

J. Korean Math. Soc. **44** (2007), No. 2, pp. 487–498

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Reprinted from the
Journal of the Korean Mathematical Society
Vol. 44, No. 2, March 2007

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GENERALIZED HARMONIC NUMBER IDENTITIES AND A RELATED MATRIX REPRESENTATION

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ABSTRACT. In this paper, we obtain important combinatorial identities of generalized harmonic numbers using symmetric polynomials. We also obtain the matrix representation for the generalized harmonic numbers whose inverse matrix can be computed recursively.

1. Introduction and preliminaries

The ordinary harmonic numbers are denoted by H_n and are defined as

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^n \frac{1}{k}, \quad n = 1, 2, \dots$$

The first few harmonic numbers are $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \dots$. These harmonic numbers were studied in antiquity and are important in various branches of number theory and combinatorial problems. They are closely related to the *Riemann zeta function* defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1},$$

where the product is over all primes p , and appear in various expressions for various special functions. It is well known that

$$(1) \quad H_n = \frac{|s(n+1, 2)|}{n!},$$

where $s(n, k)$ denotes the *Stirling numbers of the first kind* defined by

$$(x)_n := \prod_{r=1}^n (x - r + 1) = \sum_{k=0}^n s(n, k) x^k.$$

For a notational convenience, we denote $c(n, k) = |s(n, k)|$, i.e., $c(n, k)$ is the unsigned Stirling number of the first kind which counts the permutations of n

Received June 1, 2006.

2000 *Mathematics Subject Classification.* 05A30.

Key words and phrases. harmonic numbers, Riemann zeta function, Stirling numbers, Bernoulli numbers, symmetric polynomials.

elements that are the product of k disjoint cycles. Further, It is known [9] that

$$\zeta(k+1) = \sum_{n=k}^{\infty} \frac{c(n, k)}{n \cdot n!}.$$

In many recent works (see for example [1]-[5],[7]-[10]), the harmonic numbers have been generalized by several ways and the related identities were obtained. Our observations suggest that the generalized harmonic numbers can be viewed combinatorially.

For instance, the generalized harmonic numbers $H_n^{(r)}$ of order r are defined to be partial sums of the Riemann zeta function:

$$(2) \quad H_0^{(r)} = 0 \quad \text{and} \quad H_n^{(r)} = \sum_{k=1}^n \frac{1}{k^r}, \quad n, r \geq 1.$$

It is known (p.217 in [6]) that the numbers $H_n^{(r)}$ and $c(n, r)$ are connected by

$$\begin{aligned} c(n+1, 1) &= n!, \\ c(n+1, 2) &= n!H_n, \\ c(n+1, 3) &= \frac{n!}{2}(H_n^2 - H_n^{(2)}), \\ c(n+1, 4) &= \frac{n!}{6}(H_n^3 - 3H_nH_n^{(2)} + 2H_n^{(3)}), \end{aligned}$$

and so on. In [1], Adamchik obtained the general formula for $c(n, m)$ in terms of generalized harmonic numbers $H_n^{(r)}$:

$$(3) \quad c(n, m) = \frac{(n-1)!}{(m-1)!} w(n, m-1),$$

where the w -sequence is defined recursively by

$$(4) \quad w(n, 0) = 1, \quad w(n, m) = \sum_{k=0}^{m-1} (-1)^k (m-1)_k H_{n-1}^{(k+1)} w(n, m-1-k).$$

And he showed the w -sequence can be rewritten through a multiple sum:

$$w(n, m) = \sum_{i_1=1}^{n-1} \sum_{i_2=i_1+1}^{n-1} \cdots \sum_{i_m=i_{m-1}+1}^{n-1} \frac{m!}{i_1 i_2 \cdots i_m}.$$

In [5], Chu and Donno defined the generalized harmonic numbers $H_n(r)$ by

$$(5) \quad H_0(r) = 0 \quad \text{and} \quad H_n(r) = \sum_{k=1}^n \frac{1}{k+r}, \quad n \geq 1,$$

and they obtained several striking identities on the ordinary harmonic numbers.

