(s)$ having a nonempty fundamental strip $\langle \alpha, \beta \rangle$.*

(i) *Assume that $f^*(s)$ admits a meromorphic continuation to the strip $\langle \gamma, \beta \rangle$ for some $\gamma < \alpha$ with a finite number of poles there, and is analytic on $\Re(s) = \gamma$. Assume also that there exists a real number $\eta \in (\alpha, \beta)$ such that*

$$f^*(s) = \mathrm{O}(|s|^{-r}) \quad \textit{with } r > 1, \tag{23}$$

when $|s| \to \infty$ in $\gamma \leqslant \Re(s) \leqslant \eta$. If $f^*(s)$ admits the singular expansion for $s \in \langle \gamma, \alpha \rangle$,

$$f^*(s) \asymp \sum_{(\xi, k) \in A} d_{\xi,k} \frac{1}{(s-\xi)^k}, \tag{24}$$

$f^*(s)$	$f(x)$
Pole at ξ	Term in asymptotic expansion $\approx x^{-\xi}$
left of fund. strip	expansion at 0
right of fund. strip	expansion at $+\infty$
Multiple pole	Logarithmic factor
left: $\dfrac{1}{(s-\xi)^{k+1}}$	$\dfrac{(-1)^k}{k!} x^{-\xi}(\log x)^k$ at 0
right: $\dfrac{1}{(s-\xi)^{k+1}}$	$-\dfrac{(-1)^k}{k!} x^{-\xi}(\log x)^k$ at ∞
Pole with imaginary part: $\xi = \sigma + it$	Fluctuations: $x^{-\xi} = x^{-\sigma} e^{it \log x}$
Regularly spaced poles	Fourier series in $\log x$

Fig. 4. The fundamental correspondence: aspects of the converse mapping (Theorem 4 and Corollary 1).

then an asymptotic expansion of $f(x)$ at 0 is

$$f(x) = \sum_{(\xi,k)\in A} d_{\xi,k} \left(\frac{(-1)^{k-1}}{(k-1)!} x^{-\xi} (\log x)^k \right) + O(x^{-\gamma}).$$

(ii) *Similarly assume that $f^*(s)$ admits a meromorphic continuation to $\langle \alpha, \gamma \rangle$ for some $\gamma > \beta$ and is analytic on $\Re(s) = \gamma$. Assume also that the growth condition* (23) *holds in $\langle \eta, \gamma \rangle$ for some $\eta \in (\alpha, \beta)$. If $f^*(s)$ admits the singular expansion* (24) *for $s \in \langle \eta, \gamma \rangle$, then an asymptotic expansion of $f(x)$ at ∞ is*

$$f(x) = -\sum_{(\xi,k)\in A} d_{\xi,k} \left(\frac{(-1)^{k-1}}{(k-1)!} x^{-\xi} (\log x)^k \right) + O(x^{-\gamma}).$$

Proof. The proof makes use of the inversion theorem and of a residue computation using large rectangular contours in the extended strip of $f^*(s)$, see Fig. 5. As before, it suffices to consider case (i) corresponding to continuation to the left.

Let $\mathscr{S}$ be the set of poles in $\langle \gamma, \beta \rangle$. Consider the integral

$$J(T) = \frac{1}{2i\pi} \int_{\mathscr{C}} f^*(s) x^{-s} \, ds,$$

where $\mathscr{C} \equiv \mathscr{C}(T)$ denotes the rectangular contour defined by the segments

$$[\eta - iT, \eta + iT], \quad [\eta + iT, \gamma + iT], \quad [\gamma + iT, \gamma - iT], \quad [\gamma - iT, \eta - iT].$$

Assume that T is larger than $|\Im(s_0)|$ for all poles $s_0 \in \mathscr{S}$. By Cauchy's theorem, $J(T)$ is equal to the sum of residues, which is by a direct computation

$$R = \sum_{(\xi,k)\in A} d_{\xi,k} \operatorname{Res} \left(\frac{x^{-s}}{(s-\xi)^k} \right)_{s=\xi} = \sum_{(\xi,k)\in A} d_{\xi,k} \left(\frac{(-1)^{k-1}}{(k-1)!} x^{-\xi} (\log x)^k \right).$$

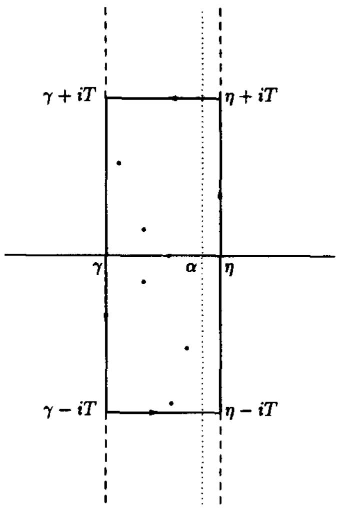

Fig. 5. The contour $\mathscr{C}$ used in the proof of Theorem 4. Dots represent poles of $f^*(s)$.

Let now T tends to $+\infty$. The integral along the two horizontal segments is $O(T^{-r})$ and thus tends to 0 as $T \to \infty$. The integral along the vertical line $\Re(s) = \eta$ that lies inside the fundamental strip tends to the inverse Mellin integral which converges given the decrease assumption on f^* and equals $f(x)$ by the inversion theorem (since $f(x)$ is continuous). The integral along the vertical line $\Re(s) = \gamma$ is bounded by a quantity of the form

$$\frac{1}{2\pi}\int_{\gamma - i\infty}^{\gamma + i\infty} |f^*(s)|\,|x^{-s}|\,|ds| = O(1)\int_0^{\infty} \frac{x^{-\gamma}\,dt}{(1+t)^r} = O(x^{-\gamma}),$$

given the growth assumption on f^*.

Thus, in the limit, $J(\infty)$ equals $f(x)$ plus a remainder term that is $O(x^{-\gamma})$ plus the sum of residues that is of the stated form in x and $\log x$. $\square$

In many cases, $f^*(s)$ is meromorphic in a complete left or right half-plane and satisfies the conditions of Theorem 4; then a complete asymptotic expansion for $f^*(s)$ results. Such an expansion may be either convergent or divergent. If divergent, the expansion is by necessity only asymptotic. If convergent it may in some cases represent the functions exactly, but this cannot be a general phenomenon decidable from the series alone[2] as $f(x)$ and $f(x) + \varpi(x)$ have the same asymptotic expansions at 0 and ∞ whenever $\varpi(x)$ is a "flat" function like $\varpi(x) = e^{-(x+1/x)}$.

[2] A proof of an exact representation requires that the remainder integrals along vertical lines tend to zero; this needs a fast enough and uniform decrease of $f^*(s)$ along vertical lines.

Example 5. *The transform of* $(1+x)^{-v}$. Take $v>0$, and consider the function

$$\phi(s)=\frac{\Gamma(s)\Gamma(v-s)}{\Gamma(v)}$$

that is analytic in the strip $\langle 0, v\rangle$. The singular expansion to the left of $\Re(s)=0$ is

$$\phi(s)\asymp\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(v+k-1)}{\Gamma(v)}\frac{1}{(s+k)}.$$

The problem here is the behavior of the original function

$$f(x)=\frac{1}{2\pi}\int_{v/2-\mathrm{i}\infty}^{v/2+\mathrm{i}\infty}\phi(s)x^{-s}\,\mathrm{d}s$$

as $x\to 0$. The decrease of $\phi(s)$ along vertical lines results from the decrease of the Gamma function. Thus, the conditions of Theorem 5 are satisfied. In this way, one finds, for any integer M,

$$f(x)=\sum_{k=0}^{M}\frac{(-1)^k}{k!}\frac{\Gamma(v+k-1)}{\Gamma(v)}x^k+\mathrm{O}(x^{M+1/2}).$$

The series on the right-hand side is a truncation of the Taylor series of $(1+x)^{-v}$, by virtue of the binomial theorem. We have thus proved that $f(x)=(1+x)^{-v}+\varpi(x)$ for a function $\varpi(x)$ that decreases at 0 faster than any power of x.

Sharper estimates of inverse Mellin integrals show that the remainder integral tends uniformly to 0 when $0<x<1$, so that the representation is in fact exact. We have thus found indirectly the Mellin pair

$$f(x)=\frac{1}{(1+x)^v},\qquad f^*(s)=\frac{\Gamma(s)\Gamma(v-s)}{\Gamma(v)}.$$

Example 6. *A classical divergent series.* The function

$$\phi(s)=\Gamma(1-s)\frac{\pi}{\sin\pi s}$$

is analytic in the strip $\langle 0, 1\rangle$. In $\Re(s)<1$, it admits the singular expansion

$$\phi(s)\asymp\sum_{n=0}^{\infty}(-1)^n n!\frac{1}{s+n},$$

which encodes for the original function the asymptotic expansion

$$f(x)\sim\sum_{n=0}^{\infty}(-1)^n n!\,x^n.$$

In fact $\phi(s)$ is the Mellin transform of the "harmonic integral"

$$f(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\,dt,$$

whose consideration goes back to Euler.

Periodicities. A pole of f^* at a point $\xi = \sigma + it$ with a nonzero imaginary part induces a term of the form

$$x^{-\xi} = x^{-\sigma} e^{-it\log x},$$

which contains an oscillating component that is periodic in $\log x$ with period $2\pi/t$. Mellin transforms are precisely useful for quantifying many nonelementary fluctuation phenomena. In this context, there sometimes occurs a vertical line of regularly spaced poles (see Section 6). A direct extension of Theorem 4 allows for the presence of infinitely many poles in a finite strip.

Corollary 1. *The conclusions of Theorem* 4 *remain valid assuming only a weaker form of the growth condition* (23) *along a denumerable set of horizontal segments* $|\Im(s)| = T_j$ *where* $T_j \to +\infty$.

Proof. Use the proof of Theorem 4 but restrict T to belong to the discrete set T_j which, by the growth condition imposed, must avoid the poles of $f^*(s)$. □

In particular *regularly spaced poles of* $f^*(s)$ *along a vertical line correspond to a Fourier series in* $\log x$. For instance, simple poles of $f^*(s)$ at the $\chi_k = \sigma + 2ik\pi/\log B$ will arise from a component of the form $(1 - B^{-s})^{-1}$ in $f^*(s)$. The corresponding residues r_k induce for $f(x)$ the infinite sum

$$\pm \sum_{k\in\mathbb{Z}} r_k x^{-\xi_k} = \pm x^{-\sigma} \sum_{k\in\mathbb{Z}} r_k \exp(-2ik\pi \log_B x),$$

where the sign is + for poles left of the fundamental strip (expansion at 0) and − otherwise (expansion at ∞).

3. Harmonic sums

This section builds upon the functional properties of the Mellin transform and the fundamental correspondence to develop a framework adapted to the analysis of harmonic sums.

Definition 3 (*Harmonic sums*). *A sum of the form*

$$G(x) = \sum_k \lambda_k g(\mu_k x) \tag{25}$$

is called a *harmonic sum*. The λ_k are the *amplitudes*, the μ_k are the *frequencies*, and $g(x)$ is called the *base function*.

The *Dirichlet series* of the harmonic sum is the sum

$$\Lambda(s) = \sum_k \lambda_k \mu_k^{-s}. \tag{26}$$

In this paper, the frequencies either decrease to zero or increase to ∞. By possibly considering $G(1/x)$ and changing the base function accordingly, we may always reduce ourselves to the single case $\mu_k \to +\infty$, which we assume in this section, unless otherwise stated.

The Mellin transform of a harmonic sum factors, as already mentioned in the introduction, into a product of the Dirichlet series $\Lambda(s)$ and of the transform $g^*(s)$ of the base function. This is true unconditionally for finite harmonic sums. Elementary arguments provide a first extension to infinite sums.

