#A46 INTEGERS 9 (2009), 605-619

SOME RESULTS FOR GENERALIZED HARMONIC NUMBERS[1]

Cong-Jiao Feng
School of Mathematical Sciences, Dalian University of Technology, Dalian, China.

Feng-Zhen Zhao
School of Mathematical Sciences, Dalian University of Technology, Dalian, China.

Received: 7/8/07, Revised: 6/13/09, Accepted: 8/4/09, Published: 11/27/09

Abstract

In this paper, we discuss the properties of a class of generalized harmonic numbers $H(n,r)$. By means of the method of coefficients, we establish some identities involving $H(n,r)$. We obtain a pair of inversion formulas. Furthermore, we investigate certain sums related to $H(n,r)$, and give their asymptotic expansions. In particular, we obtain the asymptotic expansion of certain sums involving $H(n,r)$ and the inverse of binomial coefficients by Laplace's method.

1. Introduction

It is well-known that the harmonic numbers H_n are defined by

$$H_0 = 0, \quad H_n = \sum_{k=1}^{n} \frac{1}{k},$$

and the generating function of H_n is

$$\sum_{n=1}^{\infty} H_n z^n = -\frac{\ln(1-z)}{1-z}.$$

The harmonic number H_n plays an important role in number theory and has been generalized by many authors (see[1], [2], [5], [7], [8], [11]). In this paper, we consider a class of generalized harmonic numbers $H(n,r)$. The definition of $H(n,r)$[7] is

$$H(n,r) = \sum_{1\le n_0+n_1+\cdots+n_r\le n} \frac{1}{n_0 n_1 \cdots n_r}, \quad \text{for} \quad n \ge 1, \quad r \ge 0.$$

It is clear that $H(n,0) = H_n$. The generating function of $H(n,r)$ is (see [4])

$$\sum_{n=r+1}^{\infty} H(n,r) z^n = \frac{(-1)^{r+1} \ln^{r+1}(1-z)}{1-z}. \tag{1}$$

[1]This work was supported by the Science Research Foundation of Dalian University of Technology.

From (1) we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} H(n+r+1,r)z^n &= \frac{(-1)^{r+1}\ln^{r+1}(1-z)}{z^{r+1}(1-z)}, \quad (2)\\ \sum_{n=r+1}^{\infty} \frac{H(n,r)}{n+1}z^{n+1} &= \frac{(-1)^{r}\ln^{r+2}(1-z)}{r+2},\\ \sum_{n=0}^{\infty} \frac{H(n+r+1,r)}{n+r+2}z^{n} &= \frac{(-1)^{r}\ln^{r+2}(1-z)}{(r+2)z^{r+2}}.\end{aligned}$$

There are many relations between $H(n,r)$ and H_n. For instance (see [4]),

$$\begin{aligned}\sum_{r=0}^{n} \frac{1}{(r+1)!}H(n,r) &= n,\\ \sum_{r=0}^{n-1} \frac{(-1)^r}{(r+1)!}H(n,r) &= 1,\\ \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r!}H(n+1,r) &= H_n.\end{aligned}$$

The numbers $H(n,r)$ can be computed by the formula (see [4])

$$H(n,r) = \frac{(-1)^{r+1}}{n!}\left(\frac{d^n}{dx^n}\frac{[\ln(1-x)]^{r+1}}{1-x}\bigg|_{x=0}\right).$$

Some initial values of $H(n,r)(n \geq r+1)$ are given in Table 1.

$n \setminus r$	0	1	2	3	4	5
1	1					
2	$\frac{3}{2}$	1				
3	$\frac{11}{6}$	2	1			
4	$\frac{25}{12}$	$\frac{35}{12}$	$\frac{5}{2}$	1		
5	$\frac{137}{60}$	$\frac{15}{4}$	$\frac{17}{4}$	3	1	
6	$\frac{49}{20}$	$\frac{203}{45}$	$\frac{49}{8}$	$\frac{35}{6}$	$\frac{7}{2}$	1

Table 1: Initial Values of $H(n,r)$

In this paper, we investigate the properties of $H(n,r)$. The paper is organized as follows. In Section 2, we obtain some identities for $H(n,r)$ and Cauchy numbers of the first kind (associated Stirling numbers of the first kind) by means of the method of coefficients [10]. In Section 3,we obtain a pair of inversion formulas.In Section 4, we give the asymptotic expansion of certain sums related to $H(n,r)$ and Cauchy numbers of the second kind (binomial coefficients) when r is fixed.

