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Harmonic Number Identities Via Euler's Transform

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Abstract

We evaluate several binomial transforms by using Euler's transform for power series. In this way we obtain various binomial identities involving power sums with harmonic numbers.

1 Introduction and prerequisites

Given a sequence $\{a_k\}$, its *binomial transform* $\{b_k\}$ is the sequence defined by

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k, \text{ with inversion } a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k,$$

or, in the symmetric version

$$b_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1} a_k \text{ with inversion } a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1} b_k$$

(see [7, 12, 14]). The binomial transform is related to the *Euler transform* of series defined in the following lemma. Euler's transform is used sometimes for improving the convergence of certain series [1, 8, 12, 13].

Lemma 1. *Given a function analytical on the unit disk*

$$f(t) = \sum_{n=0}^{\infty} a_n t^n, \tag{1}$$

then the following representation is true

$$\frac{1}{1-t}\, f\left(\frac{t}{1-t}\right) = \sum_{n=0}^{\infty} t^n \left(\sum_{k=0}^{n} \binom{n}{k} a_k\right). \tag{2}$$

(Proof can be found in the Appendix.)

If we have a convergent series

$$s = \sum_{n=0}^{\infty} a_n, \tag{3}$$

we can define the function

$$f(t) = \sum_{n=0}^{\infty} a_n t^n, \quad |t| < 1. \tag{4}$$

Then, with $t = \frac{1}{2}$ in (2) we obtain

$$s = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_k\right) \frac{1}{2^{n+1}}. \tag{5}$$

This formula is a classical version of Euler's series transformation. Sometimes the new series converges faster, sometimes not – see the examples in [10].

We shall use Euler's transform for the evaluation of several interesting binomial transformations, thus obtaining binomial identities of combinatorial and analytical character. Evaluating a binomial transform is reduced to finding the Taylor coefficients of the function on the left hand side of (2). In Section 2 we obtain several identities with harmonic numbers. In Section 3 we prove Dilcher's formula via Euler's transform.

This paper is close in spirit to the classical article [7] of Henry Gould.

Remark 2. *The representation (2) can be put in a more flexible equivalent form*

$$\frac{1}{1-\lambda t}\, f\left(\frac{\mu t}{1-\lambda t}\right) = \sum_{n=0}^{\infty} t^n \left(\sum_{k=0}^{n} \binom{n}{k} \mu^k \lambda^{n-k} a_k\right), \tag{6}$$

where λ, μ are appropriate parameters.

To show the equivalence of (2) and (6) we first write

$$f\left(\frac{\mu t}{\lambda}\right) = \sum_{n=0}^{\infty} a_n \left(\frac{\mu}{\lambda}\right)^n t^n, \tag{7}$$

and then apply (2) to the function $g(t) = f(\frac{\mu}{\lambda} t)$. This provides

$$\frac{1}{1-t}\, f\left(\frac{\mu}{\lambda}\frac{t}{1-t}\right) = \sum_{n=0}^{\infty} t^n \left(\sum_{k=0}^{n} \binom{n}{k} \left(\frac{\mu}{\lambda}\right)^k a_k\right). \tag{8}$$

Replacing here t by λt yields (6).

With $\lambda = 1$ and $\mu = -1$ we have

$$\frac{1}{t-1}\, f\left(\frac{t}{t-1}\right) = \sum_{n=0}^{\infty} t^n \left(\sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1} a_k\right), \tag{9}$$

corresponding to the symmetrical binomial transform.

Lemma 3. *Given a formal power series*

$$g(t) = \sum_{n=0}^{\infty} b_n t^n, \tag{10}$$

we have

$$\frac{g(t)}{1-t} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} b_k\right) t^n. \tag{11}$$

This is a well-known property. To prove it we just need to multiply both sides of (11) by $1-t$ and simplify the right hand side.

2 Identities with harmonic numbers

Proposition 4. *The following expansion holds in a neighborhood of zero*

$$\frac{\log(1-\alpha t)}{1-\beta t} = -\sum_{n=1}^{\infty} \left(\alpha \beta^{n-1} + \frac{1}{2}\alpha^2 \beta^{n-2} + \cdots + \frac{1}{n}\alpha^n\right) t^n \tag{12}$$

where α, β are appropriate parameters.