Lemma 2. *Assume that $g(x)$ is bounded over any interval $[a, b] \subset (0, +\infty)$, and that it satisfies $g(x) = O(x^u)$ when $x \to 0$ and $g(x) = O(x^v)$ when $x \to +\infty$, with $u > v$. Assume that the Dirichlet series $\Lambda(s)$ has a half-plane of absolute convergence $\Re(s) > \sigma_a$ that has a nonempty intersection Δ with the strip $\langle -u, -v\rangle$.*

Then the harmonic sum $G(x)$ of (25) is defined for all x in $(0, +\infty)$. The transform $G^*(s)$ is well-defined in Δ where it factors as

$$G^*(s) = \Lambda(s) \cdot g^*(s).$$

We recall that the theory of generalized Dirichlet series [41] guarantees that a Dirichlet series like $\Lambda(s)$ has a half-plane of absolute convergence $\Re(s) > \sigma_a$ and a half-plane of simple convergence $\Re(s) > \sigma_c$ where

$$\sigma_a - \sigma_c \geqslant 0.$$

Proof. First consider the special case where $g(x)$ is $O(1)$ at 0, is $O(x^{-1})$ at ∞, and $\Lambda(s)$ convergence absolutely for $\Re(s) > -\delta$ for some $\delta > 0$.

Then over $(0, +\infty)$, we have both $|g(x)| < C/x$ and $|g(x)| < D$, for some constants $C, D > 0$. Thus, one has

$$|G(x)| < D \cdot \sum_k |\lambda_k| < \infty,$$

by the assumption that $\Lambda(0)$ converges absolutely. Consequently, $G(x) = O(1)$ everywhere and in particular at 0. By summation, $G(x)$ is also $O(x^{-1})$ everywhere, and in particular as $x \to +\infty$. By the dominated convergence theorem, $G^*(s)$ therefore exists in the strip $\langle 0, 1\rangle$.

The general case reduces to the special case by normalizing $g(x)$ and considering

$$\hat{g}(x) = x^{-u/(v-u)} g(x^{1/(v-u)}). \qquad \square$$

Our treatment throughout this paper relies on the assumption that $g^*(s)$ and $\Lambda(s)$ in Lemma 2 are continuable as meromorphic functions in regions of the complex plane larger than what their definitions imply, and additionally satisfy controlled growth conditions towards $\pm i\infty$.

Definition 4. (i) A function $\phi(s)$ is said to be of *fast decrease* in the closed strip $\sigma_1 \leqslant \Re(s) \leqslant \sigma_2$ if for any $r > 0$,

$$\phi(s) = O(|s|^{-r}),$$

as $|s| \to \infty$ in the strip.

(ii) A function is said to be of *slow increase* in the closed strip $\sigma_1 \leqslant \Re(s) \leqslant \sigma_2$ if for some $r > 0$,

$$\phi(s) = O(|s|^{r}),$$

as $|s| \to \infty$ in the strip.

The property of fast decrease means that $\phi(s)$ decays faster than any negative power of $|s|$. For instance, the function $\pi/\sin \pi s$ is of fast decrease in any finite closed strip since the complex exponential representation yields

$$\frac{\pi}{\sin \pi s} = O(e^{-\pi|t|}), \quad s = \sigma + it,$$

which exhibits an exponential decrease along vertical lines. A similar exponential decay holds for $\Gamma(s)$ by the complex version of Stirling's formula [1, p. 257]:

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi}\,|t|^{\sigma - 1/2}\, e^{-\pi|t|/2} \quad (t \to +\infty). \tag{27}$$

The property of slow increase holds for many Dirichlet series and, for $\sigma < 1$, one has [83, p. 94]

$$\zeta(\sigma + it) = O(|t|^{1-\sigma}). \tag{28}$$

(Uniform versions of (27), (28) also exist, see the cited references.)

The property of slow increase is also called "finite order". It means that $\phi(s)$ is of at most polynomial growth in the strip. The function $\zeta(s)$ is bounded in any closed strip of the form $[1 + \varepsilon, M]$ and thus is of slow increase there. Classical theorems [83] that extend (28) near $\Re(s) = 1$ show that it is of slow increase in any finite strip of the complex plane.

The theorem below represents the basic paradigm for the analysis of harmonic sums. It presupposes that *the Mellin transform of the base function is of fast decrease and the Dirichlet series of the harmonic sum is of moderate growth in an extended region of the complex plane.* In the statement, only the abscissa of simple convergence is needed, a handy improvement for many applications.

Theorem 5 (Harmonic sums). *Consider the general harmonic sum $G(x)$. Let the transform of the base function $g^*(s)$ have a fundamental strip $\langle \alpha, \beta \rangle$. Let the Dirichlet series $\Lambda(s)$ admit the half-plane of simple convergence $\Re(s) > \sigma_c$. Assume that*
- *the half-plane of convergence of $\Lambda(s)$ intersects the fundamental strip of $g^*(s)$: $\sigma_c < \beta$ and let $\alpha' = \max(\alpha, \sigma_c)$;*
- *the functions $g^*(s)$ and $\Lambda(s)$ admit a meromorphic continuation in a strip $\langle \gamma, \beta \rangle$ and are analytic on $\Re(s) = \gamma$, for some $\gamma < \alpha$;*
- *on the closed strip $\gamma \leqslant \Re(s) \leqslant (\alpha' + \beta)/2$, the function $g^*(s)$ is of fast decrease and the function $\Lambda(s)$ is of slow increase.*

Then the harmonic sum $G(x)$ converges for all $x > 0$ on $(0, +\infty)$. An asymptotic expansion of $G(x)$ as $x \to 0$ till an error term $O(x^{-\gamma})$ is obtained by termwise translation of the singular expansion of $G^*(s) = \Lambda(s) g^*(s)$ according to the rule

$$\frac{A}{(s-\xi)^{k+1}} \mapsto A\frac{(-1)^k}{k!}x^{-\xi}(\log x)^k.$$

Proof. First select an arbitrary integration abscissa σ in (α', β) and take some σ_0 such that $\alpha' < \sigma_0 < \sigma$. The inversion theorem provides

$$\sum_{n=1}^{N} \lambda_n g(\mu_n x) = \frac{1}{2\mathrm{i}\pi}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \sum_{n=1}^{N} \frac{\lambda_n}{\mu_n^s} g^*(s) x^{-s}\,\mathrm{d}s. \tag{29}$$

There is a well-known bound on the growth of a (generalized) Dirichlet series in any fixed half-plane strictly interior to its simple convergence domain that reads

$$|\Lambda(s)| \leqslant C(|s|+1),$$

for some constant C, see [41, pp. 3–4]. From this, we have

$$\left|\sum_{n=1}^{N} \frac{\lambda_n}{\mu_n^s} g^*(s) x^{-s}\right| \leqslant C(|s|+1)\cdot|g^*(s)|\cdot x^{-\sigma}, \tag{30}$$

with permits to apply the dominated convergence theorem and establishes the convergence of $G(x)$ expressed as an inverse Mellin integral:

$$G(x) = \frac{1}{2\mathrm{i}\pi}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \Lambda(s) g^*(s) x^{-s}\,\mathrm{d}s. \tag{31}$$

In addition, the bound (30) shows that $G(x)$ is $O(x^{-\sigma})$ where σ was chosen arbitrarily in (α', β). Thus, the strip $\langle \alpha', \beta \rangle$ is included in the fundamental strip of $G(x)$.

The upper bound of $O(x^{-\sigma})$ also holds for all the partial sums $\sum_{n=1}^{N} \lambda_n g(\mu_n x)$ which are therefore dominated by an integrable function over $(0, +\infty)$. Thus, the dominated convergence theorem applies once more and yields

$$G^*(s) = \lim_{N\to\infty} \int_0^\infty \sum_{n=1}^{N} \lambda_n \cdot g(\mu_n x) x^{s-1}\,\mathrm{d}x = \left(\lim_{N\to\infty} \sum_{n=1}^{N} \frac{\lambda_n}{\mu_n^s}\right)\cdot g^*(s),$$

that is to say the fundamental relation $G^*(s) = \Lambda(s) g^*(s)$.

The proof is now completed by a straight application of the *Converse Mapping Theorem* (Theorem 4). □

A symmetric result holds near $x \to \infty$, assuming now the meromorphic extensions to hold to the left of the fundamental strip; the sign is reversed due to the orientation of the contour. This discussion is summarized by the following informal statement.

Mellin summation formula. *Under the conditions of Theorem* 5,

$$\sum_k \lambda_k g(\mu_k x) \sim \pm \sum_{s \in H} \operatorname{Res}\,(g^*(s) \Lambda(s) x^{-s}),$$

- *for an expansion near* 0, *the sum is over the set* H *of poles to the left of the fundamental strip, and the sign is* $+$;
- *for an expansion near* ∞, *the sum is over poles to the right of the fundamental strip, and the sign is* $-$.

In addition, the relation (31) shows that $G(x)$ is of C^∞ class since $\Lambda(s) g^*(s)$ is of fast decrease so that derivation under the integral sign is permitted.

Example 7. *Harmonic numbers.* The function

$$h(x) = \sum_{k=1}^{\infty} \left[\frac{1}{k} - \frac{1}{k+x} \right] = \sum_{k=1}^{\infty} \frac{1}{k} \frac{x/k}{1+x/k}$$

satisfies $h(n) = H_n$ where H_n is the harmonic number. (It is closely related to $(\mathrm{d}/\mathrm{d}x) \log \Gamma(x)$.) Its Mellin transform is

$$h^*(s) = -\frac{\pi}{\sin \pi s} \zeta(1-s),$$

with fundamental strip $\langle -1, 0 \rangle$ and singular expansion to the right of this fundamental strip

$$h^*(s) \asymp \left[\frac{1}{s^2} - \frac{\gamma}{s} \right] - \sum_{k=1}^{\infty} (-1)^k \frac{\zeta(1-k)}{s-k}.$$

Thus (with a minus sign corresponding to the expansion at ∞)

$$H_n \sim \log n + \gamma + \frac{1}{2n} + \sum_{k \geqslant 2} \frac{(-1)^k B_k}{k} \frac{1}{n^k} \sim \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} + \cdots.$$

Generalized harmonic numbers can be analyzed in the same way, see [37, p. 300] for a table giving $H_n^{(2)}$.

Example 8. *Stirling's formula.* From the product decomposition of the Gamma function, one has

$$\ell(x) := \log \Gamma(x+1) - \gamma x = \sum_{n=1}^{\infty} \left[\frac{x}{n} - \log\left(1 + \frac{x}{n}\right) \right] \quad (s \in \langle -2, +\infty \rangle).$$

This example is similar in spirit to the previous one. The Mellin transform is

$$\ell^*(s) = -\zeta(-s) \frac{\pi}{s \sin \pi s},$$

with fundamental strip $\langle -2, -1 \rangle$. There are double poles at $s = -1$, $s = 0$ and simple poles at the positive integers,

$$\ell^*(s) \asymp \left[\frac{1}{(s+1)^2} + \frac{1-\gamma}{(s+1)} \right] + \left[\frac{1}{2s^2} - \frac{\log \sqrt{2\pi}}{s} \right] + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \zeta(-n)}{n(s+n)}.$$

Hence Stirling's formula

$$\log(x!) \sim \log(x^x e^{-x} \sqrt{2\pi x}) + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)} \frac{1}{x^{2n-1}}.$$

Example 9. *Polylogarithms.* The function

$$L_w(x) = \sum_{n=1}^{\infty} \frac{e^{-nx}}{n^w}$$

is related to polylogarithms that are defined for integer w by $\mathrm{Li}_w(z) = \sum_{n=1}^{\infty} z^n n^{-w}$. For $w \in \mathbb{N}$, one has $L_w(x) = \mathrm{Li}_w(e^{-x})$. We assume first that w is a positive integer, $w = k$.