For convenience, we recall some definitions involved in the paper. Throughout, we denote the Cauchy numbers of the first kind and the second kind by a_n and b_n, respectively. Let $s(n,k)$, $s_2(n,k)$, and $S(n,k)$ stand for Stirling numbers of the first kind, associated Stirling numbers of the first kind, and Stirling numbers of the second kind, respectively. Their definitions are respectively (see [3]):

$$\sum_{n=0}^{\infty} a_n \frac{z^n}{n!} = \frac{z}{\ln(1+z)}, \qquad \sum_{n=0}^{\infty} b_n \frac{z^n}{n!} = \frac{-z}{(1-z)\ln(1-z)},$$

$$\sum_{n=k}^{\infty} s(n,k) \frac{z^n}{n!} = \frac{\ln^k(1+z)}{k!}, \qquad \sum_{n=k}^{\infty} S(n,k) \frac{z^n}{n!} = \frac{(e^z-1)^k}{k!},$$

$$\sum_{n=k}^{\infty} s_2(n,k) \frac{z^n}{n!} = \frac{[\ln(1+z)-z]^k}{k!}.$$

Throughout this paper, the binomial coefficients $\binom{n}{m}$ are defined by

$$\binom{n}{m} = \begin{cases} \dfrac{n!}{m!(n-m)!}, & n \geq m, \\ 0, & n < m, \end{cases}$$

where n and m are nonnegative integers.

Let $[z^n]f(z)$ denote the coefficient of z^n for the formal power series of $f(z)$. The $[t^n]$ are called the "coefficient of" functionals [10]. If $f(t)$ and $g(t)$ are formal power series, the following relations hold [10]:

$$[t^n](\alpha f(t) + \beta g(t)) = \alpha[t^n]f(t) + \beta[t^n]g(t), \tag{3}$$

$$[t^n]tf(t) = [t^{n-1}]f(t), \tag{4}$$

$$[t^n]f(t)g(t) = \sum_{k=0}^{n}([y^k]f(y))[t^{n-k}]g(t). \tag{5}$$

2. Some Identities Involving $H(n,r)$

In this section, we establish some identities involving $H(n,r)$ by using (3)-(5).

Cauchy numbers of the first kind a_n and Cauchy numbers of the second kind b_n play important roles in approximate integrals and difference-differential equations (see [9]). Some values of a_n and b_n are:

n	0	1	2	3	4	5	6	7	8	9
a_n	1	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{4}$	$-\frac{19}{30}$	$\frac{9}{4}$	$-\frac{863}{84}$	$\frac{1375}{24}$	$-\frac{33953}{90}$	$\frac{57281}{20}$
b_n	1	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{9}{4}$	$\frac{251}{30}$	$\frac{475}{12}$	$\frac{19087}{84}$	$\frac{36799}{24}$	$\frac{1070017}{90}$	$\frac{2082753}{20}$

In Section 4, we give the asymptotic expansion of the the sum involving $H(n,r)$ and b_n. In this section, we establish some identities related to $H(n,r)$ and a_n. In [9], there is an identity involving Cauchy numbers of the first kind a_n and harmonic numbers H_n, namely

$$1+\sum_{n=1}^{\infty}\frac{(-1)^n a_n H_n}{n!n} \quad = \quad \frac{\pi^2}{6}.$$

From the generating functions of $H(n,r)$ and Cauchy numbers of the first kind a_n, we have

Theorem 1 *Let $n \geq 1$ and $r \geq 1$. Then*

$$\sum_{j=0}^{n}\frac{(-1)^j a_j H(n-j+r+1,r)}{j!} \quad = \quad H(n+r,r-1), \tag{6}$$

$$\sum_{j=0}^{n}\frac{(-1)^j a_j H(n-j+r+1,r)}{j!(n-j+r+2)} \quad = \quad \frac{(r+1)H(n+r,r-1)}{(r+2)(n+r+1)}. \tag{7}$$

