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GENERATING FUNCTIONS OF CONVOLUTION MATRICES

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1. INTRODUCTION

Hoggatt and Bergum [2] studied the general expression for the entry in the i^{th} row and the j^{th} column of a convolution matrix and obtained row generating functions for the convolution matrix of the sequence $\{1, u_2, u_3, u_4, \ldots\}$. In this paper, we extend the Strong Convolution Decomposition Theorem [3] to a more general case. Based on this extension, we decompose a convolution matrix into a product of a lower trianglular matrix and the upper triangular Pascal-like matrix. This interesting decomposition of a convolution matrix leads a novel approach to the subject proposed in [2]. Using this new method, we obtain a simple explicit formula for entries of a convolution matrix and row generating functions of the convolution matrix of the sequences $\{v_n\}$ and $\{u_n\}$. Moreover, the approach developed here can be easily extended to a rather broad category of integer matrices.

To review, the convolution of two sequences $\{a_n\}$ and $\{b_n\}$, $(n = 1, 2, 3, \ldots)$, is the sequence $\{c_n\}$ where $c_n = \sum_{k=1}^{n} a_k b_{n-k+1}$. The convolution matrix of two sequences $\{a_n\}$ and $\{b_n\}$ is the matrix whose first column is $\{a_n\}$ and whose i^{th} column $(i = 2, 3, \ldots)$ is the convolution sequence of the $(i-1)^{th}$ column with $\{b_n\}$. We say that the convolution matrix of the sequences $\{a_n\}$ and $\{a_n\}$ is the convolution matrix of the sequence $\{a_n\}$. There are many well-known integer matrices which can be written as convolution matrices of some sequences. The rectangular Pascal triangle matrix, for instance, is the convolution matrix of the sequence $\{1, 1, 1, 1, \ldots\}$ and the lower triangular Pascal matrix is the convolution matrix of the sequences $\{1, 1, 1, 1, \ldots\}$ and $\{0, 1, 1, 1, \ldots\}$. Furthermore, the convolution matrices of some particular sequences may have very interesting properties. For example, the convolution matrix of the sequence $\{1, 2, 3, 4, 5, 6, \ldots\}$ is

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$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 2 & 4 & 6 & 8 & 10 & 12 & \dots \\ 3 & 10 & 21 & 36 & 55 & 78 & \dots \\ 4 & 20 & 56 & 120 & 220 & 364 & \dots \\ 5 & 35 & 126 & 330 & 715 & 1365 & \dots \\ 6 & 56 & 252 & 792 & 2002 & 4368 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad (1)$$

It is not difficult to show: (1) The sum of the antidiagonal of the matrix M is the even term of the Fibonacci sequence $\{1, 3, 8, 21, 55, 144, \dots\}$. (2) Each row of the convolution matrix M is clearly the row of the triangle of coefficients of shifted Chebyshev's $S(n, x-2) = U(n, x/2-1)$ polynomials (exponents of x in decreasing order). (3) The determinant of an upper left corner $n \times n$ submatrix of M is $2^{(n-1)n/2}$. We left the proofs of results (1) and (2) as exercises for the interested reader.

Therefore, studying the properties of convolution matrices can be important for understanding the structure of a class of integer matrices.

2. GENERALIZED CONVOLUTION DECOMPOSITION THEOREM

To examine convolution matrices in general, it is convenient to represent a convolution matrix C of two sequences $\{v_n\}$ and $\{u_n\}$ in terms of multiplication of matrices as following:

$$C = [\bar{V}, U\bar{V}, U^2\bar{V}, \dots, U^n V, \dots], \qquad (2)$$

where

$$U = \begin{pmatrix} u_1 & 0 & 0 & 0 & 0 & 0 & \dots \\ u_2 & u_1 & 0 & 0 & 0 & 0 & \dots \\ u_3 & u_2 & u_1 & 0 & 0 & 0 & \dots \\ u_4 & u_3 & u_2 & u_1 & 0 & 0 & \dots \\ u_5 & u_4 & u_3 & u_2 & u_1 & 0 & \dots \\ u_6 & u_5 & u_4 & u_3 & u_2 & u_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad (3)$$

and $\bar{V}$ is a vector $[v_1, v_2, v_3, v_4, \dots]^T$.

In [3] we obtain the Strong Convolution Decomposition Theorem for a convolution matrix of a sequence $\{u_1, u_2, u_3, \dots\}$, where u_1 is a positive integer. We restate the theorem as following:

Strong Convolution Decomposition Theorem: Let $\{u_n\}$ be a sequence whose first term is a positive integer u_1, and let V be the convolution matrix of that sequence. Then $V = SP_U^{u_1}$ for some lower triangular matrix S and the upper triangular Pascal matrix P_U. Moreover, successive columns of S are successive convolutions of the sequence $\{u_n\}$ with $\{0, u_2, u_3, u_4, \dots\}$.