The function $L_k(x)$ is a typical harmonic sum with transform

$$L_k^*(s) = \zeta(s+k) \Gamma(s),$$

valid in $\Re(s) > 1$ at least. The singular expansion obtains from the singular expansions of $\zeta(s)$ (shifted) and $\Gamma(s)$:

$$L_k^*(s) \asymp \sum_{\substack{n=0 \\ n \neq k-1}}^{\infty} (-1)^n \frac{\zeta(k-n)}{n!} \frac{1}{s+n}$$
$$+ \frac{(-1)^{k-1}}{(k-1)!} \left[\frac{1}{(s+k-1)^2} + \frac{H_{k-1}}{s+k-1} \right] \quad (s \in \langle -\infty, +\infty \rangle).$$

The double pole at $s = k-1$ induces a logarithmic factor and

$$L_k(x) = \frac{(-1)^{k-1}}{(k-1)!} x^{k-1} [-\log x + H_{k-1}] + \sum_{\substack{n=0 \\ n \neq k-1}}^{\infty} (-1)^n \frac{\zeta(k-n)}{n!} x^n.$$

The right-hand side is a priori only an asymptotic representation valid as $x \to 0$, but here the representation is exact as seen from uniform bounds deriving from growth formulae for the Gamma and zeta functions ((27), (28)). We have obtained in this way the Cohen–Zagier representation formula for polylogarithms [58, p. 387].

When w is not an integer, there is no confluence of singularities and, accordingly, the logarithmic term disappears. For instance

$$L_{1/2}(x) \equiv \sum_{n=1}^{\infty} \frac{e^{-nx}}{\sqrt{n}} = \sqrt{\frac{\pi}{x}} + \sum_{n=0}^{\infty} (-1)^n \frac{\zeta(\frac{1}{2}-n)}{n!} x^n.$$

Example 10. *Modified theta functions.* The sum

$$\Theta_w(x) = \sum_{n=1}^{\infty} \frac{e^{-n^2x^2}}{n^w}$$

is related to theta functions. The general case can be treated like before but we restrict attention to $w = 0, 1$. The case $w = 1$ arises in the analysis of bubble sort, see [56, p. 129 and Ex. 5.2.2–5]. One has

$$\Theta_1^*(s) = \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\zeta(s+1).$$

There is a double pole at $s = 0$ and poles of $\Gamma(s/2)$ are the even negative integers. This gives the expansion as $x \to 0$:

$$\Theta_1^*(s) \asymp \left[\frac{1}{s^2} + \frac{\gamma}{2s}\right] + \frac{1}{12}\frac{1}{s+2} + \frac{1}{240}\frac{1}{s+4} + \cdots,$$

$$\Theta_1(x) = -\log x + \frac{\gamma}{2} + \frac{1}{12}x^2 + \frac{1}{240}x^4 + \cdots.$$

The situation for $\Theta_0(x)$ is different since the poles of $\Gamma(s/2)$ at the even integers are cancelled by the values of the zeta function: $\zeta(-2) = \zeta(-4) = \cdots = 0$. Thus

$$\Theta_0(x) = \frac{1}{2}\frac{\sqrt{\pi}}{x} - \frac{1}{2} + O(x^M) \quad (x \to 0)$$

for any $M > 0$. The degenerate asymptotic series can only be asymptotic.

This last example is related to the modular transformation of theta functions (see also Poisson's summation formula in Section 4), and various related forms have been investigated from the asymptotic standpoint by Ramanujan, Berndt, De Bruijn, Knuth, and Gonnet, see [8; 15, p. 43; 36; 56, p. 129]. Ramanujan discusses some criteria that entail the existence of finite expansions [8, Ch. XV] for similar functions.

In general, the absence of singularities in a left half-plane indicates a terminating expansion with exponentially small error term. (In many practical situations estimates of the error term result from applying the saddle point method to remainder integrals.)

The two examples that follow are motivated by the two types of harmonic sums mentioned in the introduction. One involves strongly fluctuating amplitudes resulting though in a smooth global behavior. The other has geometrically increasing frequencies so that the Dirichlet series has a line of poles resulting, by Corollary 1, in an oscillating asymptotic behavior expressed by a Fourier series.

Example 11. *A divisor sum.* The analysis of the height of trees (see the introduction) suggests considering the sum

$$D(x)=\sum_{k=1}^{\infty} d(k)\,\mathrm{e}^{-kx},$$

with $d(k)$ being the number of divisors of k. Given the equality

$$\sum_{k=1}^{\infty}\frac{d(k)}{k^s}=\zeta^2(s),$$

the Mellin transform of $D(x)$ is

$$D^*(s)=\Gamma(s)\zeta^2(s),$$

with fundamental strip $\langle 1, +\infty\rangle$. There are singularities at $s=1, s=0$, then at all the odd negative integers, so that

$$D^*(s)\asymp\left[\frac{1}{(s-1)^2}+\frac{\gamma}{s-1}\right]+\left[\frac{1}{4s}\right]_{s=0}-\sum_{k=0}^{\infty}\frac{(\zeta(-2k-1))^2}{(2k+1)!}\frac{1}{s+2k+1}.$$

This translates into the asymptotic expansion of $D(x)$ as $x\to 0$:

$$D(x)\sim\frac{1}{x}(-\log x+\gamma)+\frac{1}{4}-\sum_{k=0}^{\infty}\frac{(\zeta(-2k-1))^2}{(2k+1)!}x^{2k+1}.$$

Clearly sums of divisors can be treated in a similar way as their Dirichlet series are expressible in terms of the Riemann zeta function. Ramanujan discovered a number of related formulae later treated by Berndt and Evans using Mellin transforms [8].

From the proof of Theorem 5, what essentially matters is the *balance* between the growth of $\Lambda(s)$ and the decrease of $g^*(s)$. The asymptotic expansion remains valid as long as $G^*(s)$ is globally $\mathrm{O}(s^{-r})$ for some $r>1$ in the extended strip (the C^∞ character mentioned above may get lost, though). Also, in accordance with Corollary 1, it is sufficient that the decrease of $G^*(s)$ hold along a discrete set of horizontal segments. This permits to cope with situations involving infinitely many imaginary poles, corresponding to an oscillatory behavior.

Example 12. *A doubly exponential sum and periodicities.* The prototype of harmonic sums with a fluctuating behavior is the function

$$G(x)=\sum_{k=0}^{\infty}\mathrm{e}^{-x2^k},$$

whose behavior is sought as $x \to 0$. The Mellin transform is

$$G^*(s) = \Gamma(s) \cdot \left(\sum_{k=0}^{\infty} (2^k)^{-s} \right) = \frac{\Gamma(s)}{1 - 2^{-s}}.$$

There are infinitely many poles of $\Lambda(s)$ at the points

$$\chi_k = \frac{2ik\pi}{\log 2}.$$

The Gamma function decreases fast as $\Im(s) \to \pm\infty$. The Dirichlet series $\Lambda(s) = (1 - 2^{-s})^{-1}$ stays bounded along horizontal segments that pass in between poles at $|\Im(s)| = (2k+1)i\pi/\log 2$, for k an integer. Thus, Corollary 1 to Theorem 4 applies here.

The residue of $\Lambda(s)$ is $1/(\log 2)$ by virtue of periodicity. Thus,

$$\frac{1}{1 - 2^{-s}} \asymp \sum_{k \in \mathbb{Z}} \frac{1}{\log 2} \frac{1}{s - \chi_k} \quad (s \in \mathbb{C}). \tag{32}$$

Globally, the poles of $G^*(s)$ are thus a double pole at 0, single poles induced by $\Lambda(s)$ at the χ_k, and poles at the negative integers. Translating to $G(x)$ yields

$$G(x) = -\log_2(x) - \frac{\gamma}{\log 2} + \frac{1}{2} + Q(\log_2 x) + \sum_{n=1}^{\infty} \frac{1}{1 - 2^n} \frac{(-x)^n}{n!},$$

again an exact representation. There $Q(\log_2 x)$ condenses the contribution from the non-zero imaginary poles that reflects (32):

$$Q(\log_2 x) = \frac{1}{\log 2} \sum_{k \in \mathbb{Z} \setminus \{0\}} \Gamma(\chi_k) x^{-\chi_k} = \frac{1}{\log 2} \sum_{k \in \mathbb{Z} \setminus \{0\}} \Gamma(\chi_k) \mathrm{e}^{-2ik\pi \log_2 x}.$$

Thus, after the logarithmic term, $G(x)$ contains a fluctuating term of order $\mathrm{O}(1)$ that is expressed as a Fourier series in $\log_2 x$ with explicit coefficients of a Gamma type.

An entirely similar method permits to extract fluctuations of sum like

$$K(x) = \sum_{k=0}^{\infty} (1 - \mathrm{e}^{-x/2^k}),$$

when $x \to \pm\infty$, see Proposition 2 below.

We observe that $|\Gamma(\pm 2i\pi/\log 2)| \doteq 0.54521 \times 10^{-6}$, and due to the fast decrease of $\Gamma(s)$ along imaginary lines, the fluctuating function $Q(\log_2 x)$ stays bounded by 10^{-6}. Such minute fluctuations constitute a common feature of many Mellin analyses.

A closely related example is analyzed by Hardy (his function $\phi_2(x)$ in [40, p. 37]) who notes that Ramanujan failed to obtain an elementary proof of the prime number theorem because his argument neglected these tiny "wobbles" arising from imaginary poles.

4. Summatory formulae

What we call here an *asymptotic* summatory formula is a formula that gives the asymptotic form of a class of harmonic sums when either the amplitude–frequency pair stays fixed (and the base function is taken in a wide class of functions) or the base function stays fixed (and the amplitude–frequency pair varies). The generality afforded by Theorem 5 permits to deduce a large number of summatory formulae, a typical case being generalizations of the Euler–Maclaurin formula.

Euler–Maclaurin summation. The classical Euler–Maclaurin formula provides a full asymptotic expansion expressing the convergence of a Riemann sum to the corresponding integral. The expansion assumes the base function to be C^∞ and then involves the Bernoulli numbers. The formula has been extended by Barnes [6] and later revisited by Gonnet [36] in the case of certain functions singular at the origin.