For the rising factorial $[x]_n = x(x + 1) \cdots (x + n - 1)$ ($n \geq 1$), by writing $\frac{1}{[r]_{p+1}}$ as the sum of $p + 1$ partial fractions we obtain:

$$(6) \quad \frac{1}{[r]_{p+1}} = \sum_{k=0}^p \frac{A_k^{(p)}}{k+r} = \frac{1}{p!} \sum_{k=0}^p \binom{p}{k} \frac{(-1)^k}{k+r},$$

where $A_k^{(p)} = 1/\prod_{t=0, t \neq k}^p (t - k)$, $t \neq k$. It is easy to show that (6) can also be obtained by using the fact that

$$\Delta^n \left(\frac{1}{x}\right) = (E - 1)^n \left(\frac{1}{x}\right) = (-1)^n \frac{n!}{[x]_{n+1}},$$

where Δ and E are the forward and the shift operators with unit step, respectively. The rising factorial $[r]_{p+1}$ satisfies the following identity, typically proved by induction or telescoping sums [11]:

$$(7) \quad \sum_{r=1}^n \frac{1}{[r]_{p+1}} = \frac{\binom{n+p}{p} - 1}{p(p!) \binom{n+p}{p}}, \quad p \geq 1.$$

Using (6) and (7), we can establish an interesting identity for the generalized harmonic numbers $H_n(r)$:

$$\sum_{r=0}^p (-1)^r \binom{p}{r} H_n(r) = \frac{\binom{n+p}{p} - 1}{p \binom{n+p}{p}}.$$

Besides, in [2], Benjamin et. al. showed that the generalized harmonic numbers $H_n^{<r>}$ defined by

$$(8) \quad H_n^{<0>} = \frac{1}{n} \quad \text{and} \quad H_n^{<r>} = \sum_{k=1}^n H_k^{<r-1>}, \quad n, r \geq 1$$

can be expressed in terms of r -Stirling numbers. In 1997, Santmyer [10] defined the generalized harmonic numbers $H_{n,r}$ of rank r by

$$(9) \quad H_{n,r} = \sum_{k_0 + \dots + k_r = n} \frac{1}{k_0 k_1 \cdots k_r}, \quad n \geq 1, r \geq 0.$$

Note that these generalized harmonic numbers $H_n^{(r)}$, $H_n(r)$, $H_n^{<r>}$ and $H_{n,r}$ reduce to the ordinary harmonic numbers H_n when $r = 1$ or $r = 0$.

In this paper, we define other generalized harmonic numbers

$$H(n, r) := \frac{c(n + 1, r + 1)}{n!}$$

which are direct generalization of (1). The purpose of this paper is to obtain some interesting identities involving $H(n, r)$. These results are derived from symmetric polynomials. Further, we give a matrix representation for $H(n, r)$.

2. Symmetric polynomials

In this section, we are going to consider the so-called *symmetric polynomials* which are not only important tool in mathematics but also play considerable roles in computer science, physics and statistics.

A polynomial $P(x_1, x_2, \dots, x_n)$ in the variables x_1, x_2, \dots, x_n is called a symmetric polynomial or symmetric function if it is invariant under all possible permutations of the variables x_1, x_2, \dots, x_n . Especially, important symmetric polynomials that will be considered in the current paper are the *elementary symmetric polynomial* $\sigma_k^{(n)}$, the *complete symmetric polynomial* $\tau_k^{(n)}$, and the *power sum symmetric polynomial* $S_k^{(n)}$ on the variables x_1, x_2, \dots, x_n . These polynomials for integers n, k with $n \geq k \geq 0$ are defined by:

$$(10) \quad \bullet \sigma_k^{(n)}(x_1, x_2, \dots, x_n) = \sum_{1 \leq r_1 < \dots < r_k \leq n} x_{r_1} x_{r_2} \cdots x_{r_k},$$

$$(11) \quad \bullet \tau_k^{(n)}(x_1, x_2, \dots, x_n) = \sum_{1 \leq r_1 \leq \dots \leq r_k \leq n} x_{r_1} x_{r_2} \cdots x_{r_k},$$

$$(12) \quad \bullet S_k^{(n)}(x_1, x_2, \dots, x_n) = \sum_{r=1}^n x_r^k,$$

where $\sigma_0^{(n)}(x_1, x_2, \dots, x_n) = 1$ and $\tau_0^{(n)}(x_1, x_2, \dots, x_n) = 1$.

Lemma 1. (Newton-Girard identity [13]) *For positive integers m, n such that $1 \leq m \leq n$, the following holds:*

$$(13) \quad m \sigma_m^{(n)}(x_1, \dots, x_n) = \sum_{r=1}^m (-1)^{r+1} S_r^{(n)}(x_1, \dots, x_n) \sigma_{m-r}^{(n)}(x_1, \dots, x_n).$$

It is worthy to mention that two polynomials $\sigma_k^{(n)}$ and $\tau_k^{(n)}$ can be written through multiple sums as follow:

$$(14) \quad \sigma_k^{(n)}(x_1, x_2, \dots, x_n) = \sum_{r_k=k}^n \sum_{r_{k-1}=k-1}^{r_k-1} \cdots \sum_{r_1=1}^{r_2-1} x_{r_1} x_{r_2} \cdots x_{r_k},$$

$$(15) \quad \tau_k^{(n)}(x_1, x_2, \dots, x_n) = \sum_{r_1=1}^n \sum_{r_2=r_1}^n \cdots \sum_{r_k=r_{k-1}}^n x_{r_1} x_{r_2} \cdots x_{r_k}.$$

It is known that two generating functions $E(t)$ and $F(t)$ for $\sigma_k^{(n)}$ and $\tau_k^{(n)}$ are given respectively by

$$(16) \quad E(t) = \prod_{i=1}^n (1 + x_i t) = \sum_{r=0}^n \sigma_r^{(n)}(x_1, x_2, \dots, x_n) t^r,$$

$$(17) \quad F(t) = \prod_{i=1}^n \frac{1}{1 - x_i t} = \sum_{r=0}^{\infty} \tau_r^{(n)}(x_1, x_2, \dots, x_n) t^r.$$

These functions satisfy $E(t)F(-t) = F(t)E(-t) = 1$ and

$$(18) \quad \sum_{m=0}^r (-1)^m \sigma_m^{(r)} \tau_{r-m}^{(r)} = \delta_{r0}$$

for specific non-negative integers n and r such that $0 \leq r \leq n$, where δ_{nk} is the Kronecker symbol. The unsigned Stirling numbers of the first kind $c(n, k)$ and the Stirling numbers of the second kind $S(n, k)$ defined by $x^n = \sum_{k=0}^n S(n, k)(x)_k$ are related to $\sigma_k^{(n)}$ and $\tau_k^{(n)}$ by

$$(19) \quad c(n, k) = \sigma_{n-k}^{(n-1)}(1, 2, \dots, n-1),$$

$$(20) \quad S(n, k) = \tau_{n-k}^{(k)}(1, 2, \dots, k).$$

Before we consider more generalized harmonic number identities it may be useful to give the following result.