Here we propose to generalize this theorem to any convolution matrix of two sequences $\{v_n\}$ and $\{u_n\}$ using notations developed by Call and Velleman [1].

Let $P_U[x]$ be the infinite dimensional matrix defined by

$$(P_U[x])_{i,j} = \begin{cases} x^{j-i}\binom{j-1}{i-1}, & \text{if } j \geq i \\ 0, & \text{otherwise} \end{cases}, \tag{4}$$

where x is any real number. Then it is proved in [1] that

$$(P_U)^r = P_U[r] \quad \text{and} \quad (P_U^r)^{-1} = P_U[-r] = P_U^{-r}, \tag{5}$$

where $P_U = P_U[1]$, the standard upper triangular Pascal matrix, and r is any real number. If $r = 0$, then we define $P_U^0 = I$, where I is an identity matrix.

Using these facts, we can prove the following generalization of the Strong Convolution Decomposition Theorem in [3]:

Theorem 1(Generalized Convolution Decomposition Theorem): Let $\{v_n\}$ and $\{u_n\}$ be any two sequences of real numbers and let C be the infinite dimensional convolution matrix of $\{v_n\}$ and $\{u_n\}$. Then $C = SP_U^{u_1}$ for some lower triangular matrix S and the upper triangular Pascal matrix P_U. Moreover, successive columns of S are successive convolutions of the sequences $\{v_n\}$ and $\{0, u_2, u_3, u_4, \ldots\}$.

Proof: Using equations (2) and (5),

$$S = C \times P_U^{-u_1}$$

$$= [\bar{V}, U\bar{V}, U^2\bar{V}, \ldots, U^n\bar{V}, \ldots] \begin{pmatrix} \binom{0}{0} & \binom{1}{0}(-u_1)^1 & \binom{2}{0}(-u_1)^2 & \binom{3}{0}(-u_1)^3 & \binom{4}{0}(-u_1)^4 & \cdots \\ 0 & \binom{1}{1} & \binom{2}{1}(-u_1)^1 & \binom{3}{1}(-u_1)^2 & \binom{4}{1}(-u_1)^3 & \cdots \\ 0 & 0 & \binom{2}{2} & \binom{3}{2}(-u_1)^1 & \binom{4}{2}(-u_1)^2 & \cdots \\ 0 & 0 & 0 & \binom{3}{3} & \binom{4}{3}(-u_1)^1 & \cdots \\ 0 & 0 & 0 & 0 & \binom{4}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= [\bar{V}, (U - u_1 I)\bar{V}, (U - u_1 I)^2\bar{V}, \ldots, (U - u_1 I)^n\bar{V}, \ldots]$$
$$= [\bar{V}, B\bar{V}, B^2\bar{V}, \ldots, B^n\bar{V}, \ldots] \tag{6}$$

where $B = (U - u_1 I)$. Thus, $C = SP_U^{u_1}$ where $S = [\bar{V}, B\bar{V}, B^2\bar{V}, \ldots, B^n\bar{V}, \ldots]$ is the convolution matrix of the sequences $\{v_1, v_2, v_3, v_4, \ldots\}$ and $\{0, u_2, u_3, u_4, \ldots\}$. This completes the proof of the theorem.

When the first term v_1 of the sequence $\{v_1, v_2, v_3, v_4, \ldots\}$ is a positive integer and $\{u_n\} = \{v_n\}$, this theorem agrees with the Strong Convolution Decomposition Theorem in [3]. The following corollary is an immediate conclusion of Theorem 1.

Corollary 1: Let C_m be the upper left corner $m \times m$ submatrix of the convolution matrix C of the sequences $\{v_n\}$ and $\{u_n\}$. Then $|C_m| = v_1^m u_2^{m(m-1)/2}$.

Proof: By Theorem 1, $C = SP_U^{u_1}$. Now, $|P_U^{u_1}| = 1^{u_1} = 1$, so $|C| = |S|$. Since S is a lower triangular with diagonal elements $v_1, v_1u_2, v_1u_2^2, \ldots, |S_m| = v_1^m u_2^{1+2+\cdots+(m-1)}$, where S_m is the upper left corner $m \times m$ submatrix of S. Hence, $|C_m| = |S_m| = v_1^m u_2^{m(m-1)/2}$.

Remark: The determinant of any convolution matrix of the sequences $\{v_n\}$ and $\{u_n\}$ is wholly determined by the first term of the sequence $\{v_n\}$ and the second term of the sequence $\{u_n\}$. Corollary 1 is a generalization of the corollary in [3].