Proof. From the definitions of a_n and $H(n,r)$, we have

$$\begin{aligned}\sum_{j=0}^{n}\frac{(-1)^j a_j H(n-j+r+1,r)}{j!} &= \sum_{j=0}^{n}\left([z^j]\frac{-z}{\ln(1-z)}\right)[z^{n-j}]\left(\frac{(-1)^{r+1}\ln^{r+1}(1-z)}{z^{r+1}(1-z)}\right)\\ &= [z^n]\frac{(-1)^r\ln^r(1-z)}{z^r(1-z)}\\ &= H(n+r,r-1),\end{aligned}$$

$$\begin{aligned}\sum_{j=0}^{n}\frac{(-1)^j a_j H(n-j+r+1,r)}{j!(n-j+r+2)} &= \sum_{j=0}^{n}\left([z^j]\frac{-z}{\ln(1-z)}\right)[z^{n-j}]\left(\frac{(-1)^{r}\ln^{r+2}(1-z)}{(r+2)z^{r+2}}\right)\\ &= [z^n]\frac{(-1)^{r+1}\ln^{r+1}(1-z)}{(r+2)z^{r+1}}\\ &= \frac{(r+1)H(n+r,r-1)}{(r+2)(n+r+1)}.\end{aligned}$$

□

Identities (6)-(7) relate $H(n,r)$ and Cauchy numbers of the first kind.

It is well-known that Stirling numbers play an important role in combinatorial analysis, and associated Stirling numbers are significant in enumerative combinatorics (see [3]). We know that associated Stirling numbers of the first kind $s_2(n,k)$ are related to the number of a set, and the value of $|s_2(n,k)|$ is the number of derangements of a set $N(|N|=n)$ with k orbits. By the generating functions of $H(n,r)$ and the Stirling numbers of the first kind $s(n,r)$, we immediately get

$$H(n,r)=\frac{(r+1)!}{n!}(-1)^{n+r+1}s(n+1,r+2). \tag{8}$$

The associated Stirling numbers of the first kind $s_2(n,k)$ and harmonic numbers H_n satisfy [13]:

$$\sum_{j=0}^{n}\frac{(-1)^j H_{j+1}s_2(n-j+k,k)}{(j+2)(n-j+k)!}=\frac{(-1)^k}{2}\sum_{j=0}^{k}\frac{(-1)^j(j+1)(j+2)s(n+j+2,j+2)}{(k-j)!(n+j+2)!}.$$

For $s_2(n,k)$ and $H(n,r)$, we have the following result.

Theorem 2 *Let $k\geq 1$, $n\geq 1$ and $r\geq 0$. Then*

$$\sum_{j=0}^{n}\frac{(-1)^j s_2(j+k,k)H(n-j+r+1,r)}{(j+k)!}=\frac{(-1)^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}H(n+j+r+1,j+r).$$

Proof. From the generating functions of $s_2(n,k)$ and $H(n,r)$, we get

$$\begin{aligned}
&\sum_{j=0}^{n}\frac{(-1)^j s_2(j+k,k)H(n-j+r+1,r)}{(j+k)!}\\
&= \sum_{j=0}^{n}\left([z^j]\frac{[\ln(1-z)+z]^k}{(-1)^k k! z^k}\right)[z^{n-j}]\frac{(-1)^{r+1}\ln^{r+1}(1-z)}{z^{r+1}(1-z)}\\
&= [z^n]\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^{r+1}\ln^{j+r+1}(1-z)}{(-1)^k k! z^{j+r+1}(1-z)}\\
&= \frac{(-1)^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}[z^n]\frac{(-1)^{j+r+1}\ln^{j+r+1}(1-z)}{z^{j+r+1}(1-z)}\\
&= \frac{(-1)^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}H(n+j+r+1,j+r).
\end{aligned}$$