Lemma 2. *For all nonzero real numbers x_1, x_2, \dots, x_n and for all nonnegative integer i , we have*

$$(21) \quad \sigma_i^{(n)}\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}\right) = \frac{1}{x_1 x_2 \dots x_n} \sigma_{n-i}^{(n)}(x_1, x_2, \dots, x_n).$$

Proof. Replacing each x_i by $\frac{1}{x_i}$ in (16) gives:

$$\sum_{r=0}^n \sigma_r^{(n)}\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}\right) t^r = \frac{1}{x_1 x_2 \dots x_n} \prod_{i=1}^n (t + x_i).$$

Hence

$$\sum_{i=0}^n \sigma_i^{(n)}\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}\right) t^i = \frac{1}{x_1 x_2 \dots x_n} \sum_{i=0}^n \sigma_{n-i}^{(n)}(x_1, x_2, \dots, x_n) t^i.$$

The result follows. □

As a direct consequence of Lemma 2, we see that for specific positive integer n , the special case $x_i = i$ for each $i = 1, 2, \dots, n$ yields:

$$(22) \quad \sigma_i^{(n)}\left(1, \frac{1}{2}, \dots, \frac{1}{n}\right) = \frac{1}{n!} \sigma_{n-i}^{(n)}(1, 2, \dots, n).$$

Lemma 3. *For integers n, m with $n \geq m \geq 2$, we have*

$$(23) \quad c(n, m) = (n-1)! \sigma_{m-1}^{(n-1)}\left(1, \frac{1}{2}, \dots, \frac{1}{n-1}\right).$$

Proof. With the help of (22) and (19) we obtain

$$\frac{c(n, m)}{(n-1)!} = \frac{1}{(n-1)!} \sigma_{n-m}^{(n-1)}(1, 2, \dots, n-1) = \sigma_{m-1}^{(n-1)}\left(1, \frac{1}{2}, \dots, \frac{1}{n-1}\right),$$

which proves (23). □

From Lemma 3 and (3), we immediately obtain a simple representation for the w -sequence:

$$(24) \quad w(n, m) = m! \sigma_m^{(n-1)} \left(1, \frac{1}{2}, \dots, \frac{1}{n-1} \right).$$

Further, using the recurrence relation (25) enables us to compute the numbers $w(n, w)$ recursively as follows:

$$w(i, j) = w(i-1, j) + \frac{j}{i-1} w(i-1, j-1), \quad i \geq 2, j \geq 1$$

so that $w(i, i) = 0$ and $w(i, 0) = 1$ for $i \geq 0$, and $w(i, 1) = H_{i-1}$ for $i \geq 1$.

3. Generalized harmonic numbers $H(n, r)$

We begin with the recurrence relation for the unsigned Stirling numbers of the first kind $c(n, r)$ in [6]:

$$(25) \quad c(n, r) = c(n-1, r-1) + (n-1)c(n-1, r), \quad n, r \geq 1$$

with $c(n, 0) = \delta_{n0}$, $c(n, n) = 1$, $c(n, 1) = (n-1)!$ for $n \geq 1$. The relation (25) also gives $c(n+1, 2) = c(n, 1) + nc(n, 2)$. Dividing both sides by $n!$ taking into account the fact that $c(n, 1) = (n-1)!$, we obtain

$$(26) \quad g(n) = \frac{1}{n} + g(n-1), \quad n \geq 1,$$

where $g(n) = c(n+1, 2)/n!$. Since $H_n = \frac{1}{n} + H_{n-1}$, $n \geq 1$, from (26) we obtain $g(n) = H_n$ which proves (1).

More generally, we may rewrite the recurrence relation (25) in the form:

$$(27) \quad \frac{c(n+1, r+1)}{n!} = \frac{1}{n} \frac{c(n, r)}{(n-1)!} + \frac{c(n, r+1)}{(n-1)!}.$$

Let us define the *generalized harmonic numbers* $H(n, r)$ by

$$(28) \quad H(n, 0) = 1 \quad \text{and} \quad H(n, r) = \frac{c(n+1, r+1)}{n!} \quad n, r \geq 1.$$

First note that from (27) the numbers $H(n, r)$ satisfy the recurrence relation:

$$(29) \quad H(n, r) = H(n-1, r) + \frac{1}{n} H(n-1, r-1).$$

Theorem 4. *The generalized harmonic numbers $H(n, r)$ satisfy*

$$(30) \quad H(n, r) = \sum_{1 \leq k_1 < \dots < k_r \leq n} \frac{1}{k_1 k_2 \dots k_r}.$$

