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Generalized harmonic numbers with Riordan arrays

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Abstract

By observing that the infinite triangle obtained from some generalized harmonic numbers follows a Riordan array, we obtain very simple connections between the Stirling numbers of both kinds and other generalized harmonic numbers. Further, we suggest that Riordan arrays associated with such generalized harmonic numbers allow us to find new generating functions of many combinatorial sums and many generalized harmonic number identities.

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1. Introduction

The *harmonic numbers* are defined by

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{for } n = 1, 2, \dots,$$

and it is well known that the generating function is $\frac{-\ln(1-x)}{1-x}$. The first few harmonic numbers are $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \dots$. These numbers have been generalized by several authors (see for example [2,4–7,10]), which reduce to the ordinary harmonic numbers when $r = 0$ or $r = 1$:

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- $H_0^{(r)} = 0$ and $H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r}$ for $n, r \geqslant 1$ [2,5], (1)

- $H_n^0 = \frac{1}{n}$ and $H_n^r = \sum_{k=1}^{n} H_k^{r-1}$ for $n, r \geqslant 1$ [4], (2)

- $H_{n,0} = 1$ and $H_{n,r} = \sum_{1 \leqslant n_1 < \cdots < n_r \leqslant n} \frac{1}{n_1 n_2 \cdots n_r}$ for $n, r \geqslant 1$ [7], (3)

- $H(n,r) = \sum_{1 \leqslant n_0 + n_1 + \cdots + n_r \leqslant n} \frac{1}{n_0 n_1 \cdots n_r}$ for $n \geqslant 1$, $r \geqslant 0$ [6,10]. (4)

Like the Pascal triangular array, some of these generalized harmonic numbers define infinite lower triangular arrays. If the array is characterized by two analytic functions, the first is invertible and the second has a compositional inverse, then it corresponds to a *Riordan array* of the *Riordan group* introduced by Shapiro et al. [11]. Riordan arrays constitute a practical device for solving combinatorial sums by means of generating functions. Further, many traditional applications of the Lagrange inversion formula can be approached from a Riordan array concept. Thus it is worthwhile to investigate that an infinite triangle obtained from the generalized harmonic numbers is Riordan array or not. Motivated by this concept, we are interested in triangular arrays of generalized harmonic numbers. In Section 2, we give a brief description of the concept of a Riordan array.

In this paper, by observing that the infinite triangle obtained from $H(n,r)$ given by (4) follows a Riordan array, we obtain very simple connections between $H_n^{(r)}$, H_n^r, $H_{n,r}$, $H(n,r)$ and the Stirling numbers of both kinds. Further, we suggest that Riordan arrays associated with such generalized harmonic numbers allow us to find new generating functions of many combinatorial sums and many generalized harmonic number identities. Finally, we observe the *harmonic polynomials* $H_n(z)$ with the generalized harmonic numbers $H(n,r)$ as the coefficients. As a result, we show that such polynomials may be expressed by means of the Bernoulli polynomials, and we generate a great deal of new rational sequences related to the harmonic numbers.

2. Riordan array and its structure

We begin this section by describing the concept of a Riordan array. A Riordan array is defined by a couple of analytic functions or formal power series $D = (g(x), f(x)) = [d_{n,k}]_{n,k \geqslant 0}$, $g(0) \neq 0$, such that the generic element of D is

$$d_{n,k} = [x^n] g(x) \big(x f(x)\big)^k, \tag{5}$$

where $[x^n] f(x)$ denotes the coefficient operator of x^n obtained from $f(x)$. From this definition, $D = (g(x), f(x))$ is an infinite, lower triangular array. If $f(0) \neq 0$, the Riordan array is called *proper*. A common example of a Riordan array is the Pascal triangle $\big[\binom{n}{k}\big]_{n,k \geqslant 0}$ for which we have $g(x) = f(x) = 1/(1-x)$. We denote by $\mathcal{R}$ the set of proper Riordan arrays. It is known [11] that $(\mathcal{R}, *)$ forms a group under matrix multiplication $*$ with the identity $I = (1, 1)$:

$$\big(g(x), f(x)\big) * \big(h(x), \ell(x)\big) = \big(g(x) h\big(x f(x)\big), f(x) \ell\big(x f(x)\big)\big). \tag{6}$$