□

3. Inversion Formulas

For sequences $\{f_n\}$ and $\{g_n\}$, it is well-known that

$$f_n = \sum_{k=0}^{n} S(n,k) g_k \Longleftrightarrow g_n = \sum_{k=0}^{n} s(n,k) f_k.$$

Now we prove that

Theorem 3 *Let $\{f_n\}$ and $\{g_n\}$ be two sequences. Then*

$$\begin{aligned} & f_n = \sum_{k=0}^{n} H(n+1,k) g_k \\ \Longleftrightarrow\ & g_n = \frac{1}{(n+1)!} \sum_{k=0}^{n} (-1)^{n-k} (k+1)! S(n+2,k+2) f_k. \end{aligned} \tag{9}$$

Proof. Let

$$g(z) = \sum_{m=0}^{\infty} g_m z^m, \quad f(z) = \sum_{m=0}^{\infty} f_m z^m.$$

(i) When

$$f_n = \sum_{k=0}^{n} H(n+1,k) g_k,$$

we have

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} g_k z^k \sum_{m=k}^{\infty} H(m+1,k) z^{m-k} \\ &= \sum_{k=0}^{\infty} g_k \frac{(-1)^{k+1} \ln^{k+1}(1-z)}{z(1-z)} \\ &= \frac{-\ln(1-z)}{z(1-z)} g(-\ln(1-z)). \end{aligned}$$

Let $u = \ln(1-z)$. Then $z = 1 - e^u$ and

$$\begin{aligned} g(-u) &= -\frac{(1-e^u)e^u}{u} f(1-e^u) \\ &= -\frac{1}{u} \sum_{m=0}^{\infty} (-1)^{m+1} f_m (e^u - 1)^{m+2} - \frac{1}{u} \sum_{m=0}^{\infty} (-1)^{m+1} f_m (e^u - 1)^{m+1}. \end{aligned}$$

It follows from the definition of $S(n,k)$ that

$$\begin{aligned} g(-u) &= \sum_{m=0}^{\infty}(-1)^m(m+2)!f_m\sum_{p=0}^{\infty}S(p+m+2,m+2)\frac{u^{p+m+1}}{(p+m+2)!} \\ &\quad + \sum_{m=0}^{\infty}(-1)^m(m+1)!f_m\sum_{p=0}^{\infty}S(p+m+1,m+1)\frac{u^{p+m}}{(p+m+1)!}. \end{aligned}$$

Then

$$\begin{aligned} [u^n]g(-u) &= (-1)^n g_n \\ &= \sum_{j=0}^{n}\frac{S(n+1,j+1)(-1)^{j+1}(j+1)!f_j}{(n+1)!} \\ &\quad + \sum_{j=0}^{n-1}\frac{S(n+1,j+2)(-1)^{j+1}(j+2)!f_j}{(n+1)!}. \end{aligned}$$

On the other hand,

$$S(n+1,j+1)+(j+2)S(n+1,j+2)=S(n+2,j+2), \quad S(n,n)=1. \tag{10}$$

Then we have

$$g_n=\frac{1}{(n+1)!}\sum_{k=0}^{n}(-1)^{n-k}(k+1)!S(n+2,k+2)f_k.$$

(ii) When

$$g_n=\frac{1}{(n+1)!}\sum_{k=0}^{n}(-1)^{n-k}(k+1)!S(n+2,k+2)f_k,$$

we have

$$\begin{aligned} g(z) &= \sum_{k=0}^{\infty}g_k z^k \\ &= \sum_{j=0}^{\infty}(j+1)!f_j\sum_{k=j}^{\infty}\frac{(-1)^{k-j}}{(k+1)!}S(k+2,j+2)z^k. \end{aligned}$$

It follows from (10) that

$$\begin{aligned} g(z) &= \sum_{j=0}^{\infty}(j+1)!f_j\sum_{k=j}^{\infty}\frac{(-1)^{k-j}}{(k+1)!}[S(k+1,j+1)+(j+2)S(k+1,j+2)]z^k \\ &= \sum_{j=0}^{\infty}(j+1)!f_j\sum_{k=j+1}^{\infty}\frac{(-1)^{k-j-1}}{k!}S(k,j+1)z^{k+1} \\ &\quad +\sum_{j=0}^{\infty}(j+1)!f_j(j+2)\sum_{k=j+1}^{\infty}\frac{(-1)^{k-j}}{(k+1)!}S(k+1,j+2)z^k. \end{aligned}$$

