

generating function of Laguerre polynomials*

pahio[†]

2013-03-22 3:16:14

We start from the definition of Laguerre polynomials via their Rodrigues formula

$$L_n(z) := e^z \frac{d^n}{dz^n} e^{-z} z^n \quad (n = 0, 1, 2, \dots). \quad (1)$$

The consequence

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \quad (2)$$

of Cauchy integral formula allows to write (1) as the complex integral

$$L_n(z) = \frac{n!}{2i\pi} \oint_C \frac{e^z e^{-\zeta}}{(\zeta - z)^{n+1}} d\zeta = \frac{n!}{2i\pi} \oint_C \frac{e^{z-\zeta} d\zeta}{(1 - \frac{z}{\zeta})^n (\zeta - z)},$$

where C is any closed contour around the point z and the direction is anticlockwise. The substitution

$$\zeta - z := \frac{zt}{1-t}, \quad \zeta = \frac{z}{1-t}, \quad t = 1 - \frac{z}{\zeta} \quad d\zeta = \frac{z dt}{(1-t)^2}$$

here yields

$$L_n(z) = \frac{n!}{2i\pi} \oint_{C'} \frac{e^{-\frac{zt}{1-t}} z dt}{(1-t)^2 t^n \cdot \frac{zt}{1-t}} = \frac{n!}{2i\pi} \oint_{C'} \frac{e^{-\frac{zt}{1-t}} dt}{(1-t)t^{n+1}}$$

where the contour C' goes round the origin. Accordingly, by (2) we can infer that

$$L_n(z) = \left[\frac{d^n}{dt^n} \frac{e^{-\frac{zt}{1-t}}}{1-t} \right]_{t=0},$$

* *GeneratingFunctionOfLaguerrePolynomials* created: *2013-03-2* by: *pahio* version: *42007* Privacy setting: *1* *Derivation* *33B99* *30B10* *26C05* *26A09* *33E30*

[†]This text is available under the Creative Commons Attribution/Share-Alike License 3.0. You can reuse this document or portions thereof only if you do so under terms that are compatible with the CC-BY-SA license.

whence we have found the generating function

$$\frac{e^{-\frac{zt}{1-t}}}{1-t} = \sum_{n=0}^{\infty} \frac{L_n(z)}{n!} t^n$$

of the Laguerre polynomials.