Proof. It is sufficient to prove (12) when $\beta = 1$ and then rescale the variable t, i.e. we only need

$$\frac{\log(1-\alpha t)}{1-t} = -\sum_{n=1}^{\infty} \left(\alpha + \frac{1}{2}\alpha^2 + \cdots + \frac{1}{n}\alpha^n\right) t^n. \tag{13}$$

This follows immediately from Lemma 3. □

Corollary 5. *With $\alpha = 1$ in (13) we obtain the generating function of the harmonic numbers*

$$-\frac{\log(1-t)}{1-t} = \sum_{n=0}^{\infty} H_n t^n, \qquad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}. \tag{14}$$

The next proposition is one of our main results

Proposition 6. *For every positive integer n and every two complex numbersλ, μ,*

$$\sum_{k=1}^{n} \binom{n}{k} H_k \lambda^{n-k} \mu^k = H_n (\lambda+\mu)^n - \left(\lambda(\lambda+\mu)^{n-1} + \frac{\lambda^2}{2}(\lambda+\mu)^{n-2} + \cdots + \frac{\lambda^n}{n}\right). \tag{15}$$

Proof. We apply (6) to the function

$$f(t) = -\frac{\log(1-t)}{1-t} = \sum_{n=0}^{\infty} H_n t^n. \tag{16}$$

On the left hand side we obtain

$$\frac{-1}{1-\lambda t}\frac{\log(1-\frac{\mu t}{1-\lambda t})}{1-\frac{\mu t}{1-\lambda t}} = -\frac{\log(1-(\lambda+\mu)t)}{1-(\lambda+\mu)t} + \frac{\log(1-\lambda t)}{1-(\lambda+\mu)t}, \tag{17}$$

which equals, according to Corollary 5 and Proposition 4,

$$\sum_{n=1}^{\infty} H_n(\lambda+\mu)^n t^n - \sum_{n=1}^{\infty}\left(\lambda(\lambda+\mu)^{n-1} + \frac{\lambda^2}{2}(\lambda+\mu)^{n-2} + \cdots + \frac{\lambda^n}{n}\right) t^n. \tag{18}$$

At the same time, by Euler's transform the right hand side is

$$\sum_{n=1}^{\infty} t^n \left(\sum_{k=1}^{n}\binom{n}{k} H_n \lambda^{n-k}\mu^k\right). \tag{19}$$

Comparing coefficients in (18) and (19) we obtain the desired result. □

Corollary 7. *Setting $\lambda = \mu = 1$ in (15) yields the well-known identity (see, for instance, [6, 14]):*

$$\sum_{k=1}^{n}\binom{n}{k} H_k = 2^n\left(H_n - \sum_{k=1}^{n}\frac{1}{k2^k}\right). \tag{20}$$

Corollary 8. *Setting $\lambda = 1$ in (15) reduces it to*

$$\sum_{k=1}^{n}\binom{n}{k} H_k\mu^k = H_n(1+\mu)^n - \left((1+\mu)^{n-1} + \frac{(1+\mu)^{n-2}}{2} + \cdots + \frac{1+\mu}{n-1} + \frac{1}{n}\right). \tag{21}$$

We shall use this last identity to obtain a representation for the combinatorial sum

$$\sum_{k=1}^{n}\binom{n}{k} H_k k^m \mu^k, \tag{22}$$

by applying the operator $(\mu\frac{d}{d\mu})^m$ to both sides in (21). First, however, we need the following lemma.

Lemma 9. *For every positive integer m define the quantities*

$$a(m,n,\mu) = \left(\mu\frac{d}{d\mu}\right)^m (1+\mu)^n = \sum_{k=0}^{n}\binom{n}{k} k^m \mu^k. \tag{23}$$

Then

$$a(m,n,\mu) = \sum_{k=0}^{n} \binom{n}{k} k! S(m,k) \mu^k (1+\mu)^{n-k}. \tag{24}$$

This is a known identity that can be found, for example, in [6].

From Lemma 9 we obtain another of our main results.