Proposition 1 (Generalized Euler–Maclaurin summation). *Let $g(x)$ be such that*

$$g(x) \underset{x\to 0}{\sim} \sum_{k=0}^{\infty} c_k x^{\alpha_k},$$

for some increasing sequence $\{\alpha_k\}$ with $\alpha_0 > -1$. Assume that $g^*(s)$ exists in $\langle -\alpha_0, \beta\rangle$ for some $\beta > 1$ and is meromorphically continuable to a function of fast decrease in any finite strip of $\langle -\infty, \beta\rangle$. Then

$$G(x) \equiv \sum_{k=1}^{\infty} g(kx) \underset{x\to 0}{\sim} \frac{1}{x}\int_0^\infty g(x)\,\mathrm{d}x + \sum_{k=0}^{\infty} c_k \zeta(-\alpha_k) x^{\alpha_k}.$$

Proof. The transform $G^*(s)$ is $g^*(s)\,\zeta(s)$ with residue at $s = 1$ that equals $g^*(1)$, itself the integral of $g(x)$. By Theorem 3, there are also poles of $g^*(s)$ and $G^*(s)$ at each $-\alpha_k$. The result then follows by Theorem 5 after taking into account the slow increase of $\zeta(s)$ in finite strips. □

Example 13. *A sum with half-integer exponents.* The sum

$$S_n = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}(n+k)^{3/2}}$$

is an Euler–Maclaurin sum since

$$S_n = \frac{1}{n^2} G\left(\frac{1}{n}\right) \quad \text{where} \quad G(x) = \sum_{k=1}^{\infty} g(kx), \quad g(x) = \frac{1}{x^{1/2}(1+x)^{3/2}}.$$

The expansion of $g(x)$ near $x = 0$ involves half-integer exponents:

$$g(x) \sim \sum_{k=0}^{\infty} c_k x^{k-1/2}, \quad c_k = (-1)^k (k+1) 2^{-2k} \binom{2k+1}{k}.$$

In addition, $g^*(s)$ has fundamental strip $\langle 1/2, 5/2\rangle$, and an explicit transform (deriving from the Eulerian Beta integral or from an earlier example)

$$g^*(s) = \mathrm{B}(s - \tfrac{1}{2}, 2 - s) = \frac{\Gamma(s - 1/2)\Gamma(2 - s)}{\Gamma(3/2)},$$

which guarantees its fast decrease. Thus, a simple computation yields

$$S_n \sim \frac{2}{n} + \sum_{k=0}^{\infty} c_k \zeta(-k + \tfrac{1}{2}) \frac{1}{n^{k+3/2}}.$$

Related examples like

$$\sum_{k=1}^{n-1} \frac{1}{\sqrt{k(n-k)}},$$

motivated by the analysis of interpolation sequential search, are discussed in [36] and tabulated in [37, p. 298]. (The analysis is however more delicate, with $g^*(s)$ only decreasing like $s^{-1/2}$.)

Example 14. *An exponential sum with square roots.* The function

$$G(x) = \sum_{n=1}^{\infty} \mathrm{e}^{-\sqrt{n}x}$$

admits the expansion

$$G(x) \underset{x \to 0}{\sim} \frac{2}{x^2} - \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \zeta\left(-\frac{k}{2}\right) x^k,$$

since its Mellin transform is $\Gamma(s)\zeta(s/2)$.

Many similar summatory formulae can be given under the assumptions of Proposition 1, like

$$\sum_{k=1}^{\infty} (-1)^{k-1} g(kx) \sim \sum_{k=0}^{\infty} c_k (1 - 2^{1+\alpha_k}) \zeta(-\alpha_k) x^{\alpha_k},$$

$$\sum_{k=1}^{\infty} (\log k)\, g(kx) \sim \frac{1}{x} \log \frac{1}{x} \int_0^{\infty} g(x)\,\mathrm{d}x + \frac{1}{x} \int_0^{\infty} g(x)(\log x)\,\mathrm{d}x + \sum_{k=0}^{\infty} c_k \zeta'(-\alpha_k) x^{\alpha_k}.$$

For instance, Ramanujan gives [8]

$$\sum_{k \geqslant 1} \mathrm{e}^{-kx} \log k = \frac{1}{x}\left(\log \frac{1}{x} - \gamma\right) + \log \sqrt{2\pi} + \mathrm{O}(x).$$

Several examples of Ramanujan have been treated systematically by Berndt and Evans using Mellin transforms [9]. Functions with logarithmic singularities at 0 can also be dealt with in this way; see the papers by Barnes [6] and Gonnet [36] for alternative approaches.

Note on formal Mellin analyses: In this range of problems, it is of interest to observe that asymptotic series follow *formally* from an exchange of summations in the harmonic sum, with the Dirichlet series coming out as a purely divergent series (!). This observation often constitutes a useful heuristic in the discovery of possible summation formulae then rigorously established by Mellin transforms. An illustration is provided by the formula

$$G(x) = \sum_{k=1}^{\infty} (-1)^k g((2k+1)x) \sim \sum_{k=0}^{\infty} c_k \eta(-\alpha_k) x^{\alpha_k},$$

where

$$\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)^s}$$

is an entire "L-function" associated with a character modulo 4, see [12]. In effect, one has formally

$$\begin{aligned} G(x) &\overset{!}{=} \sum_m (-1)^m \left(\sum_k c_k x^{\alpha_k} (2m+1)^{\alpha_k} \right) \\ &\overset{!}{=} \sum_k c_k x^{\alpha_k} \left(\sum_{m=0}^{\infty} (-1)^k (2m+1)^{\alpha_k} \right) \\ &\overset{!}{=} \sum_{k=0}^{\infty} c_k \eta(-\alpha_k) x^{\alpha_k}. \end{aligned}$$

The Mellin transform justifies such formal manipulations while "explaining" the occurrence of additional terms like the integral in the Euler–Maclaurin summation.

Dyadic sums. Sums involving powers of 2, of the type

$$G_w(x) = \sum_{k \geqslant 0} 2^{-kw} g\left(\frac{x}{2^k}\right),$$

are particularly frequent in the analysis of algorithms and we call them *dyadic* sums. In applications x usually represents a large parameter.

Proposition 2 (Dyadic sums). *Let $g(x)$ be such that*

$$g(x) \underset{x \to \infty}{\sim} \sum_{k=0}^{\infty} d_k x^{-\beta_k},$$

for some increasing sequence $\{\beta_k\}$ *with* $\beta_0 > 0$. *Assume that* $g^*(s)$ *exists in* $\langle \alpha, \beta_0 \rangle$ *for some* $\alpha < 0$ *and is meromorphically continuable to a function of fast decrease in any finite strip of* $\langle \alpha, +\infty \rangle$. *Then*

$$G_0(x) \equiv \sum_{k \geq 0} g\left(\frac{x}{2^k}\right) \underset{x \to +\infty}{\sim} \frac{1}{\log 2} \sum_{k \in \mathbb{Z}} g^*\left(\frac{2ik\pi}{\log 2}\right) \mathrm{e}^{-2ik\pi \log_2 x} + \sum_{k=0}^{\infty} \frac{d_k}{1 - 2^{\beta_k}} x^{-\beta_k}.$$

Proof. The transform of $G_0(x)$ is

$$G_0^*(s) = \frac{g^*(s)}{1 - 2^s},$$

and it is readily subjected to Theorem 5 and Corollary 1. □

Proposition 2 is characteristic of a large number of similar summation formulae. Confluence of singularities will in general induce logarithmic factors.

Example 15. *The standard dyadic sum.* Consider the analysis of $G_0(x)$ when now $g(\infty) \neq 0$ and

$$g(x) \underset{x \to +\infty}{\sim} \sum_{k=0}^{\infty} d_k x^{-k}.$$

(Thus $d_0 = g(\infty) \neq 0$.) There is now a double pole of $G_0^*(s)$ at $s = 0$ so that a two-term expansion is required for $g^*(s)$ there. By the fundamental splitting, one has

$$g^*(s) = \frac{g(\infty)}{s} + \int_0^1 g(x) x^{s-1}\,\mathrm{d}x + \int_1^{\infty} (g(x) - g(\infty)) x^{s-1}\,\mathrm{d}x,$$

so that

$$g^*(s) = \frac{g(\infty)}{s} + \gamma[g] + \mathrm{O}(s) \quad \text{where } \gamma[g] = \int_0^1 g(x) \frac{\mathrm{d}x}{x} + \int_1^{\infty} (g(x) - g(\infty)) \frac{\mathrm{d}x}{x}.$$

The constant $\gamma[g]$ is called the *Euler constant* of g since $\gamma[\mathrm{e}^{-x} - 1] = -\gamma$. Thus,

$$G(x) \sim g(\infty) \log_2 x + \tfrac{1}{2} g(\infty) + \frac{\gamma[g]}{\log 2} + P(\log_2 x) + \sum_{k=1}^{\infty} \frac{d_k}{1 - 2^k} x^{-k},$$

for an explicitly determined periodic function $P(\cdot)$.

This example illustrates in passing the general technique for determining terms of the series expansion of a Mellin transform outside of its convergence strip by adapting the technique of subtracted singularities of Theorem 3.

Poisson's summation formula. There is a fruitful connection between the classical Poisson summation formula [43, Vol. 2, p. 271]

$$\sum_{k=-\infty}^{+\infty} f(kx) = \frac{1}{x} \sum_{k=-\infty}^{+\infty} \hat{f}\left(\frac{2\pi}{x}\right), \qquad \hat{f}(u) = \int_{-\infty}^{+\infty} f(t) \mathrm{e}^{-2i\pi t u}\,\mathrm{d}u, \tag{33}$$

the functional equation of the Riemann zeta function, and other formulae of analysis like the modular transformation of theta functions (this was first observed by Riemann in his original memoir) or the partial fraction expansion of the cotangent. That connection can be based on the Mellin transform of well-chosen harmonic sums. We only provide here brief indications and refer for details to [55] and works cited therein.

The hyperbolic cotangent admits two equivalent expressions, one as a (harmonic) sum of exponentials, the other as a (harmonic sum) partial fraction decomposition:

$$\pi \coth(\pi x) = \frac{1+\mathrm{e}^{-2\pi x}}{1-\mathrm{e}^{-2\pi x}} = 1 + 2\sum_{n \geqslant 1} \mathrm{e}^{-2n\pi x}$$
$$= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{x^2+n^2}. \tag{34}$$

The functional equation of the Riemann zeta function,

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2}\,\Gamma(1-s)\zeta(1-s). \tag{35}$$

appears to be the "image" under the Mellin transform of the identity between the two forms of (34), see [83, p. 24].

The proof of Poisson's summation formula then results from taking the Mellin transform of both sides of (33) and inserting the functional equation of $\zeta(s)$. This approach has the merit of establishing a general equivalence between summation formulae of the Poisson type and functional equations of Dirichlet series [7, 42, 55]. Davies' book contains a direct illustration [13] with a proof of the modular transformation of the Dedekind η function.

Perron's formula. A collection of summatory formulae, usually referred to as Perron's formula, express partial sums of coefficients of a Dirichlet series as complex integrals of the inverse Mellin type applied to the Dirichlet series itself [41]. In our perspective, Perron's formula results from taking the step function $H(x)$ of (13) as the base function. The transform of $H(x)$ being $H^*(s) = 1/s$, one has formally for a Dirichlet series $\Lambda(s) = \sum_k \lambda_k \mu_k^{-s}$:

$$\sum_{\mu_k x < 1} \lambda_k \equiv \sum_k \lambda_k H(\mu_k x) = \frac{1}{2\mathrm{i}\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \Lambda(s) x^{-s} \frac{\mathrm{d}s}{s}.$$

Perron's formula is essential in analytic number theory, for instance in the proof of the prime number theorem. It has been recently employed for the analysis of cost functions of "divide-and-conquer" algorithms. There it permits to capture periodic fluctuations that are often of a fractal nature. For applications to mergesort and maxima finding in computational geometry, see [24, 25]; for digital sums that occur in the analysis of several algorithms (sorting networks, register allocation), we refer to [26]. A related class of divide-and-conquer recurrences (with maximum) has been treated via Tauberian methods by Fredman and Knuth [35], and elementarily by Pippenger [69]. Note that the use of Perron's formula is often delicate since the transform $H^*(s)$ does not have good decay properties towards $\pm\,\mathrm{i}\,\infty$.

PART II. COMBINATORIAL APPLICATIONS

In this part, we survey some of the major applications of Mellin asymptotics to combinatorial problems. Sums related to Catalan numbers form the subject of Section 5, and Section 6 develops the important "Bernoulli splitting process' that leads to dyadic sums. Additional combinatorial applications form the subject of Section 7.

5. Catalan sums

Catalan sums have the form

$$S_n = \sum_{k=1}^{n} \lambda_k \frac{\binom{2n}{n-k}}{\binom{2n}{n}}, \tag{36}$$

where the λ_k are of an arithmetical character. Such binomial sums occur in average values of characteristic parameters of combinatorial objects enumerated by the Catalan numbers,

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \tag{37}$$

like plane trees, binary trees, or ballot sequences [11, 38].