Proof. From Lemma 3, we have

$$H(n, r) = \sigma_r^{(n)} \left(1, \frac{1}{2}, \dots, \frac{1}{n} \right).$$

Hence using (10) yields

$$H(n, r) = \sum_{1 \leq k_1 < \dots < k_r \leq n} \frac{1}{k_1 k_2 \dots k_r}.$$

This completes the proof. □

The formula (30) can be used to find a formula for coefficients of powers of m in the Stirling numbers $S(m + n, m)$ of the second kind (see [8]).

At this stage it is convenient to obtain a generating function for the generalized harmonic numbers $H(n, r)$. From (16), we obtain

$$(31) \quad \sum_{r=0}^n \sigma_{n-r}^{(n)}(x_1, x_2, \dots, x_n) t^r = \prod_{i=1}^n (t + x_i).$$

Setting $x_m = m$ for each $m = 1, 2, \dots, n$ in (31) gives

$$\sum_{r=0}^n \sigma_{n-r}^{(n)}(1, 2, \dots, n) t^r = \prod_{i=1}^n (t + i).$$

Thus from (19) we have

$$(32) \quad \sum_{r=0}^n c(n + 1, r + 1) t^r = \prod_{i=1}^n (t + i).$$

Using (28) together with $(x)_n = n! \binom{x}{n}$ and (32) proves that the generating function we are looking for is given by

$$(33) \quad \sum_{r=0}^n H(n, r) t^r = \binom{n + t}{n}.$$

Since $H(n, r) = \frac{1}{r!} w(n + 1, r)$, the generating function for $w(n, k)$ in (3) can be easily obtained.

The *Bernoulli polynomials* $B_n(x)$ defined by

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_{n-k} x^k$$

are important in obtaining closed form expressions for sum of powers of integers [11] such as

$$(34) \quad \sum_{r=0}^n r^k = \frac{1}{k + 1} (B_{k+1}(n + 1) - B_{k+1}), \quad k \geq 0,$$

where $B_n = B_n(0)$ is the n -th Bernoulli number.

Theorem 5. *The generalized harmonic numbers $H(n, r)$ satisfy the following identities:*

- (i) $\sum_{r=1}^n H(n, r) = n,$
- (ii) $\sum_{r=1}^n (-1)^{r+1} H(n, r) = 1,$

- (iii) $\sum_{j=r}^n \frac{1}{j} H(j-1, r-1) = H(n, r),$
- (iv) $\sum_{r=0}^n (-1)^r H(n, r) (B_{r+1}(n+1) - B_{r+1}) \frac{1}{r+1} = 1,$
- (v) $\sum_{r=0}^n H(n, r) (B_{r+1}(n+1) - B_{r+1}) \frac{1}{r+1} = \binom{2n+1}{n},$
- (vi) $\sum_{r=1}^n r H(n, r) = (n+1)(H_{n+1} - 1),$
- (vii) $\sum_{r=0}^n (2^r - 1) H(n, r) = \binom{n+1}{2}.$

Proof. Putting $t = 1$ in (33) and taking into account the fact that $H(n, 0) = 1$ gives (i). Similarly putting $t = -1$ in (33) yields the identity (ii). To prove (iii), since

$$H(n, r) = \sigma_r^{(n)}\left(1, \frac{1}{2}, \dots, \frac{1}{n}\right) = \sum_{1 \leq k_1 < \dots < k_r \leq n} \frac{1}{k_1 k_2 \dots k_r},$$