Basically, the concept of a Riordan array is used in a constructive way to find the generating function of many combinatorial sums. For any sequence $\{h_k\}$ having $h(x)$ as its generating function, we have

$$\sum_{k=0}^{n} d_{n,k} h_k = \left[x^n\right] g(x) h\bigl(x f(x)\bigr). \tag{7}$$

For example, if $D = (\frac{-1}{1-x}, \frac{-1}{1-x})$ and $h(x) = \frac{-\ln(1-x)}{1-x}$ we obtain

$$\sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} H_k = \left[x^n\right] \left(\frac{-1}{1-x}\right) h\left(\frac{-x}{1-x}\right) = \left[x^n\right] \ln \frac{1}{1-x} = \frac{1}{n}, \quad n \geqslant 1.$$

Also, Riordan arrays have special structure. If $D = (g(x), f(x)) = [d_{n,k}]_{n,k\geqslant 0}$ is a proper Riordan array, then every element $d_{n+1,k+1}$ of D can be expressed as a linear combination of the elements in the preceding row starting from the preceding column, and every element in column 0 can be expressed as a linear combination of all the elements of the preceding row (see [8,9]):

(i) $d_{n+1,k+1} = \sum_{j=0}^{\infty} a_j d_{n,k+j}$, $k, n = 0, 1, \ldots,$
(ii) $d_{n+1,0} = \sum_{j=0}^{\infty} z_j d_{n,j}$, $n = 0, 1, \ldots.$

The coefficients $a_0, a_1, a_2, \ldots$ and $z_0, z_1, z_2, \ldots$ appearing in (i) and (ii) are called by the *A-sequence* and the *Z-sequence* of the Riordan array, respectively. If $A(x)$ and $Z(x)$ are the generating functions of the corresponding sequences then it can be proven (see [8,13]) that $f(x)$ and $g(x)$ are the solutions of the functional equations, respectively:

$$f(x) = A\bigl(x f(x)\bigr), \tag{8}$$

$$g(x) = g(0) / \bigl(1 - x Z\bigl(x f(x)\bigr)\bigr). \tag{9}$$

The relations can be inverted to formulas for the A-sequence and Z-sequence, respectively.

3. Connection with the Riordan array

Now, we consider the generating function of the generalized harmonic numbers $H(n,r)$ defined in (4). It is known that the numbers $H(n,r)$ can be calculated by the formula (see [6,10])

$$H(n,r) = \frac{(-1)^{r+1}}{n!} \left(\frac{d^n}{dx^n} \frac{[\ln(1-x)]^{r+1}}{1-x} \bigg|_{x=0} \right). \tag{10}$$

Some values of $H(n,r)$ are shown in Table 1.

From (10), the generating function of $H(n,r)$ can be expressed by

$$\frac{(-1)^{r+1} (\ln(1-x))^{r+1}}{1-x} = \left(\frac{-\ln(1-x)}{1-x} \right) \bigl(-\ln(1-x)\bigr)^r. \tag{11}$$

Since

Table 1
Generalized harmonic numbers $H(n,r)$

$n \setminus r$	0	1	2	3	4	5
1	1					
2	$\frac{3}{2}$	1				
3	$\frac{11}{6}$	2	1			
4	$\frac{25}{12}$	$\frac{35}{12}$	$\frac{5}{2}$	1		
5	$\frac{137}{60}$	$\frac{15}{4}$	$\frac{17}{4}$	3	1	
6	$\frac{49}{20}$	$\frac{203}{45}$	$\frac{49}{8}$	$\frac{35}{6}$	$\frac{7}{2}$	1

$$\hat{g}(x) := \frac{-\ln(1-x)}{1-x} = x\left(1 + \frac{3}{2}x + \frac{11}{6}x^2 + \frac{25}{12}x^3 + \cdots\right),$$

by setting $\hat{g}(x) = xg(x)$, we obtain the following proper Riordan array corresponding to the function (11):

$$\mathcal{H} := \big(g(x), f(x)\big) = \left(\frac{-\ln(1-x)}{x(1-x)}, \frac{-\ln(1-x)}{x}\right). \tag{12}$$

Hence the generic element $h_{n,k}$ of $\mathcal{H}$ is given by

$$\begin{aligned} h_{n,k} &= \big[x^n\big]g(x)\big(-\ln(1-x)\big)^k = \big[x^{n+1}\big]\left(\frac{-\ln(1-x)}{1-x}\right)\big(-\ln(1-x)\big)^k \\ &= H(n+1,k), \quad n,k \geqslant 0. \end{aligned} \tag{13}$$

We will call $\mathcal{H}$ the *generalized harmonic array*.