The historic paper of De Bruijn, Knuth, and Rice [16] provided the first analysis of this type. It concerns the expected height $\bar{H}_n$ of rooted plane trees of n nodes under the uniform distribution. The sequence λ_k is then the divisor function $d(k)$. We briefly explain here the connection between this combinatorial problem and a Catalan sum like (36). (See also [51, p. 135] for details.)

A plane tree decomposes recursively as a root node to which is attached a sequence of trees. Let A_n be the number of trees with n nodes; the ordinary generating function of the sequence $\{A_n\}$ is defined by

$$A(x) = \sum_{n=1}^{\infty} A_n z^n.$$

Then, by the classical laws of combinatorial analysis (see, e.g., [33, 38, 78, 87]), the decomposition translates into a functional equation that admits an explicit solution

$$A(z) = \frac{z}{1 - A(z)} \quad \text{and} \quad A(z) = \frac{1 - \sqrt{1 - 4z}}{2}.$$

We have $A_{n+1} = C_n$ with C_n the Catalan number of (37).

Similarly let $A_h(z)$ be the generating function of trees of height at most h. As height is inherited from subtrees, one then has the basic recurrence

$$A_{h+1}(z) = \frac{z}{1 - A_h(z)} \quad \text{with } A_1(z) = z.$$

Thus, the $A_h(z)$ are rational fractions that are also approximants to an infinite continued fraction representing $A(z)$. Solving the implied recurrence yields the closed form

$$A_h(z) = z\,\frac{(1+\sqrt{1-4z})^h - (1-\sqrt{1-4z})^h}{(1+\sqrt{1-4z})^{h+1} - (1-\sqrt{1-4z})^{h+1}}.$$

It results that the A_h can be expressed in terms of $A(z)$ alone; their Taylor expansion then derives by the Lagrange–Bürmann inversion theorem for analytic functions:

$$A_{n+1,h} - A_{n+1,h-1} = \sum_{j\geqslant 1} \omega_{jh}(n), \tag{38}$$

where

$$\omega_m(n) = \Delta^2\binom{2n}{n-m} = \binom{2n}{n+1-m} - 2\binom{2n}{n-m} + \binom{2n}{n-1-m}.$$

Thus, the number of trees of height $\geqslant h$ appears as a "sampled" sum of the $2n$th line of Pascal's triangle (upon taking second-order differences).

By a well-known form of the expectations of discrete random variables, the mean height $\bar{H}_{n+1}$ satisfies

$$\bar{H}_{n+1} = \frac{1}{A_{n+1}} \sum_h \sum_j \omega_{jh}(n).$$

Grouping terms according to the value of jh then reduces this expression to a simple sum:

$$\bar{H}_{n+1} = \frac{1}{A_{n+1}} \sum_k d(k)\,\omega_k(n). \tag{39}$$

We are therefore led to considering sums of a pattern similar to (36),

$$S_n^{(a)} = \sum_{k=1}^{n} d(k) \frac{\binom{2n}{n-k-a}}{\binom{2n}{n}},$$

since

$$\frac{1}{n+1}\bar{H}_{n+1} = S_n^{(1)} - 2S_n^{(0)} + S_n^{(-1)}, \tag{40}$$

the treatment of the central sum being typical. Stirling's formula yields the Gaussian approximation of binomial numbers: for $k = w\sqrt{n}$, and with $k = o(n^{3/4})$, one has

$$\frac{\binom{2n}{n-k}}{\binom{2n}{n}} \sim e^{-w^2}\left(1 - \frac{w^4 + 3w^2}{6n} + \frac{5w^8 + 6w^6 - 45w^4 - 60}{360n^2} + \cdots\right).$$

This leads to introducing the continuous harmonic sum

$$G(x) = \sum_{s} d(k) e^{-k^2x^2},$$

and an elementary argument (domination of the central terms) justifies the use of the Gaussian approximation inside $S_n^{(0)}$:

$$S_n^{(0)} = G\left(\frac{1}{\sqrt{n}}\right) + o(1). \tag{41}$$

The asymptotic analysis of $G(x)$ for small x (now, $x = n^{-1/2}$) is then similar to Example 11 above. From

$$G^*(s) = \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\zeta^2(s),$$

one gets

$$G(x) \sim \frac{1}{4}\sqrt{\frac{\pi}{x}}(-2\log(2x) + 3\gamma) + \frac{1}{4} + \cdots. \tag{42}$$

The other sums $S_n^{(\pm 1)}$ are treated similarly. From (40), (41), (42) (and their analogues), it is found that *the expected height of a random plane rooted tree of n nodes is*

$$\sqrt{\pi n} - \tfrac{1}{2} + o(1).$$

Full asymptotic expansions could also be determined by this technique.

The basic method here consists in approximating Catalan sums (36) by Gaussian sums of the form

$$G(x) = \sum_{k=1}^{\infty} \lambda_k e^{-k^2x^2},$$

and treating the latter by Mellin transforms. Related Catalan sums surface in the analysis of Batcher's odd–even merge sorting network [75] and in register allocation [30, 50], where the arithmetic function λ_k involved in (36) is either a function of the Gray code representation of k or the function $v_2(k)$ representing the exponent of 2 in the prime number decomposition of k.

6. The Bernoulli splitting process and dyadic sums

The Bernoulli splitting process is a general model of the random allocation of resource either in the time domain (such as stations sharing a common communication channel) or in the space domain (such as keys sharing some primary or secondary storage) whose analysis usually leads to a variety of dyadic sums. The process takes a set G of individuals and splits them recursively as follows:

- If $\operatorname{card}(G) \leqslant 1$ then the process stops and no splitting occurs.
- Otherwise $\operatorname{card}(G) \geqslant 2$, and each $g \in G$ flips a coin. Let G_0 and G_1 be the two subsets of G corresponding to the groups of individuals having flipped heads (0) and tails (1). Then the process is recursively applied to the two subsets G_0 and G_1.

A realization of the process may be described by a tree $\tau(G)$ whose internal binary nodes correspond to splittings of more than 1 element; the external nodes either contain a single individual or the empty set.

If one views the elements of G as having predetermined an infinite sequence of bits, the tree $\tau(G)$ is nothing but the *digital trie* associated to G viewed as a set of "keys"; see [37, 56, 62, 76]. Retrieval of an element g in $\tau(G)$ is achieved by following an access path dictated by g. A sequential execution of the splitting process also constitutes a way to regulate access to a common shared channel (groups consisting of single individuals may deliver their message without interference); this is the *tree communication protocol* of Capetanakis–Tsybakov; see [60, 63].

Given that the cardinality of the original group G is n, there are two basic random variables: the number I_n of nontrivial separation stages corresponding to the number of internal nodes in $\tau(G)$; the total number L_n of coin flippings corresponding to an internal path length in $\tau(G)$. The expectations $i_n = \mathrm{E}\{I_n\}$ and $\ell_n = \mathrm{E}\{L_n\}$ satisfy recurrences that reflect the nature of the splitting process; for $n \geqslant 2$, one has

$$i_n = 1 + \sum_{k=0}^{n} \pi_{n,k}(i_k + i_{n-k}), \quad \ell_n = n + \sum_{k=0}^{n} \pi_{n,k}(\ell_k + \ell_{n-k}), \qquad \pi_{n,k} = \frac{1}{2^n}\binom{n}{k}, \tag{43}$$

with initial conditions $i_0 = i_1 = \ell_0 = \ell_1 = 0$. The splitting probabilities $\pi_{n,k}$ are specific of the Bernoulli splitting process and they represent the probability of turning k heads out of n coin flips.

The basic technique to solve (43) consists in introducing the exponential generating functions

$$I(z) = \sum_{n=0}^{\infty} i_n \frac{z^n}{n!}, \qquad L(z) = \sum_{n=0}^{\infty} \ell_n \frac{z^n}{n!}, \tag{44}$$

with which (43) transforms into

$$I(z) = 2\mathrm{e}^{z/2}\, I\left(\frac{z}{2}\right) + (\mathrm{e}^z - 1 - z), \qquad L(z) = 2\mathrm{e}^{z/2}\, L\left(\frac{z}{2}\right) + z(\mathrm{e}^z - 1). \tag{45}$$

A functional equation of the form

$$\phi(z) = 2e^{z/2}\phi\left(\frac{z}{2}\right) + a(z), \tag{46}$$

with $a(z)$ a known function and $\phi(z)$ the unknown, is solved by iteration:

$$\begin{aligned}\phi(z) &= a(z) + 2e^{z/2}\,\phi\left(\frac{z}{2}\right)\\ &= a(z) + 2e^{z/2}\,a\left(\frac{z}{2}\right) + 4e^{3z/4}\,\phi\left(\frac{z}{4}\right)\\ &= \cdots\\ &= \sum_{k=0}^{\infty} 2^k e^{z(1-2^{-k})} a\left(\frac{z}{2^k}\right).\end{aligned} \tag{47}$$

This principle applies to $I(z)$ and $L(z)$ with $a(z) = e^z - 1 - z$ and $a(z) = z(e^z - 1)$, respectively. Upon expanding the exponentials, one finds the explicit forms

$$\begin{aligned} i_n &= \sum_{k=0}^{\infty} 2^k \left[1 - \left(1 - \frac{1}{2^k}\right)^n - \frac{n}{2^k}\left(1 - \frac{1}{2^k}\right)^{n-1}\right],\\ \ell_n &= n\sum_{k=0}^{\infty}\left[1 - \left(1 - \frac{1}{2^k}\right)^{n-1}\right].\end{aligned} \tag{48}$$

From there, the most direct route is the exponential approximation

$$(1-a)^n = e^{-n\log(1-a)} = e^{-na + O(na^2)} \approx e^{-na}.$$

It is legitimate to use it in (48) (see [56, p. 131] for a justification based on splitting the sum). With

$$F(x) = \sum_{k=0}^{\infty} 2^k\left[1 - \left(1 + \frac{x}{2^k}\right)e^{-x/2^k}\right], \qquad G(x) = x\sum_{k=0}^{\infty}[1 - e^{-x/2^k}],$$

one finds elementarily $i_n = F(n) + O(\sqrt{n})$ and $\ell_n = G(n) + O(\sqrt{n})$. The functions $F(x)$ and $G(x)$ are dyadic sums of a type already considered. Hence

$$F(x) = \frac{x}{\log 2} + xP(\log_2 x) + O(\sqrt{x}),$$

$$G(x) = x\log_2 x + \left(\frac{\gamma}{\log 2} + \frac{1}{2}\right)x + xQ(\log_2 x) + O(\sqrt{x}),$$

where $P(u)$ and $Q(u)$ are absolutely convergent Fourier series.

Returning to tries and neglecting the periodic fluctuations that are of amplitude less than 10^{-5}, we find that the number of binary nodes is on average about $1.44n$, *a 44% waste in storage*, while the average depth ℓ_n/n of a random external node is about $\log_2 n$ which, in the information-theoretic sense, corresponds to *an asymptotically optimal search cost.*

The variances of these parameters have been extensively studied by Kirschenhofer, Prodinger, and Szpankowski [53, 54]. These authors note that it is not a trivial problem to obtain the exact order of the dominant terms as there are non-trivial cancellations equivalent to modular type identities discovered by Ramanujan.

Jacquet and Régnier [45] have established that path length and node size are both asymptotically normally distributed. The analysis involves the study of nonlinear bivariate difference equations of which the following is typical:

$$T(u,z) = uT^2\left(u, \frac{z}{2}\right) + (1-u)(1+z)\mathrm{e}^{-z}.$$

Information for large complex z with u taken as a parameter can be gathered by a quasi-linearization process based on considering $\log T(u,z)$ which is amenable to Mellin transform technique. Once this is done, the coefficient $[z^n]\,T(u,z)$ can be recovered asymptotically, and since it is directly related to a characteristic function of the number of nodes, the Gaussian result is then established. Mahmoud's book [62] contains a detailed description of this interesting application of Mellin transforms to nonlinear bivariate problems. A recent extension has been made to suffix trees and the Lempel–Ziv data compression scheme [47].