Then

$$\begin{aligned} g(z) &= z\sum_{j=0}^{\infty}(-1)^{j+1}f_j(e^{-z}-1)^{j+1}+z\sum_{j=0}^{\infty}(-1)^{j+1}f_j(e^{-z}-1)^{j+2} \\ &= -z(e^{-z}-1)e^{-z}f(1-e^{-z}). \end{aligned}$$

Let $v=1-e^{-z}$. Then $z=-\ln(1-v)$,

$$\begin{aligned} f(v) &= -\frac{\ln(1-v)}{v(1-v)}g(-\ln(1-v)) \\ &= \sum_{m=0}^{\infty}g_m\sum_{j=m}^{\infty}H(j+1,m)v^j, \end{aligned}$$

$$\begin{aligned} [v^n]f(v) &= f_n \\ &= \sum_{k=0}^{n}H(n+1,k)g_k. \end{aligned}$$

Hence (9) holds. □

4. Asymptotic Expansion of Certain Sums Involving $H(n,r)$

Sometimes it is difficult to compute the accurate values of sums involving $H(n,r)$. However, we give the asymptotic values of certain sums related to $H(n,r)$. In this section, we give asymptotic expansions of certain sums involving $H(n,r)$ and Cauchy numbers of the second kind (binomial coefficients). At first, we recall a lemma.

Lemma ([6]) Let α be a real number and

$$L(z)=\ln\frac{1}{1-z}.$$

When $n \to \infty$,

$$[z^n](1-z)^{\alpha}L^k(z) \sim \frac{1}{\Gamma(-\alpha)} n^{-\alpha-1}\ln^k n, \quad (\alpha \notin \mathbb{Z}_{\geq}0), \tag{11}$$

$$[z^n](1-z)^m L^k(z) \sim (-1)^m k m! n^{-m-1}\ln^{k-1} n, \quad (m \in \mathbb{Z}_{\geq 0}, \quad k \in \mathbb{Z}_{\geq 1}). \tag{12}$$

Now we give the asymptotic expansions of certain sums involving $H(n,r)$ using above lemma.

Theorem 4 *Assume that r is fixed with $r \geq 1$. For $H(n,r)$ and Cauchy numbers of the second kind b_n, we have*

$$\sum_{j=0}^{n} \frac{b_j}{j!} H(n-j+r+1,r) \sim (n+r)\ln^r(n+r), \quad (n \to \infty). \tag{13}$$

Proof. We can verify that

$$\begin{aligned}\sum_{j=0}^{n} \frac{b_j H(n-j+r+1,r)}{j!} &= \sum_{j=0}^{n}\left([z^j]\frac{-z}{(1-z)\ln(1-z)}\right)[z^{n-j}]\frac{(-1)^{r+1}\ln^{r+1}(1-z)}{z^{r+1}(1-z)} \\ &= [z^n]\frac{(-1)^r \ln^r(1-z)}{z^r(1-z)^2}.\end{aligned}$$

Then

$$\sum_{j=0}^{n} \frac{b_j}{j!} H(n-j+r+1,r) = [z^{n+r}](1-z)^{-2}L^r(z).$$

It follows from (11) that

$$[z^{n+r}](1-z)^{-2}L^r(z) \sim \frac{n+r}{\Gamma(2)}\ln^r(n+r).$$

Since $\Gamma(2) = 1$, (13) holds. □

It is well-known that the Stirling numbers of the first kind $s(n,r)$ satisfy

$$s(n,r) = \sum_{0 \leq j \leq h \leq n-r} (-1)^{j+h}\binom{h}{j}\binom{n-1+h}{n-r+h}\binom{2n-r}{n-r-h}\frac{(h-j)^{n-r+h}}{h!}.$$

Due to (8), we can express $H(n,r)$ in terms of binomial coefficients:

$$\begin{aligned} H(n,r) &= \frac{(-1)^{n+r+1}(r+1)!}{n!} \sum_{0\le j\le h\le n-r-1} (-1)^{j+h}\binom{h}{j}\binom{n+h}{n-r-1+h} \\ &\quad \times \binom{2n-r}{n-r-1-h}\frac{(h-j)^{n-r-1+h}}{h!}. \end{aligned}$$

Now we give the asymptotic expansion of certain sums involving $H(n,r)$ and binomial coefficients.