Theorem 3.1. *Let $\mathcal{H}$ be the generalized harmonic array. Then both A-sequence and Z-sequence of $\mathcal{H}$ may be expressed by means of the Bernoulli numbers B_n:*

$$\text{(i)} \quad A(x) = x + \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n,$$

$$\text{(ii)} \quad Z(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(B_{n+1} - 1)}{(n+1)!} x^n.$$

Proof. Let $\mathcal{H} = (g(x), f(x))$ be the same Riordan array as (12). Applying the functional equation (8) to $f(x)$, we have

$$\frac{-\ln(1-x)}{x} = A\big(-\ln(1-x)\big).$$

By setting $y = -\ln(1-x)$ or $x = (e^y - 1)/e^y$, we have

$$A(y) = \frac{ye^y}{e^y - 1} = y + \frac{y}{e^y - 1} = y + \sum_{n=0}^{\infty} \frac{B_n}{n!} y^n,$$

which proves (i). Similarly, applying the functional equation (9) to (12), we have

$$\frac{-\ln(1-x)}{x(1-x)} = \frac{1}{1 - xZ(-\ln(1-x))},$$

which leads us to

$$Z(y) = \frac{-y + e^{-y} - e^{-2y}}{y(e^{-y} - 1)} = \frac{1}{y}\left(\frac{ye^y}{e^y - 1} - e^{-y}\right) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(B_{n+1} - 1)}{(n+1)!} y^n.$$

Hence the proof is completed. □

It is interesting to observe that the A-sequence and Z-sequence of $\mathcal{H}$ are appearing in the Euler–Maclaurin summation formula (see 3.6.28 in [1]) given by

$$\sum_{k=1}^{n-1} f_k = \int_0^n f(k)\,dk - \frac{1}{2}\big[f(0) + f(n)\big] + \sum_{k=1}^{n-1} \frac{B_{2k}}{(2k)!}\big[f^{(2k-1)}(n) - f^{(2k-1)}(0)\big].$$

Theorem 3.2. *Let $H(n,r)$ be the generalized harmonic numbers given by* (4). *Then*

(i) $$\sum_{r=0}^{n-1} \frac{1}{(r+1)!} H(n,r) = n,$$

(ii) $$\sum_{r=0}^{n-1} \frac{(-1)^r}{(r+1)!} H(n,r) = 1,$$

(iii) $$\sum_{r=1}^{n} \frac{(-1)^{r+1}}{r!} H(n+1,r) = H_n,$$

(iv) $$\sum_{r=0}^{n-2} (-1)^{r+1} \frac{B_{r+1} - 1}{(r+1)!} H(n-1,r) = H_n.$$

Proof. Let $h(x) = \frac{e^x - 1}{x} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^n$. From (7) we have

$$\sum_{r=0}^{n-1} \frac{H(n,r)}{(r+1)!} = \big[x^{n-1}\big]\left(\frac{\ln(1-x)}{x(x-1)}\right)\left(\frac{x}{(x-1)\ln(1-x)}\right) = \big[x^{n-1}\big]\frac{1}{(x-1)^2} = n,$$

which proves (i). Similarly, if we take

$$h(x) = \frac{1-e^{-x}}{x} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(n+1)!}x^n,$$

and

$$h(x) = 1 - e^{-x} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(n+1)!}x^{n+1}$$

respectively, from (7) we can get (ii) and (iii).

The identity (iv) is an immediate consequence of Z-sequence of Theorem 3.1. □

Another interesting identities can be derived from (7) by suitable choice of a generating function $h(x)$. As noted in [13], the Riordan array concept is particularly important because identities involving Stirling numbers cannot be treated by methods related to hypergeometric functions, such as Gosper's algorithm or WZ-pairs.