7. Other combinatorial examples

Combinatorial sums may necessitate a certain amount of "preprocessing" before being reducibile to harmonic sums, as already exemplified by Catalan sums or sums related to tries. This section describes indirect applications of the asymptotic analysis of harmonic sums to two new types of combinatorial problems.

Reduction to standard harmonic sums. Longest runs in random binary strings are treated by Knuth [57] in a paper that deals with the equivalent problem of carry propagation in parallel binary adders. There the problem requires an analysis of dominant poles of a family of rational functions eventually leading to dyadic sums.

Examples 16. *Longest runs in strings.* Consider strings over a binary alphabet $\mathscr{A} = \{0, 1\}$. The problem is to estimate the expected length $\bar{L}_n$ of the longest run of 1's in a random string of length n, where all the 2^n possible strings are taken equally likely. The distribution was studied by Feller [21] and Knuth [57].

The probability that a random string of length n has no run of k consecutive 1's is

$$q_{n,k} = \frac{1}{2^n}[z^n]\,\frac{1-z^k}{1-2z+z^{k+1}}. \tag{49}$$

The set of such strings is described by the regular expression $1^{<k}\cdot(01^{<k})^*$, where $1^{<k}$ denotes a sequence of less than k 1's and $(\,)^*$ denotes arbitrary repetition of a pattern; the general principles of combinatorial analysis permit to write the ordinary

generating function of the set of strings under consideration as

$$\frac{1-z^k}{1-z}\cdot\frac{1}{1-z(1-zk)/(1-z)},$$

which justifies (49).

Let ρ_k be the smallest positive root of the denominator of (49) that lies between $\frac{1}{2}$ and 1. An application of the principle of the argument shows such a root to exist with all other roots that are of a larger modulus. By dominant pole analysis, the $q_{n,k}$ satisfy

$$q_{n,k} \sim c_k(2\rho_k)^{-n} \quad \text{with } c_k = \frac{1-\rho_k^k}{\rho_k(2-(k+1)\rho_k^k)}, \tag{50}$$

for large n but fixed k.

The denominator of the fraction in (49) behaves near $z = 1/2$ like a "perturbation" of $1-2z$ so that one expects ρ_k to be approximated by $\frac{1}{2}$ as $k \to \infty$. An elementary argument detailed in [57] shows that in fact

$$\rho_k = \tfrac{1}{2}(1 + 2^{-k-1} + \mathrm{O}(k2^{-2k})). \tag{51}$$

Accordingly $c_k = 1 + \mathrm{O}(k2^{-k})$.

By means of contour integration, Knuth justifies the use of (51) inside (50) for a wide range of values of k and n, which results in the approximate formula

$$q_{n,k} \approx (1-2^{-k-1})^n \approx \mathrm{e}^{-n2^{-k-1}}.$$

Let $\hat{q}_{n,k}$ denote the approximation $\mathrm{e}^{-n2^{-k-1}}$ to $q_{n,k}$. Following [57], one finds

$$\begin{aligned}\bar{L}_n &\equiv \sum_{k=0}^{\infty}[1-q_{n,k}] = \sum_{k=0}^{\infty}[1-\hat{q}_{n,k}] + \mathrm{O}\left(\frac{1}{\sqrt{n}}\right)\\ &= \sum_{k=0}^{\infty}[1-\mathrm{e}^{-n2^{-k-1}}] + \mathrm{O}\left(\frac{1}{\sqrt{n}}\right).\end{aligned}$$

This is a typical case of a dyadic sum studied repeatedly in previous sections and

$$\bar{L}_n = \log_2 n + \frac{\gamma}{\log 2} - \frac{1}{2} + \frac{1}{\log 2}\sum_{k\in\mathbb{Z}\setminus\{0\}} \Gamma\left(\frac{2ik\pi}{\log 2}\right)\mathrm{e}^{-2ik\pi\log_2 x} + \mathrm{O}\left(\frac{1}{\sqrt{n}}\right).$$

Thus, the expected length of the longest run $\bar{L}_n$ fluctuates around $\log_2 n + 0.33274$ with a minute amplitude.

An entirely similar analysis provides the expected size of the largest summand in a random composition of an integer n.

Nonstandard Dirichlet series. The algorithm of *Probabilistic Counting* introduced in [27] permits to estimate within a few percent the number of distinct elements of a large file using only a very small amount of auxiliary memory and is of interest in the context of query optimization in data bases. The analysis appeals to the nonstandard

Dirichlet series

$$N(s) = \sum_{j=1}^{\infty} \frac{(-1)^{\nu(j)}}{j^s},$$

where $\nu(j)$ is the sum of the digits in the binary representation of j.

Example 17. *Probabilistic counting.* The very design of the algorithm of [27] necessitates the analysis of a parameter of $\mathscr{B}^n$ where $\mathscr{B} = \{0,1\}^{\infty}$ is the set of all infinitely long binary strings. Let $u = (u_1, \ldots, u_n)$ be an element of $\mathscr{B}^n$; the parameter $R = R_n$ is defined to take the value k if the sequences $1, 01, 0^21, \ldots, 0^{k-1}1$ all occur as initial segments of some of the u_j's but no u_j starts with the pattern 0^k1. A plausible reasoning suggests that the expectation $\bar{R}_n$ of R over $\mathscr{B}^n$ taken with uniform measure should be close to $\log_2 n$, but higher-order asymptotic information is required here.

First, an inclusion–exclusion argument of [27] shows that

$$q_{n,k} \equiv \Pr\{R_n \geqslant k\} = \sum_{j=0}^{2^k} (-1)^{\nu(j)} \left(1 - \frac{j}{2^k}\right)^n. \tag{52}$$

It is proved in [27] that, in a central region near $\log_2 n$, one can use the exponential approximation $(1-a)^n \approx \mathrm{e}^{-na}$ while simultaneously extending the range of values of j to infinity. This justifies the approximation

$$q_{n,k} \approx \hat{q}_{n,k} \quad \text{where } \hat{q}_{n,k} = \sum_{j=0}^{\infty} (-1)^{\nu(j)} \mathrm{e}^{-jn/2^k} = \prod_{j=0}^{\infty} (1 - \mathrm{e}^{-2^j(n/2^k)}). \tag{53}$$

Define the function

$$\theta(x) = \sum_{j \geqslant 0} (-1)^{\nu(j)} \mathrm{e}^{-jx} = \prod_{j \geqslant 0} (1 - \mathrm{e}^{-x2^j}). \tag{54}$$

Eq. (53) means that the cumulative distribution $q_{n,k}$ is well approximated by $\theta(n/2^k)$. In fact, an elementary argument of [27] establishes that

$$\bar{R}_n \equiv \sum_{k=1}^{\infty} q_{n,k} = \Theta(n) + \mathrm{o}(1) \quad \text{where } \Theta(x) = \sum_{k=1}^{\infty} \theta\left(\frac{x}{2^k}\right), \tag{55}$$

and the asymptotic form of $\Theta(x)$ for large x is required.

The functions $\theta(x)$ and $\Theta(x)$ are both harmonic sums. From (54) and (55), one finds

$$\theta^*(s) = N(s) \cdot \Gamma(s), \qquad \Theta^*(s) = \frac{2^s \theta^*(s)}{1-2^s} = \frac{2^s}{1-2^s} N(s) \Gamma(s). \tag{56}$$

In particular, a more detailed investigation of $N(s)$ is needed.

Grouping terms by 2^m in the definition of $N(s)$ and using the binomial theorem (see also Proposition 6 below) shows that $N(s)$ is an entire function and is of moderate growth in any right half-plane, owing to cancellations afforded by the sign alternation properties of the sequence $(-1)^{\nu(j)}$. This permits to justify (55) and hence by a residue computation it gives the asymptotic form of $\Theta(x)$:

$$\Theta(x) \underset{x\to+\infty}{=} \log_2 x + \frac{\gamma}{\log 2} + \frac{N'(0)}{\log 2} - \frac{1}{2} + P(\log_2 x) + \mathrm{o}(1), \tag{57}$$

for some oscillating function $P(u)$ once more found to be of small amplitude.

The grouping technique yields $N(0) = -1$ (used in (57)) and it permits expressing $N'(0)$ as an alternating sum of logarithms. Thus, from (55), (57), one gets

$$R_n = \log_2(\varphi n) + P(\log_2 n) + \mathrm{o}(1),$$

where

$$\varphi = 2^{-1/2}\mathrm{e}^{\gamma} \prod_{m=1}^{\infty} \left(\frac{2m+1}{2m}\right)^{(-1)^{\nu(m)}} \doteq 0.77351.$$

There results an algorithm based on the idea that $(1/\varphi)2^{R_n}$ can be turned into an unbiased statistical estimator of the a priori unknown number n of distinct binary strings considered.

The sequence $\varepsilon(j) = (-1)^{\nu(j)}$ is the classical Thue–Morse sequence. Dirichlet series related to $N(s)$ have been considered in [2]. Similar techniques are used in other probabilistic estimation algorithms like *Approximate Counting* [22] and the collision resolution methods of [39].

PART III. GENERAL MELLIN ASYMPTOTICS

The last sections have shown that Mellin asymptotics is applicable to harmonic sums provided the Dirichlet series is of moderate growth while the transform of the base function is of fast decrease. The applications encountered so far have made an essential use of properties specific to the Riemann zeta function, to the Gamma function, or to the sine function.

This part shows that, thanks to general theorems presented in Section 8, Mellin asymptotics may also be applied to "implicit" harmonic sums, where it is no longer required to have closed forms for either the Dirichlet series or the transform of the base function.

8. General conditions for Mellin asymptotics

In this section, we provide general conditions under which Mellin transforms are small towards $\pm i\infty$ and describe a general class of Dirichlet series that are of moderate growth. These results extend the range of applicability of Mellin asymptotics to a much larger class of harmonic sums.

Smallness of Mellin transforms. The smallness of a Mellin transform is directly related to the degree of "smoothness" (differentiability, analyticity) of the original function.

Proposition 3 (Smallness in the fundamental strip). (i) *Let $f(x)$ be locally integrable with fundamental strip $\langle \alpha, \beta \rangle$. Then, uniformly with respect to σ in any closed subinterval of (α, β), one has*

$$f^*(\sigma + \mathrm{i}t) = \mathrm{o}(1) \quad \text{as } t \to \pm\infty.$$

(ii) *If in addition $f(x)$ is of class $\mathscr{C}^r$ and the fundamental strip of $\Theta^r f$ contains $\langle \alpha, \beta \rangle$, then*

$$f^*(\sigma + \mathrm{i}t) = \mathrm{o}(|t|^{-r}) \quad \text{as } t \to \pm\infty.$$

Proof (*sketch*). (i) From the form

$$f^*(\sigma + \mathrm{i}t) = \int_0^\infty f(x)\, \mathrm{e}^{-\sigma x}\, \mathrm{e}^{\mathrm{i}t \log x}\, \mathrm{d}x,$$

the function $f^*(s)$ is an integrable function hashed by a complex exponential. By the Riemann–Lebesgue lemma [43, 81], $f^*(s)$ tends to 0 as $t \to \pm\infty$.

(ii) The second form results from the formula $\mathscr{M}[\Theta^r f(x); s] = (-1)^r s^r f^*(s)$. □

The next proposition shows that smallness extends beyond the fundamental strip for smooth functions with smooth derivatives.