Theorem 5 *Assume that k and r are fixed with $k \ge 1$ and $r \ge 1$. When $n \to \infty$,*

$$\sum_{j=0}^{n} \binom{2j}{j} \frac{H(n-j+r+1,r)}{4^j} \sim 2\sqrt{\frac{n+r+1}{\pi}}\ln^{r+1}(n+r+1), \tag{14}$$

$$\sum_{j=0}^{n} \binom{j+k}{k} H(n-j+r+1,r) \sim \frac{(n+r+1)^{k+1}\ln^{r+1}(n+r+1)}{(k+1)!}. \tag{15}$$

Proof. We note that

$$\sum_{n=0}^{\infty} \binom{2n}{n}\frac{z^n}{4^n} = \frac{1}{\sqrt{1-z}}, \quad |z|<1, \tag{16}$$

$$\sum_{n=0}^{\infty} \binom{n+k}{k} z^n = \frac{1}{(1-z)^{k+1}}, \quad |z|<1. \tag{17}$$

From (2), (16), and (17), we can prove that

$$\begin{aligned} \sum_{j=0}^{n} \binom{2j}{j}\frac{H(n-j+r+1,r)}{4^j} &= \sum_{j=0}^{n}\left([z^j]\frac{1}{\sqrt{1-z}}\right)[z^{n-j}]\frac{(-1)^{r+1}\ln^{r+1}(1-z)}{z^{r+1}(1-z)} \\ &= [z^n]\frac{L^{r+1}(z)}{z^{r+1}(1-z)^{3/2}} \\ &= [z^{n+r+1}](1-z)^{-3/2}L^{r+1}(z), \end{aligned}$$

and

$$\begin{aligned} \sum_{j=0}^{n}\binom{j+k}{k}H(n-j+r+1,r) &= [z^n]\frac{L^{r+1}(z)}{z^{r+1}(1-z)^{k+2}} \\ &= [z^{n+r+1}](1-z)^{-k-2}L^{r+1}(z). \end{aligned}$$

Due to (11),

$$\sum_{j=0}^{n} \binom{2j}{j} \frac{H(n-j+r+1,r)}{4^j} \sim \frac{(n+r+1)^{1/2}}{\Gamma(3/2)} \ln^{r+1}(n+r+1), \quad (n \to \infty)$$

$$\sum_{j=0}^{n} \binom{j+k}{k} H(n-j+r+1,r) \sim \frac{(n+r+1)^{k+1} \ln^{r+1}(n+r+1)}{\Gamma(k+2)}, \quad (n \to \infty).$$

Noting that

$$\Gamma(3/2) = \frac{\sqrt{\pi}}{2}, \quad \text{and} \quad \Gamma(k+2) = (k+1)!,$$

we show that (14)-(15) hold. □

In particular, for $k = r = 1$ and $n \to \infty$ in (15), we get

$$\begin{aligned} \sum_{j=0}^{n} \binom{j+k}{k} H(n-j+r+1,r) &\sim \frac{(n+2)^2}{2} \ln^2(n+2) \\ &\sim \frac{n^2+4n}{2} \ln^2 n. \end{aligned}$$

From (11)-(12) and the proof of Theorem 1, we obtain

$$\sum_{j=0}^{n} \frac{(-1)^j a_j H(n-j+r+1,r)}{j!} \sim \ln^r(n+r), \quad (n \to \infty),$$

$$\sum_{j=0}^{n} \frac{(-1)^j a_j H(n-j+r+1,r)}{j!(n-j+r+2)} \sim \frac{(r+1)\ln^r(n+r+1)}{(r+2)(n+r+1)}, \quad (n \to \infty),$$

where r is fixed.