We now turn to the *Stirling numbers* of the first kind $s(n,r)$ and of the second kind $S(n,r)$, which may be defined by

$$\text{(i)}\quad (x)_n = \sum_{r=0}^{n} s(n,r)x^r,$$

$$\text{(ii)}\quad x^n = \sum_{r=0}^{n} S(n,r)(x)_r$$

where $(x)_n = \prod_{r=1}^{n}(x-r+1)$, $(x)_0 = 1$, is the Pochhammer symbol. The $s(n,r)$ are not all positive, their sign is given by $|s(n,r)| = (-1)^{n-r}s(n,r)$. For convenience, we use $c(n,r)$ as the notation for $|s(n,r)|$.

By applying (6), the generalized harmonic array $\mathcal{H}$ can be factored by

$$\mathcal{H} = \left(\frac{-\ln(1-x)}{x(1-x)}, 1\right) * \left(1, \frac{-\ln(1-x)}{x}\right). \tag{14}$$

Noticing that $(1, \frac{-\ln(1-x)}{x})$ is the Riordan array associated to the unsigned Stirling numbers $\frac{k!}{n!}c(n,k)$, (14) suggests that the harmonic numbers may be expressed by means of the Stirling numbers of both kinds (also see, for example, [2,10]).

4. Connections between the generalized harmonic numbers

In this section, by using the concept of Riordan arrays we obtain very simple connections between different generalized harmonic numbers $H(n,r)$, $H_n^{(r)}$, H_n^r and $H_{n,r}$ given in Section 1.

First, we observe that the generalized harmonic numbers $H_{n,r}$ defined in (3) can be expressed by

$$H_{n,r}=\sigma_r^{(n)}\left(1,\frac{1}{2},\dots,\frac{1}{n}\right)=\frac{1}{n!}\sigma_{n-r}^{(n)}(1,2,\dots,n), \tag{15}$$

where $\sigma_r^{(n)}$ is the rth elementary symmetric function on $\{x_1,x_2,\dots,x_n\}$ given by

$$\sigma_r^{(n)}(x_1,x_2,\dots,x_n)=\sum_{1\leqslant s_1<s_2<\cdots<s_r\leqslant n} x_{s_1}x_{s_2}\cdots x_{s_r}.$$

Since $c(n,r)=\sigma_{n-r}^{(n-1)}(1,2,\dots,n-1)$, from (15) we obtain

$$H_{n,r}=\frac{c(n+1,r+1)}{n!}. \tag{16}$$

Now, let us define $\mathcal{H}_1=[H_{n,r}]_{n\geqslant r\geqslant 1}$ to be an infinite lower triangular array. It is easy to show that $\mathcal{H}_1$ does not constitute a Riordan array but

$$\bar{\mathcal{H}}_1=\left[\frac{(r+1)!}{(n+1)}H_{n,r}\right]_{n,r\geqslant 0}=\left(\left(\frac{-\ln(1-x)}{x}\right)^2,\frac{-\ln(1-x)}{x}\right)$$

is a Riordan array. Thus we obtain

$$H_{n,r}=\frac{n+1}{(r+1)!}\left[x^{n+1}\right]\left(-\ln(1-x)\right)^{r+1}. \tag{17}$$

Theorem 4.1. *Let $H_{n,r}$ be generalized harmonic numbers given by* (3). *Then*

$$H(n,r)=(r+1)!H_{n,r+1},\quad r\geqslant 0. \tag{18}$$

Proof. Since

$$\frac{d}{dx}\left(-\ln(1-x)\right)^{r+1}=\frac{(r+1)\left(-\ln(1-x)\right)^r}{1-x},$$

from (17) we have

$$\left[x^n\right]\frac{(-\ln(1-x))^r}{1-x}=r!H_{n,r}.$$

Hence we obtain

$$H(n,r)=\left[x^n\right]\frac{(-\ln(1-x))^{r+1}}{1-x}=(r+1)!H_{n,r+1},$$

which completes the proof. □

From (16) and (18), we obtain immediately

$$H(n,r)=\frac{(r+1)!}{n!}c(n+1,r+2), \tag{19}$$

which proves Theorem 3.5 in [10]. The formula (19) can be viewed as the general formula for the Stirling numbers of the first kind in terms of the generalized harmonic numbers $H(n,r)$.