applying (14) gives

$$\begin{aligned} H(n, r) &= \sum_{k_r=r}^n \sum_{k_{r-1}=r-1}^{k_r-1} \dots \sum_{k_1=1}^{k_2-1} \frac{1}{k_1 k_2 \dots k_r} \\ &= \frac{1}{r} \sum_{k_{r-1}=r-1}^{r-1} \sum_{k_{r-2}=r-2}^{k_{r-1}-1} \dots \sum_{k_1=1}^{k_2-1} \frac{1}{k_1 k_2 \dots k_{r-1}} \\ &\quad + \frac{1}{r+1} \sum_{k_{r-1}=r-1}^r \sum_{k_{r-2}=r-2}^{k_{r-1}-1} \dots \sum_{k_1=1}^{k_2-1} \frac{1}{k_1 k_2 \dots k_{r-1}} + \dots \\ &\quad + \frac{1}{n} \sum_{k_{r-1}=r-1}^{n-1} \sum_{k_{r-2}=r-2}^{k_{r-1}-1} \dots \sum_{k_1=1}^{k_2-1} \frac{1}{k_1 k_2 \dots k_{r-1}} \\ &= \frac{1}{r} H(r-1, r-1) + \frac{1}{r+1} H(r, r-1) + \dots \\ &\quad + \frac{1}{n} H(n-1, r-1) \\ &= \sum_{j=r}^n \frac{1}{j} H(j-1, r-1), \end{aligned}$$

as required. To prove (iv), putting $m = n$ and $x_k = k$ for each $k = 1, 2, \dots, n$ in (13) gives

$$n \sigma_n^{(n)}(1, 2, \dots, n) = \sum_{r=1}^n (-1)^{r+1} S_r^{(n)}(1, 2, \dots, n) \sigma_{n-r}^{(n)}(1, 2, \dots, n).$$

Hence we get

$$\sum_{r=1}^n (-1)^{r+1} S_r^{(n)}(1, 2, \dots, n) \sigma_{n-r}^{(n)}(1, 2, \dots, n) \frac{1}{n!} = n.$$

Consequently, by using (34) and (13) we obtain

$$(35) \quad \sum_{r=1}^n (-1)^{r+1} H(n, r) (B_{r+1}(n+1) - B_{r+1}) \frac{1}{r+1} = n.$$

The identity (35) can also be written in the form:

$$\sum_{r=0}^n (-1)^r H(n, r) (B_{r+1}(n+1) - B_{r+1}) \frac{1}{r+1} = 1.$$

To prove (v), from (33) we have

$$\sum_{r=0}^n H(n, r) t^r = \binom{n+t}{n} = \frac{(n+t)_n}{n!}.$$

Hence

$$(36) \quad \sum_{t=0}^n \sum_{r=0}^n H(n, r) t^r = \sum_{r=0}^n H(n, r) \left(\sum_{t=0}^n t^r \right) = \sum_{t=0}^n \binom{n+t}{n} = \binom{2n+1}{n}.$$

Using (34) then (36) yields

$$\sum_{r=0}^n H(n, r) (B_{r+1}(n+1) - B_{r+1}) \frac{1}{r+1} = \binom{2n+1}{n},$$

which is the required result. To prove (vi), using (33) we obtain

$$(37) \quad \sum_{r=0}^n H(n, r) t^r = \frac{(t+n)_n}{n!} = \frac{1}{n!} \prod_{r=1}^n (t+r).$$

Differentiating both sides of (37) with respect to t gives

$$(38) \quad \sum_{r=1}^n r H(n, r) t^{r-1} = \binom{n+t}{n} \sum_{r=1}^n \frac{1}{t+r} = \binom{n+t}{n} (H_{n+t} - H_t).$$

Putting $t = 1$ in (38) yields

$$\sum_{r=1}^n r H(n, r) = (n+1)(H_{n+1} - 1),$$

as required. To prove (vii), by using the forward operator Δ with unit step, then (37) yields

$$\Delta \left(\sum_{r=0}^n H(n, r) t^r \right) = \Delta \left(\frac{(t+n)_n}{n!} \right).$$