Now we compare the asymptotic values of

$$\sum_{j=0}^{n} \frac{(-1)^j a_j H(n-j+r+1,r)}{j!} \quad \text{and} \quad \sum_{j=0}^{n} \frac{(-1)^j a_j H(n-j+r+1,r)}{j!(n-j+r+2)}$$

with their accurate ones, when $r = 1$ and $n \to \infty$. For $r = 1$,

$$\sum_{j=0}^{n} \frac{(-1)^j a_j H(n-j+r+1,r)}{j!} = H_{n+1},$$

$$\sum_{j=0}^{n} \frac{(-1)^j a_j H(n-j+r+1,r)}{j!(n-j+r+2)} = \frac{2H_{n+1}}{3(n+2)}.$$

It follows from Euler-Maclaurin's formula that

$$H_n = \ln n + \gamma + \frac{1}{2n} + \mathrm{O}\left(\frac{1}{n^2}\right), \quad n \geq 1,$$

where $\gamma = 0.57721\cdots$ is Euler's constant. Hence we have

$$\begin{aligned}\sum_{j=0}^{n} \frac{(-1)^j a_j H(n-j+2,1)}{j!} &= \ln(n+1) + \gamma + \frac{1}{2(n+1)} + \mathrm{O}\left(\frac{1}{n^2}\right),\\ \sum_{j=0}^{n} \frac{(-1)^j a_j H(n-j+r+1,r)}{j!(n-j+r+2)} &= \frac{2\ln(n+1)}{3(n+2)} + \frac{2\gamma}{3(n+2)} + \mathrm{O}\left(\frac{1}{n^2}\right),\end{aligned}$$

where $n \geq 1$.

It is evident that the harmonic numbers H_n satisfy that

$$H_n - H_{n-1} = \frac{1}{n}.$$

For $H(n,r)$, we derive an asymptotic recurrence relation:

Theorem 6 *Let r be fixed with $r \geq 1$. When $n \to \infty$,*

$$H(n,r) - H(n-1,r) \sim \frac{(r+1)\ln^r n}{n}.$$

Proof. It follows from (1) and (12) that

$$\begin{aligned}H(n,r) - H(n-1,r) &= [z^n]L^{r+1}(z)\\ &\sim \frac{(r+1)\ln^r n}{n}, \quad (n \to \infty).\end{aligned}$$

□

In the final result of this section, we give the asymptotic expansion of certain sums for inverses of binomial coefficients and $H(n,r)$ by Laplace's method.

Theorem 7 *Let $r \geq 1$. When $r \to \infty$,*

$$\sum_{n=r+1}^{\infty} \frac{(-1)^n H(n,r)}{(2n+1)\binom{2n}{n}} \sim (-1)^{r+1}\frac{2}{5}\sqrt{\frac{5\pi}{r+1}}\left(\ln\frac{5}{4}\right)^{r+3/2}, \tag{18}$$

$$\sum_{n=r+1}^{\infty} \frac{H(n,r)}{(2n+1)\binom{2n}{n}} \sim \frac{2}{3}\sqrt{\frac{3\pi}{r+1}}\left(\ln\frac{4}{3}\right)^{r+3/2}. \tag{19}$$

Proof. We know that the inverse of a binomial coefficient is related to an integral [12] as follows:

$$\binom{n}{m}^{-1} = (n+1)\int_0^1 z^m(1-z)^{n-m}dz. \tag{20}$$

Owing to (20),

$$\begin{aligned}\sum_{n=r+1}^{\infty}\frac{(-1)^n H(n,r)}{(2n+1)\binom{2n}{n}} &= \sum_{n=r+1}^{\infty} H(n,r)\int_0^1 (-z)^n(1-z)^n dz,\\ \sum_{n=r+1}^{\infty}\frac{H(n,r)}{(2n+1)\binom{2n}{n}} &= \sum_{n=r+1}^{\infty} H(n,r)\int_0^1 z^n(1-z)^n dz.\end{aligned}$$