In [2], Adamchik obtained the general formula for $c(n,r)$ in terms of the generalized harmonic numbers $H_n^{(r)}$ given by (1):

$$c(n,r) = \frac{(n-1)!}{(r-1)!} w(n,r-1), \tag{20}$$

where $w(n,r)$ is defined recursively by

$$w(n,r) = \sum_{k=0}^{r-1} [1-r]_k H_{n-1}^{(k+1)} w(n,r-1-k), \quad w(n,0) = 1, \tag{21}$$

and

$$[x]_n = \prod_{r=1}^{n} (x+r-1) = \left[\frac{t^n}{n!}\right] \frac{1}{(1-t)^x}, \quad [x]_0 = 1. \tag{22}$$

Comparing (19) with (20), we obtain immediately

$$H(n,r) = w(n+1,r+1).$$

Hence (21) can be expressed by

$$H(n,r) = \sum_{k=0}^{r} [-r]_k H_n^{(k+1)} H(n,r-k-1), \quad H(n,-1) = 1, \tag{23}$$

which means $H(n,r)$ can be connected with $H_n^{(r)}$.

Next, we consider the hyperharmonic numbers H_n^r in (2) defined by taking repeated partial sums of harmonic numbers. In [4], Benjamin et al. expressed the numbers H_n^r in terms of r-Stirling numbers leading to combinatorial interpretations of many interesting identities. We note that the hyperharmonic numbers H_n^r are defined only for $r \geqslant 1$. In Section 5, we will generalize the numbers in polynomials defined for any real number r.

Let us define $\mathcal{H}_2 = [H_{n-r+1}^r]_{n \geqslant r \geqslant 1}$ to be an infinite lower triangular array. It is easy to show that $\mathcal{H}_2$ is the Riordan array with

$$\mathcal{H}_2 = \left(\frac{-\ln(1-x)}{x(1-x)}, \frac{1}{1-x} \right). \tag{24}$$

Theorem 4.2. *Let H_n^r be hyperharmonic numbers given by* (2). *Then*

$$\text{(i)} \quad H(n,r) = \sum_{k=r}^{n-1} \frac{r!}{k!} s(k,r) H_{n-k}^{k+1},$$

$$\text{(ii)} \quad H_n^r = \sum_{k=r-1}^{n+r-2} \frac{r!}{k!} S(k,r-1) H(n+r-1,k).$$

Proof. Applying (6), we have

$$\left(\frac{-\ln(1-x)}{x(1-x)}, \frac{-\ln(1-x)}{x}\right) = \left(\frac{-\ln(1-x)}{x(1-x)}, \frac{1}{1-x}\right) * \left(1, \frac{\ln(1+x)}{x}\right). \tag{25}$$

Noticing that $(1, \frac{\ln(1+x)}{x})$ and its inverse $(1, \frac{e^x-1}{x})$ are the Riordan arrays associated to the Stirling numbers $\frac{k!}{n!}s(n,k)$ of the first kind and $\frac{k!}{n!}S(n,k)$ of the second respectively, the formulas (i) and (ii) are immediate consequences of the matrix multiplication from (24) and (25). □

Theorem 4.3. *Let H_n^r be hyperharmonic numbers given by* (2). *Then*

$$\sum_{r=1}^{n}(-1)^r r H_{n-r+1}^r = \frac{1}{n(n-1)} \quad (n \geqslant 2). \tag{26}$$

Proof. Let $\mathcal{H}_2$ be the generalized harmonic array given by (24) and let

$$h(x) := \frac{-1}{(x+1)^2} = \sum_{k=0}^{\infty}(-1)^{k+1}(k+1)x^k. \tag{27}$$

By setting $k+1=r$ in (27), from (7) we obtain

$$\sum_{r=1}^{n}(-1)^r r H_{n-r+1}^r = \left[x^n\right]\frac{(1-x)\ln(1-x)}{x} = \frac{1}{n(n-1)} \quad (n \geqslant 2),$$

which completes the proof. □

Note that the sequence $2, 6, 12, 20, 30, 42, \dots$ appearing in the denominator of (26) are called by *oblong numbers*, $n(n+1)$ (see A002378 in [12]).

5. Further generalization of the harmonic numbers

In this section, we observe the polynomials in z with the generalized harmonic number coefficients $H(n,r)$ given in (4). These polynomials generate the most generalized harmonic numbers presented in this paper.