Hence

$$\sum_{r=0}^n H(n, r) \Delta(t^r) = \frac{1}{n!} \Delta((t+n)_n).$$

Consequently, we have

$$(39) \quad \sum_{r=0}^n H(n, r)((t + 1)^r - t^r) = \frac{n}{n!}(t + n)_{n-1} = \frac{(t + n)_{n-1}}{(n - 1)!} = \binom{n + t}{n - 1}.$$

Setting $t = 1$ in (39), we obtain

$$\sum_{r=0}^n (2^r - 1)H(n, r) = \binom{n + 1}{n - 1} = \binom{n + 1}{2}.$$

This completes the proof of the theorem. □

We list more new identities for the generalized harmonic numbers $H(n, r)$ without proofs.

- (i) $H(n, r) = \frac{1}{n} \sum_{k=r}^n \binom{k}{r} H(n - 1, k - 1),$
- (ii) $\sum_{r=k}^n H(n, r)S(r + 1, k + 1) = \binom{n}{k} \frac{n+1}{(k+1)!},$
- (iii) $\sum_{k=0}^n (-1)^{k+1} H(n, k)B_{k+1} = \frac{n+1}{n+2},$
- (iv) $\sum_{k=1}^{n-r+1} \frac{H(n-k, r-1)}{k} = rH(n, r).$

4. Matrix representation

For the generalized harmonic numbers $H(n, k)$ defined by (28) we define the $n \times n$ matrix $H = [h_{ij}]_{1 \leq i, j \leq n}$ as follows:

$$h_{ij} = \begin{cases} H(i, j) & \text{if } i \geq j, \\ 0 & \text{if } i < j. \end{cases}$$

For example, the 5×5 matrix H is given by

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{3}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{11}{6} & 1 & \frac{1}{6} & 0 & 0 \\ \frac{25}{12} & \frac{35}{24} & \frac{5}{12} & \frac{1}{24} & 0 \\ \frac{137}{60} & \frac{15}{8} & \frac{17}{24} & \frac{1}{8} & \frac{1}{120} \end{bmatrix}.$$

Here the elements of the matrix H are computed recursively using (29). Using (18) we see that the inverse matrix $Q = [q_{ij}]_{1 \leq i, j \leq n}$ of H is given by

$$q_{ij} = \begin{cases} (-1)^{i+j} j! \tau_{i-j}^{(j+1)}(1, 2, \dots, j + 1) & \text{if } i \geq j, \\ 0 & \text{if } i < j. \end{cases}$$

Using (21) yields

$$q_{ij} = \begin{cases} (-1)^{i+j} j! S(i + 1, j + 1) & \text{if } i \geq j, \\ 0 & \text{if } i < j. \end{cases}$$

For example, we obtain the 5×5 matrix Q as follows:

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -3 & 2 & 0 & 0 & 0 \\ 7 & -12 & 6 & 0 & 0 \\ -15 & 50 & -60 & 24 & 0 \\ 31 & -180 & 390 & -360 & 120 \end{bmatrix}.$$

It is worthy to mention that the elements of the inverse matrix Q can also be computed recursively using the recurrence relation

$$q_{ij} = jq_{i-1,j-1} - (j+1)q_{i-1,j}, \quad i = 3, 4, \dots, n; j = 2, 3, \dots, i-1,$$

with

$$q_{i1} = (-1)^{i+1}(2^i - 1) \quad \text{for } i \geq 1; \quad q_{ii} = i! \quad \text{for } i \geq 1.$$

Due to the identity (i) of Theorem 5, the product DH is a stochastic matrix, where $D = \text{diag}(1, \frac{1}{2}, \dots, \frac{1}{n})$.

We conclude this paper by describing that the matrix representation for the generalized harmonic numbers may be used to get more identities and some combinatorial connections with other combinatorial numbers.

Acknowledgment. The authors would like to thank the referee for valuable comments and suggestions.

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