Proposition 4 (Smallness beyond the fundamental strip). *Let $f(x)$ be of class $\mathscr{C}^r$ with fundamental strip $\langle \alpha, \beta \rangle$. Assume that $f(x)$ admits an asymptotic expansion as $x \to 0^+$ (resp. $x \to \pm\infty$) of the form*

$$f(x) = \sum_{(\xi, k) \in A} c_{\xi,k}\, x^\xi (\log x)^k + \mathrm{O}(x^\gamma), \tag{58}$$

where the ξ satisfy $-\alpha \leqslant \xi < \gamma$ (resp. $\gamma < \xi \leqslant -\beta$). Assume also that each derivative $(\mathrm{d}^j/\mathrm{d}x^j)\, f(x)$ for $j = 1, \ldots, r$ satisfies an asymptotic expansion obtained by termwise differentiation of (58). Then the continuation of $f^*(s)$ satisfies

$$f^*(\sigma + \mathrm{i}t) = \mathrm{o}(|t|^{-r}) \quad \text{as } |t| \to \infty \tag{59}$$

uniformly for σ in any closed subinterval of $(-\gamma, \beta)$ (resp. of $(\alpha, -\gamma)$).

Proof. It suffices to consider extension to the left of $f^*(s)$. Choose some positive number $p > \gamma$ and define

$$a(x) = \left(\sum_{(\xi, k) \in A} c_{\xi,k}\, x^\xi (\log x)^k \right) \exp(-x^p).$$

The function $g(x) = f(x) - a(x)$ satisfies the assumptions of Proposition 3 and its transform $g^*(s)$ is thus $o(|s|^{-r})$ in its fundamental strip $\langle -\gamma, \beta \rangle$. The transform $a^*(s)$ is itself exponentially small given growth properties of the Gamma function and its derivatives. Thus $f^*(s) = a^*(s) + g^*(s)$ satisfies the stated bounds. □

Analyticity is the strongest possible form of smoothness for a function $f(x)$; in that case the transform $f^*(s)$ decays exponentially in a quantifiable way.

Proposition 5 (Exponential smallness in the analytic case). *Let $f(x)$ be analytic in S_θ where S_θ is the sector*

$$S_\theta = \{z \in \mathbb{C} \mid 0 < |t| < +\infty \text{ and } |\arg(z)| \leqslant \theta\} \quad \text{with } 0 < \theta < \pi.$$

Assume that $f(x) = O(x^{-\alpha})$ as $x \to 0$ in S_θ, and $f(x) = O(x^{-\beta})$ as $x \to \infty$ in S_θ. Then

$$f^*(\sigma + it) = O(e^{-\theta|t|})$$

uniformly for σ in every closed subinterval of (α, β).

Proof (*sketch*). The integral defining Mellin transforms in this case applied to an analytic function. By Cauchy's theorem, the integration contour may be taken as the half-line of slope θ:

$$f^*(s) = \int_0^{e^{i\theta}\infty} f(t)\, t^{s-1}\, dt.$$

The change of variable $t = \rho e^{i\theta}$ gives

$$f^*(s) = e^{i\theta s} \int_0^\infty f(\rho e^{i\theta})\, \rho^{s-1}\, d\rho.$$

The result follows as the integral converges. □

Smallness extends outside of the fundamental strip by an argument similar to that of Proposition 4 and based on subtracting suitable combinations of exponentials.

Example 18. *A sum with an implicit transform.* The harmonic sum

$$G(x) = \sum_{n=1}^{\infty} \frac{d(n)}{\sqrt{\cosh nx}},$$

with $d(n)$ the divisor function, admits the transform $\zeta^2(s) \cdot g^*(s)$ where $g^*(s)$ is the transform of $g(x) = (\cosh x)^{-1/2}$. The function $g^*(s)$ is not simply expressible in terms of standard functions (it is "implicitly" defined as the transform of $g(x)$). However, the fundamental correspondence provides its complete singular expansion (see Example 4) while Propositions 4 and 5 show that $g^*(s)$ is of fast decrease in any right half-plane. Thus, Mellin analysis applies and the asymptotics of $G(x)$ as $x \to 0$ obtains from the singular expansions of $g^*(s)$ and $\zeta^2(s)$. One gets

$$G(x) \underset{x\to 0}{\sim} -g^*(1)\frac{\log x}{x} + \frac{(g^*)'(1) + 2\gamma g^*(1)}{x} + \frac{1}{4} + \mathrm{O}(x^M),$$

for any $M > 0$. There $g^*(1) = \int_0^\infty g(x)\,\mathrm{d}x$ and $(g^*)'(1)$ is an "Euler constant" of $g(x)$ which is expressible as an integral as discussed in Example 15.

Example 19. *A lattice sum.* The goal here is to discuss certain (2-dimensional) "lattice sums" of the form

$$F(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f((m^2 + n^2)x^2). \tag{60}$$

(See also [10] for related developments.) Such a sum may be rewritten as $\sum_{k \geqslant 1} r(x) f(kx^2)$, where $r(k)$ is the number of representations of k as a sum of two squares. The corresponding Dirichlet series[3] is then

$$\rho(s) = \sum_{k=1}^{\infty} \frac{r(k)}{k^s} = \sum_{m,n \geqslant 1} \frac{1}{(m^2 + n^2)^s},$$

and everything rests on its behavior at $\pm \mathrm{i}\infty$ and its special values.

Let $\Theta(x) = \sum_{m=1}^{\infty} \mathrm{e}^{-m^2x^2}$. The Mellin transform of $\Theta^2(x)$ is

$$\mathcal{M}(\Theta^2(x), s) = \rho(s) \cdot \frac{1}{2} \Gamma\left(\frac{s}{2}\right).$$

An already discussed Mellin analysis of $\Theta(x)$ yields its asymptotic expansion near $x = 0$:

$$\Theta(x^{1/2}) = \sqrt{\frac{\pi}{2x}} - \frac{1}{2} + R(x), \tag{61}$$

the remainder term $R(x)$ being exponentially small (this alternatively results from Poisson's formula). Thus, $\Theta^2(x)$ admits an explicit three-term asymptotic expansion,

$$\Theta^2(x^{1/2}) = \frac{\pi}{2x} - \sqrt{\frac{\pi}{2x}} + \frac{1}{4} + R_2(x),$$

with again $R_2(x)$ exponentially small.

Consequently, $\rho(s)$ is meromorphic in the whole of $\mathbb{C}$ with poles at $s = 2, 1$ only and singular expansion

$$\rho(s) \asymp \frac{\pi}{2(s-2)} - \frac{1}{s-1} + \left[\frac{1}{4s}\right]_{s=0} + [0]_{s=-2} + [0]_{s=-4} + \cdots.$$

[3] In order to allow for extensions to higher dimensions or higher powers, we do not make use here of special properties of $\rho(s)$. See [40, Ch. IX] for a vivid account of these aspects.

In addition, Proposition 5 quantifies the growth of the transform of $\Theta^2(x)$ in the form $e^{-(\pi/4-\varepsilon)|t|}$. Thus, $\rho(s)$ is itself of growth at most $e^{\varepsilon|s|}$ for any $\varepsilon > 0$ in any finite strip of the complex plane.

This knowledge permits to analyze lattice sums of the form (60). For instance,

$$L(p) = \sum_{m,n\geqslant 1} \frac{1}{(m^2+n^2+p^2)^3}$$

satisfies $L(p) = p^{-6}F(1/p)$ provided one takes $f(x) = (1+x)^{-3}$. One has in $\langle 0, 3\rangle$

$$f^*(s) = \frac{\pi}{\sin \pi s} \frac{(s-1)(s-2)}{2}.$$

There results an asymptotic form of $F(x)$ that induces a corresponding estimate of $L(p)$:

$$L(p) \sim \frac{\pi}{8p^4} - \frac{3\pi}{16p^5} + \frac{5-\pi}{4p^6},$$

the error term being exponentially small since $\rho(s)$ vanishes at the even negative integers. The method clearly extends to higher dimensional sums.

Growth of Dirichlet series. Dirichlet series whose components λ_k, μ_k admit descending asymptotic expansions in powers of k have the property of being meromorphically continuable with well individuated poles.

Proposition 6 (Growth of special Dirichlet series). *Let λ_k and μ_k admit asymptotic expansions in descending powers of k as $k \to \infty$:*

$$\lambda_k \sim \sum_{r=0}^{\infty} \frac{a_r}{k^{\alpha_r}}, \qquad \mu_k \sim k^w \left(1 + \sum_{r=1}^{\infty} \frac{b_r}{k^{\beta_r}}\right).$$

Then the Dirichlet series $\sum_k \lambda_k \mu_k^{-s}$ can be continued to a meromorphic function $\Lambda(s)$ in the whole of the complex plane. The function $\Lambda(s)$ is of moderate growth in any right half-plane of the complex plane.

Proof (*sketch*). The particular case $\alpha_r = \beta_r = r$ gives the essential idea. The binomial theorem yields

$$\begin{aligned}\frac{\lambda_n}{\mu_n^s} &= \frac{1}{n^{ws}}\left[a_0 + \frac{a_1}{n} + \cdots\right]\left[1 - \frac{s}{1!}\left(\frac{b_1}{n} + \cdots\right) + \frac{s(s+1)}{2!}\left(\frac{b_1}{n} + \cdots\right)^2 + \cdots\right] \\ &= \frac{a_0}{n^{ws}} + \sum_{k\geqslant 1} \frac{P_k(s)}{n^{ws+k}},\end{aligned} \tag{62}$$

for a family of polynomials P_k. By summation, one finds that $\Lambda(s)$ has, in any finite strip, the same singularities as a partial sum of

$$\zeta(ws) + \sum_{k\geqslant 1} P_k(s)\zeta(ws+k).$$

(This is easily justified rigorously by terminating forms of the binomial theorem.) Also, $\Lambda(s)$ is of moderate growth since each component zeta function shares this characteristic.

In the general case, one finds similarly (with a slight abuse of notations)

$$\Lambda(s) \asymp \zeta(ws + \alpha_0) + \sum_{\xi \in \Xi} P_\xi(s)\zeta(ws + \xi), \tag{63}$$

where ξ ranges over the set Ξ of all linear combinations of the form $\alpha_{r_0} + \beta_{r_1} + \cdots + \beta_{r_j}$. $\square$

For instance, from the expansion

$$\frac{1}{4^n}\binom{2n}{n} = \frac{1}{\sqrt{\pi n}}\left[1 - \frac{1}{8n} + \frac{1}{128n^2} + O\left(\frac{1}{n^3}\right)\right],$$

one finds by summation that

$$\Lambda(s) = \sum_{n=1}^{\infty} \frac{1}{4^n}\binom{2n}{n}\frac{1}{n^s} = \frac{1}{\sqrt{\pi}}\left[\zeta\left(s+\frac{1}{2}\right) - \frac{1}{8}\zeta\left(s+\frac{3}{2}\right) + \frac{1}{128}\zeta\left(s+\frac{5}{2}\right) + R(s)\right], \tag{64}$$

where $R(s)$ converges in the same half-plane as $\zeta(s+\frac{7}{2})$, namely in $(-\frac{5}{2}, +\infty)$.

In a similar vein, if $Q(u)$ is a polynomial that does not vanish at the positive integers, and $P(u)$ is an arbitrary polynomial, then

$$\Lambda(s) = \sum_{n=1}^{\infty} \frac{P(n)}{(Q(n))^s}$$

has poles only at a discrete set of rational numbers that is bounded from the right.