Let us define the *harmonic polynomials* $H_n(z)$ of degree n in z by

$$\frac{-\ln(1-x)}{x(1-x)^{1-z}} = \sum_{n=0}^{\infty} H_n(z)x^n, \tag{28}$$

with the alternative representation

$$H_n(z) = \sum_{k=0}^{n} \frac{(-1)^k H(n+1,k)}{k!} z^k. \tag{29}$$

By setting $z=0$ and $z=1-r$ for $r\geqslant 1$, the harmonic polynomials $H_n(z)$ are deduced to ordinary harmonic numbers H_{n+1} and the hyperharmonic numbers H_n^r defined by (2), respectively. The numbers $H(n,r)$ and $H_{n,r}$ defined by (3) may be obtained from (18) by $H(n+1,r)=(-1)^r\frac{d^r}{dz^r}H_n(z)|_{z=0}$ and $H_{n+1,r+1}=\frac{(-1)^r}{(r+1)!}\frac{d^r}{dz^r}H_n(z)|_{z=0}$ for $n,k\geqslant 0$, respectively.

First we observe that the polynomials $H_n(z)$ are closely related to the *generalized Stirling polynomials of the first kind* $P_{n,k}(z)$ (see [3]) defined by

$$P_{n,k}(z)=\sum_{j=k+1}^{n}(-z)^{j-k-1}\binom{j-1}{k}c(n,j) \tag{30}$$

where $c(n,j)$ are the unsigned Stirling numbers of the first kind.

In [3], Adamchik expressed the multiple gamma function in terms of the derivatives of the Hurwitz zeta function together with the polynomials $P_{n,k}(z)$. Further, he obtained the following two alternative forms of representation:

$$P_{n,k}(z)=\frac{(-1)^k}{k!}\left(\frac{\partial^{n-1}}{\partial x^{n-1}}\frac{[\ln(1-x)]^k}{(1-x)^{1-z}}\bigg|_{x=0}\right) \tag{31}$$

and

$$P_{n,k}(z)=\sum_{j=k+1}^{n}(-1)^{n-j}\binom{z}{n-j}\frac{(n-1)!}{(j-1)!}c(j,k+1), \tag{32}$$

where $\binom{z}{n-j}=(z)_{n-j}/(n-j)!$.

One can establish that the polynomials $H_n(z)$ and $P_{n,k}(z)$ are related by $H_n(z)=\frac{1}{(n+1)!}P_{n+2,1}(z)$ from (28) and (31). Hence by setting $j-2=k$, it follows immediately from (30) that the polynomials $H_n(z)$ may be expressed by means of the unsigned Stirling numbers of the first kind

$$H_n(z)=\frac{1}{(n+1)!}\sum_{k=0}^{n}(k+1)c(n+2,k+2)(-z)^k.$$

Again, by setting $j-2=k$, it follows from (19) and (32) that the polynomials $H_n(z)$ may be expressed by means of the harmonic numbers

$$H_n(z)=\sum_{k=0}^{n}(-1)^{n-k}\binom{z}{n-k}H_{k+1}. \tag{33}$$

The first few harmonic polynomials are

$$H_0(z)=1,$$
$$H_1(z)=\frac{3}{2}-z,$$

$$H_2(z) = \frac{11}{6} - 2z + \frac{1}{2}z^2,$$
$$H_3(z) = \frac{25}{12} - \frac{35}{12}z + \frac{5}{4}z^2 - \frac{1}{6}z^3.$$

The basic properties for the harmonic polynomials are obtained from (28):

$$\text{(i)} \quad H_n(z+1) = H_n(z) - H_{n-1}(z),$$
$$\text{(ii)} \quad \int_0^1 H_n(z)\,dz = 1,$$
$$\text{(iii)} \quad \int_0^1 H_n(-z)\,dz = n+1.$$

Interestingly, these polynomials are closely related to the Bernoulli polynomials $B_n(z)$ defined by

$$\frac{xe^{zx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(z)\frac{x^n}{n!}.$$

Theorem 5.1. *The harmonic polynomials $H_n(z)$ may be expressed by means of the Bernoulli polynomials*

$$H_n(z) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k c(n+1,k+1)B_k(z). \tag{34}$$