Proposition 10. *For every two positive integers m and n,*

$$\sum_{k=1}^{n} \binom{n}{k} H_k k^m \mu^k = a(m,n,\mu) H_n - \sum_{p=1}^{n-1} \frac{1}{n-p} a(m,p,\mu). \tag{25}$$

Proof. Apply $(\mu \frac{d}{d\mu})^m$ to both sides of (21) and note that $(\mu \frac{d}{d\mu})^m \mu^k = k^m \mu^k$. $\square$

The sums (22) were recently studied by M. Coffey [3] by using a different method (a recursive formula) and a representation was given in terms of the hypergeometric function

3 Stirling functions of a negative argument. Dilcher's formula

Some time ago Karl Dilcher obtained the nice identity

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k^m} = \sum \frac{1}{j_1 j_2 \cdots j_m}, \quad 1 \le j_1 \le j_2 \le \cdots \le j_m \le n, \tag{26}$$

as a corollary from a certain multiple series representation [4, Corollary 3]; see also a similar result in [5]. As this is one binomial transform, it is good to have a direct proof by Euler's transform method. Before giving such a proof, however, we want to point out one interesting interpretation of the sum on the left hand side in (26).

Let $S(m,n)$ be the Stirling numbers of the second kind [9]. Butzer et al. [2] defined an extension $S(\alpha,n)$ for any complex number $\alpha \ne 0$. The functions $S(\alpha,n)$ of the complex variable α are called Stirling functions of the second kind. The extension is given by the formula

$$S(\alpha,n) = \frac{1}{n!} \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} k^\alpha, \tag{27}$$

with $S(\alpha,0) = 0$. Thus, for $m,n \ge 1$,

$$(-1)^{n-1} n! S(-m,n) = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k^m}. \tag{28}$$

For the next proposition we shall need the polylogarithmic function [11]

$$\mathrm{Li}_m(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^m}. \tag{29}$$

Proposition 11. *For any integer $m \geq 1$ we have*

$$(-1)^{n-1} n! S(-m, n) = \sum \frac{1}{j_1 j_2 \cdots j_m}, \quad 1 \leq j_1 \leq j_2 \leq \cdots \leq j_m \leq n. \tag{30}$$

Proof. The proof is based on the representation

$$\mathrm{Li}_m\left(\frac{-t}{1-t}\right) = -\sum \frac{t^{j_m}}{j_1 j_2 \cdots j_m}, \quad 1 \leq j_1 \leq j_2 \leq \cdots \leq j_m, \tag{31}$$

(see [15]) from which, in view of Lemma 2,

$$\frac{-1}{1-t}\,\mathrm{Li}_m\left(\frac{-t}{1-t}\right) = \sum_{n=1}^{\infty} A_n t^n, \tag{32}$$

with coefficients

$$A_n = \sum \frac{1}{j_1 j_2 \cdots j_m}, \quad 1 \leq j_1 \leq j_2 \leq \cdots \leq j_m \leq n. \tag{33}$$

The assertion now follows from (9). □

In conclusion, many thanks to the referee for a correction and for some interesting comments.

4 Appendix

We prove Euler's transform representation (2) by using Cauchy's integral formula, both for the Taylor coefficients of a holomorphic function and for the function itself. Thus, given a holomorphic function f as in (1), we have

$$a_k = \frac{1}{2\pi i} \oint_L \frac{1}{\lambda^k} \frac{f(\lambda)}{\lambda} d\lambda, \tag{34}$$

for an appropriate closed curve L around the origin. Multiplying both sides by $\binom{n}{k}$ and summing for k we find

$$\sum_{k=0}^{n} \binom{n}{k} a_k = \frac{1}{2\pi i} \oint_L \left(\sum_{k=0}^{n} \binom{n}{k} \frac{1}{\lambda^k} \right) \frac{f(\lambda)}{\lambda} d\lambda = \frac{1}{2\pi i} \oint_L \left(1 + \frac{1}{\lambda}\right)^n \frac{f(\lambda)}{\lambda} d\lambda. \tag{35}$$

Multiplying this by t^n (with t small enough) and summing for n we arrive at the desired representation (2), because

$$\sum_{n=0}^{\infty} t^n \left(1 + \frac{1}{\lambda}\right)^n = \frac{1}{1 - t(1 + \frac{1}{\lambda})} = \frac{1}{1-t} \frac{\lambda}{\lambda - \frac{t}{1-t}}, \tag{36}$$

and therefore,

$$\sum_{n=0}^{\infty} t^n \left(\sum_{k=0}^{n} \binom{n}{k} a_k \right) = \frac{1}{1-t} \frac{1}{2\pi i} \oint_L \frac{f(\lambda)}{\lambda - \frac{t}{1-t}} d\lambda = \frac{1}{1-t} \, f\left(\frac{t}{1-t} \right). \tag{37}$$

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