Amplitudes involving the harmonic numbers $H_n = 1 + 1/2 + \cdots + 1/n$ can be treated in this way by considering $\hat{H}_n = H_n - \log n$. From

$$H_n = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} + O(n^{-6}),$$

one deduces

$$\Lambda(s) = \sum_{n=1}^{\infty} \frac{H_n}{n^s} \asymp \zeta'(s) + \gamma\zeta(s) + \tfrac{1}{2}\zeta(s+1) - \tfrac{1}{12}\zeta(s+2) + \tfrac{1}{120}\zeta(s+4) \qquad (s \in \langle -5, \infty \rangle)$$

$$\asymp \left[-\frac{1}{(s-1)^2} + \frac{\gamma}{s-1}\right]_{s=1} + \frac{1}{2s} - \frac{1}{12(s+1)} + \frac{1}{120(s+3)}.$$

For instance, $\Lambda(s)$ has a simple pole with residue 1/120 at $s = -3$. More information on this function may be found in [5].

Example 20. *A sum with binomial amplitudes.* The function

$$G(x) = \left(\frac{1}{2}\right)\frac{1}{1+x} + \left(\frac{1\cdot 3}{2\cdot 4}\right)\frac{1}{1+2x} + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\frac{1}{1+3x} + \cdots$$

has the transform $\Lambda(s)\pi/\sin \pi s$ where $\Lambda(s) = \sum_{n=1}^{\infty} 4^{-n}\binom{2n}{n} n^{-s}$. The singularities of $\Lambda(s)$ are given by Eq. (64), hence

$$G(x) \underset{x\to 0}{=} \sqrt{\frac{\pi}{x}} + \Lambda(0) + \frac{1}{8}\sqrt{\pi x} + \mathrm{O}(x),$$

where $\Lambda(0)$ is obtained by the analytic continuation technique of Proposition 6:

$$\Lambda(0) = \sum_{n=1}^{\infty}\left[\frac{1}{4^n}\binom{2n}{n} - \frac{1}{\sqrt{\pi n}}\right] + \frac{1}{\sqrt{\pi}}\zeta\left(\frac{1}{2}\right).$$

A full asymptotic expansion in powers of $x^{1/2}$ can be determined in this way.

Example 21. *Stirling's formula for modified Gamma functions.* Consider the problem of evaluating asymptotically as $x \to +\infty$

$$H(x) = \prod_{n=1}^{\infty}\left(1 + \frac{x}{n(n+1)}\right).$$

One estimates instead the harmonic sum $G(x) = \log H(x)$ that is a harmonic sum with transform valid in $\langle -1, -\frac{1}{2}\rangle$;

$$G^*(s) = \Lambda(-s)\cdot\left(\frac{\pi}{s \sin \pi s}\right) \quad \text{where } \Lambda(s) = \sum_{n=1}^{\infty}\frac{1}{(n^2+n)^s}.$$

From Proposition 6, $\Lambda(s)$ has a singular expansion induced by the expansion of $(1+n^{-1})^{-s}$. One has

$$\Lambda(s) = \zeta(2s) - \frac{s}{1}\zeta(2s+1) + R(s)$$

where $R(s)$ converges like $\zeta(2s+2)$ in $\Re(s) > -\frac{1}{2}$.

The Mellin transform $G^*(s)$ thus has a simple pole at $s = -\frac{1}{2}$ and a triple pole at 0. This provides "Stirling's formula" for $H(x)$:

$$H(x) \underset{x\to\infty}{\sim} \mathrm{e}^{\pi\sqrt{x}}\frac{C}{x},$$

where "Stirling's constant" C is here $C = 1/\pi$.

In this particular case, $H(x)$ is explicitly expressible in terms of trigonometric functions (by the infinite product formula for $\cos(x)$); the example is only meant to demonstrate the general approach to the analysis of such modified Gamma functions.

A similar technique works for Dirichlet series $\Lambda(s)$ whose elements λ_k, μ_k admit expansions in powers of 2^k. Assume that

$$\lambda_k \sim 1 + \sum_{j=1}^{\infty} a_j 2^{-kj}, \qquad \mu_k \sim 2^k\left(1 + \sum_{j=1}^{\infty} b_j 2^{-kj}\right).$$

A technique entirely similar to that of Proposition 6 applies. From

$$\lambda_k(\mu_k)^{-s} \sim 2^{-ks}\cdot(1 + (a_1 - b_1 s)2^{-k} \\ + (a_2 + \tfrac{1}{2}(b_1^2 - 2a_1b_1 - 2b_2)s + \tfrac{1}{2}b_1^2s^2)2^{-2k} + \cdots),$$

one gets by summation

$$\Lambda(s) = \frac{2^{-s}}{1-2^{-s}} + (a_1 - b_1 s)\frac{2^{-s-1}}{1-2^{-s-1}} + \cdots.$$

Thus, $\Lambda(s)$ admits a pole at all points of a half-lattice

$$\chi_k^{(m)} = -m + \frac{2ik\pi}{\log 2} \quad \text{for } m \geqslant 0,\ k \in \mathbb{Z},$$

and is in addition of controlled growth away from poles in any finite strip of $\mathbb{C}$.

An immediate case of application is to modified dyadic sums involving frequencies of the form $2^k \pm 1$. This technique also applies to sums related to the Bernoulli splitting model of Section 6; for instance, the Mellin transform of

$$F(x) = \sum_{k=1}^{\infty}\left[1 - \left(1 - \frac{1}{2^k}\right)^x\right]$$

is

$$F^*(s) = \Lambda(s)\cdot\Gamma(s) \quad \text{where } \Lambda(s) = \sum_{k=1}^{\infty}(\log(1-2^{-k})^{-1})^{-s}.$$

By the technique above one finds

$$\Lambda(s) \asymp \frac{2^s}{1-2^s} - \frac{s}{2}\frac{2^{s-1}}{1-2^{s-1}} \quad (s \in \langle -\infty, 2\rangle).$$

This permits to compute complete asymptotic expansions without a recourse to the exponential approximation. The half-lattice of poles implies the presence of fluctuating functions at each level of the asymptotic expansion of $F(x)$.

The transfer technique. Our final example illustrates the general approach seen so far in conjunction with an important transfer technique that consists in going back and forth between various harmonic sums involving related amplitude and frequencies but different base functions. This "zigzag" method appeals to the common occurrence of a Dirichlet series in two harmonic sums

$$\sum a_k f(kx) \quad \text{and} \quad \sum a_k g(kx),$$

with $g(x) = \mathrm{e}^{-x}$ playing a special role because of the relation it establishes with ordinary generating functions.

The transfer technique has already been used implicitly: for instance the study of the harmonic sum $\mathrm{e}^{-x}(1-\mathrm{e}^{-x})^{-1}$ yields properties of $\zeta(s)$ (Example 2) that can in turn be used to analyze the theta function $\sum_k \mathrm{e}^{-k^2x^2}$ (Example 10) and eventually lattice sums (Example 19).

The principle is as follows. Assume that the behavior of

$$F(x) = \sum_{k=1}^{\infty} a_k f(kx)$$

is sought. This requires knowledge of the Dirichlet series

$$\alpha(s) = \sum_{k=1}^{\infty} \frac{a_k}{k^s}.$$

If the ordinary generating function

$$A(x) = \sum_{k=1}^{\infty} a_k x^k,$$

is known and in addition $A(x)$ is analytic at $x = 1$ (this often happens for alternating series with elementary coefficients), then

$$A(\mathrm{e}^{-x}) = \sum_{m=0}^{\infty} b_m x^m,$$

which corresponds to a singular expansion

$$\mathcal{M}(A(\mathrm{e}^{-x}), s) = \alpha(s)\Gamma(s) \asymp \sum_{m=0}^{\infty} \frac{b_m}{m+s} \quad (s \in \mathbb{C}).$$

Comparison with the singular expansion of the product $\alpha(s)\Gamma(s)$, namely

$$\alpha(s) \asymp \sum_{m=0}^{\infty} (-1)^m \frac{\alpha(-m)}{m!} \frac{1}{s+m},$$

shows by identification that

$$\alpha(-m) = (-1)^m m!\, b_m = (-1)^m m!\, [x^m]\, A(\mathrm{e}^{-x}). \tag{65}$$

In this way, coefficients in many expansions can be determined explicitly. In general, the formulae obtained are nontrivial[4] since the original series defining $\alpha(s)$ stops being convergent at negative integers.

[4] The transfer process often provides a rigorous counterpart of formal computations with purely divergent series. See the remarks relative to Euler–Maclaurin summations in Section 4 and the theory of "regularized" sums and products in [49].

Example 22. *An alternating sum with harmonic numbers.* Define

$$G(x)=\sum_{k=1}^{\infty}(-1)^k H_k \mathrm{e}^{-k^2x^2},$$

that arises as an approximation to the combinatorial sum

$$\sum_{k=1}^{n}(-1)^k H_k \binom{2n}{n-k}.$$

The Mellin transform of $G(x)$ is

$$G^*(s)=h(s)\cdot\frac{1}{2}\Gamma\left(\frac{s}{2}\right) \quad \text{where } h(s)=\sum_{k=1}^{\infty}(-1)^k\frac{H_k}{k^s}.$$

Thus, $G^*(s)$ has poles at $x=0,\ -2,\ -4,\dots$ and determining the residues of $G^*(s)$ requires the *values* of $h(s)$ at the negative even integers.

The Dirichlet series $h(s)$ is an entire function. This results from a simple extension of Proposition 6 to alternating series, as alternating zeta functions are entire. The special values of $h(s)$ are then obtained from the expansion

$$\begin{aligned}\sum_{k=1}^{\infty}(-1)^k H_k \mathrm{e}^{-kx} &= \frac{1}{1+\mathrm{e}^{-x}}\log\frac{1}{1+\mathrm{e}^{-x}}\\ &= -\frac{\log 2}{2}+\left(\frac{1}{4}-\frac{\log 2}{4}\right)x+\frac{1}{16}x^2\\ &\quad+\left(-\frac{1}{32}+\frac{\log 2}{48}\right)x^3-\frac{1}{128}x^4+\cdots,\end{aligned}$$

so that

$$h(0)=-\tfrac{1}{2}\log 2,\quad h(-2)=\tfrac{1}{8},\quad h(-4)=-\tfrac{3}{16},\dots$$

Thus finally

$$G(x)\underset{x\to 0}{\sim}-\tfrac{1}{2}\log 2-\tfrac{1}{8}x^2-\tfrac{3}{32}x^4+\cdots.$$

9. Conclusions

This paper has demonstrated the basic technology of Mellin asymptotics of harmonic sums. There, a crucial role is played by the separation property and the fundamental correspondence. The method is likely to be applicable as soon as the expressions involved are of an analytic character (in the old sense of the term) affording analytic continuation which is the most important requirement of the method. Combinatorial expressions are thus very natural candidates for Mellin asymptotics. Related techniques in connection with finite differences are explored in [34].

As mentioned in Section 4, Perron's formula belongs to the galaxy of Mellin-related techniques. We have not developed this classical aspect here as it is extensively discussed elsewhere. In [25] it is shown how divide-and-conquer recurrences lead to fluctuations of a fractal nature that can be quantified via a Mellin analysis; digital sequences are studied along similar lines in [26].

An interesting offspring of Mellin asymptotic of harmonic sums is the class of *two-stage* methods. In many cases, a generating function can be analyzed asymptotically as a harmonic sum in the complex realm, near a singularity or a critical point. It then becomes possible to estimate the coefficients of the generating function by either singularity analysis or the saddle-point method. For a conjunction of Mellin and saddle-point methods, the standard example is De Bruijn's pioneering work on the enumeration of binary partitions [14] which was follows by the general approach of Meinardus in the theory of integer partitions [3]. Singularity analysis used in conjunction with Mellin asymptotics may be used as a basis for the analysis of Catalan sums [28] as well as many other combinatorial sums. We hope to return to these questions in a future paper.

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