Proof. Applying (7), we have

$$\begin{aligned}\sum_{k=0}^{n}\left(\frac{(-1)^k}{n!}c(n+1,k+1)k!\right)\frac{B_k(z)}{k!} &= \sum_{k=0}^{n}\left(\frac{1}{1-x}, \frac{\ln(1-x)}{x}\right)_{n,k}\left[x^k\right]\frac{xe^{zx}}{e^x-1} \\ &= \left[x^n\right]\left(\frac{1}{1-x}\right)\frac{\ln(1-x)e^{z\ln(1-x)}}{e^{\ln(1-x)}-1} \\ &= \left[x^n\right]\frac{-\ln(1-x)}{x(1-x)^{1-z}} = H_n(z),\end{aligned}$$

which completes the proof. □

Corollary 5.2. *The Bernoulli polynomials $B_n(z)$ may be expressed by means of the harmonic polynomials*

$$B_n(z) = \sum_{k=0}^{n}(-1)^k k!S(n+1,k+1)H_k(z). \tag{35}$$

By setting $z = 0$ in (35), we deduce immediately that the Bernoulli numbers $B_n := B_n(0)$ may be expressed by means of the harmonic numbers

$$B_n = \sum_{k=0}^{n} (-1)^k k! S(n+1, k+1) H_{k+1}.$$

Finally, we observe the specific values of the harmonic polynomials $H_n(z)$. As a result, we obtain a plenty of interesting rational sequences related to the harmonic numbers. Further, we give a combinatorial interpretation for those sequences.

Lemma 5.3. *For any real number m, we have*

$$H_n(m) = \sum_{k=0}^{n} H_k(m+1). \tag{36}$$

Proof. Applying the convolution for coefficient operator, we have

$$\begin{aligned} H_n(m) &= \left[x^n\right] \frac{-\ln(1-x)}{x(1-x)^{1-m}} = \sum_{k=0}^{n} \left[x^k\right] \frac{-\ln(1-x)}{x(1-x)^{-m}} \left[x^{n-k}\right] \frac{1}{1-x} \\ &= \sum_{k=0}^{n} \left[x^k\right] \frac{-\ln(1-x)}{x(1-x)^{-m}} = \sum_{k=0}^{n} H_k(m+1), \end{aligned}$$

which proves (36). □

The formula (36) suggests us to find a combinatorial interpretation for the sequence $\{H_n(m)\}_{n \geqslant 0}$. In fact, the kth term of the sequence $\{H_n(m)\}$ is obtained successively from first k-partial sum of the sequence $\{H_n(m+1)\}$. Consequently, each sequence $\{H_n(m)\}$ for any real number m may be expressed in terms of the harmonic numbers from (33). Of course, from (28) such sequences have the ordinary generating function

$$\mathcal{G}\left(\left\{H_n(m)\right\}\right) = \frac{-\ln(1-x)}{x(1-x)^{1-m}}.$$

Further, we have the following theorem.

Theorem 5.4. *For any real number m, the sequence $\{H_n(m)\}_{n \geqslant 0}$ has the closed form representation*

$$H_{n-1}(m) = \sum_{k=1}^{n} \binom{n-m-k}{n-k} \frac{1}{k}, \quad n \geqslant 1, \tag{37}$$

where $\binom{n-m-k}{n-k} = (n-m-k)_{n-k}/(n-k)!$.

Proof. Applying the convolution for coefficient operator, we have

$$H_n(m) = [x^n]\frac{-\ln(1-x)}{x(1-x)^{1-m}} = \sum_{k=0}^{n}[x^k]\frac{-\ln(1-x)}{x}[x^{n-k}]\frac{1}{(1-x)^{1-m}}$$
$$= \sum_{k=0}^{n}\binom{n-m-k}{n-k}\frac{1}{k+1},$$

which leads to (37). □

By setting $m = 1 - r$, $r \geqslant 1$, in (37), we deduce immediately Theorem 1 in [4].

6. Concluding remarks

A plenty of harmonic number identities including generalized harmonic numbers of several types have been obtained by several authors. This paper addresses new approach to the study of such harmonic numbers. It suggests that the concept of a Riordan array is used in a constructive way to find the generating function of many combinatorial sums associated with generalized harmonic numbers via (7), and also can be applied in the area of closed form summation of certain classes of infinite series.

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