For $z \in [0,1]$,

$$\begin{aligned}\sum_{n=r+1}^{\infty} H(n,r)\int_0^1 (-z)^n(1-z)^n dz &= \int_0^1 \left(\sum_{n=r+1}^{\infty} H(n,r)(-z)^n(1-z)^n\right)dz,\\ \sum_{n=r+1}^{\infty} H(n,r)\int_0^1 z^n(1-z)^n dz &= \int_0^1 \left(\sum_{n=r+1}^{\infty} H(n,r)z^n(1-z)^n\right)dz.\end{aligned}$$

It follows from (1) that

$$\begin{aligned}\sum_{n=r+1}^{\infty} H(n,r)\frac{(-1)^n}{(2n+1)\binom{2n}{n}} &= (-1)^{r+1}\int_0^1 \frac{\ln^{r+1}[1+z(1-z)]}{1+z(1-z)}dz,\\ \sum_{n=r+1}^{\infty}\frac{H(n,r)}{(2n+1)\binom{2n}{n}} &= \int_0^1 \frac{\{-\ln[1-z(1-z)]\}^{r+1}}{1-z(1-z)}dz.\end{aligned}$$

Put

$$g(z) = \begin{cases} e^{\ln\ln[1+z(1-z)]}, & z \in (0,1),\\ 0, & z = 0,\\ 0, & z = 1,\end{cases}$$

and

$$\phi(z) = \frac{1}{1+z(1-z)}, \quad z \in [0,1].$$

Then $g(z)$ reaches the maximum at $z = 1/2$, $g'(1/2) = 0$, and $g''(1/2) < 0$. By applying Laplace's method, we have

$$(-1)^{r+1}\int_0^1 \frac{\ln^{r+1}[1+z(1-z)]}{1+z(1-z)}dz$$
$$\sim \phi(1/2)\Big(g(1/2)\Big)^{r+3/2}\sqrt{\frac{-2\pi}{(r+1)g''(1/2)}} \qquad (r\to\infty).$$

Then (18) holds.

Using the same method, we obtain (19). □

Acknowledgement The authors would like to thank the anonymous referees for their helpful comments.

References

[1] V. S. Adamick, On Stirling numbers and Euler sums, J. Comput. Appl. Math. **79**(1997): 119-130.

[2] A. T. Benjamin, D. Gaebler, R. Gaebler, A combinatorial approach to hyperharmonic numbers, Integers, **3**(2003): A15.

[3] L. Comtet, Advanced Combinatorics, D. Reidel Publication Company, 1974.

[4] Gi-Sang Cheon, M. A. El-Mikkawy, Generalized harmonic numbers with Riordan arrays, J. Number Theory, **128**(2008): 413-425.

[5] W. Chu, Harmonic number identities and Hermite-Padé approximations to the logarithm function, J. Approx. Theory, **137**(2005): 42-56.

[6] P. Flajolet, E. Fusy, X. Gourdon, D. Panario, and N. Pouyanne, A hybrid of Darboux's method and singularity analysis in combinatorial asymptotics, The Electronic J. Combinatorics, **13**(2006).

[7] A. Gertsch, Generalized harmonic numbers, in: Number Theory, C. R. Acad. Sci. Paris Ser. I **324**(1997): 7-10.

[8] I. M. Gessel, On Miki's identity for Bernoulli numbers, J. Number Theory, **110**(2005): 75-82.

[9] D. Merlini, R. Sprugnoli, and M. C. Verri, The Cauchy numbers, Discrete Math. **306**(2006): 1906-1920.

[10] D. Merlini, R. Sprugnoli, and M. C. Verri, The method of coefficients, American Math. Month. **114**(2007): 40-57.

[11] J. M. Santmyer, A Stirling like sequence of rational numbers, Discrete Math. **171**(1997): 229-239.

[12] T. Tiberiu, Combinatorial sums and series involving inverses of binomial coefficients, Fibonacci Quart. **38.1**(2000): 79–84.

[13] Feng-Zhen Zhao, Some properties of associated Stirling numbers, J. Integer Sequences, **11**(2008): A08.1.7.