3. GENERAL TERM OF A CONVOLUTION MATRIX

In this section, we employ the Generalized Convolution Decomposition Theorem to find the general term of a convolution matrix C of sequence $\{u_n\}$. This approach can be easily extended to the convolution matrix of two sequences $\{v_n\}$ and $\{u_n\}$. Let S_n be the matrix consisting of the first n rows of the matrix S defined previously. It is easy to see that S_n is an $n \times \infty$ convolution matrix of the finite sequences $\{u_1, u_2, u_3, u_4, \ldots, u_n\}$ and $\{0, u_2, u_3, u_4, \ldots, u_n\}$. If $f(x) = \sum_{i=1}^n u_i x^{i-1}$ is the generating function of the finite sequence $\{u_1, u_2, u_3, u_4, \ldots, u_n\}$ and $g(x) = \sum_{i=2}^n u_i x^{i-1}$ is the generating function of the finite sequence $\{0, u_2, u_3, u_4, \ldots, u_n\}$ then the generating function for the l^{th} column of S_n is $f(x)g^{l-1}(x) = u_1 g^{l-1}(x) + g^l(x)$. Thus, the entry in the i^{th} row and the j^{th} column of S_n, denoted by $s_{i,j}$, is the coefficient of x^{i-1} in $f(x)g^{j-1}(x)$. By noting that the coefficient of x^k in $(u_2x + u_3x^2 + \cdots + u_nx^{n-1})^m$ is

$$\sum_{\substack{l_2+l_3+\cdots+l_n=m \\ l_i \geq 0;\ i=2,3,\ldots,n; \\ 1l_2+2l_3+\cdots+(n-1)l_n=k}} \frac{m!}{l_2!l_3!\ldots l_n!} u_2^{l_2} u_3^{l_3} \ldots u_n^{l_n}, \tag{7}$$

we have

$$s_{n,m} = u_1 \sum_{\substack{l_2+l_3+\cdots+l_n=m-1 \\ l_i \geq 0;\ i=2,3,\ldots,n; \\ 1l_2+2l_3+\cdots+(n-1)l_n=n-1}} \frac{(m-1)!}{l_2!l_3!\ldots l_n!} u_2^{l_2} u_3^{l_3} \ldots u_n^{l_n},$$

$$+ \sum_{\substack{l_2+l_3+\cdots+l_n=m \\ l_i \geq 0;\ i=2,3,\ldots,n; \\ 1l_2+2l_3+\cdots+(n-1)l_n=n-1}} \frac{m!}{l_2!l_3!\ldots l_n!} u_2^{l_2} u_3^{l_3} \ldots u_n^{l_n}. \tag{8}$$

Let $c_{n,m}$ be the entry in the n^{th} row and the m^{th} column of C. From the Generalized Convolution Decomposition Theorem, we know that

$$c_{n,m} = \sum_{k=1}^{\min\{n,m\}} s_{n,k} \binom{m-1}{k-1} u_1^{m-k}$$

$$= \sum_{k=1}^{\min\{n,m\}} \{u_1 \sum_{\substack{l_2+l_3+\cdots+l_n=k-1 \\ l_i \geq 0;\ i=2,3,\ldots,n; \\ 1l_2+2l_3+\cdots+(n-1)l_n=n-1}} \frac{(k-1)!}{l_2!l_3!\ldots l_n!} u_2^{l_2} u_3^{l_3} \ldots u_n^{l_n}$$

$$+ \sum_{\substack{l_2+l_3+\cdots+l_n=k \\ l_i \geq 0;\ i=2,3,\ldots,n; \\ 1l_2+2l_3+\cdots+(n-1)l_n=n-1}} \frac{k!}{l_2!l_3!\ldots l_n!}\} \binom{m-1}{k-1} u_1^{m-k}. \tag{9}$$

4. ROW GENERATING FUNCTIONS FOR A CONVOLUTION MATRIX

Now we are in a position to find the row generating functions of a convolution matrix. Let $F_m(x)$ be the generating function for the m^{th} row of a convolution matrix C of the sequences $\{v_n\}$ and $\{u_n\}$. It is easy to see that

$$F_m(x) = \sum_{k=1}^{\infty} c_{m,k} x^{k-1}. \tag{10}$$

Also, by Theorem 1, $F_m(x)$ equals

$$F_m(x) = [s_{m,1}, s_{m,2}, \ldots, s_{m,m}, 0, 0, \ldots] P_U^{u_1} \left[1, x, x^2, \ldots, x^m, \ldots\right]^T. \tag{11}$$

It is not difficult to prove that

$$P_U^{u_1} \left[1, x, x^2, \ldots, x^m, \ldots\right]^T$$

$$= \left[\frac{1}{(1-u_1x)}, \frac{x}{(1-u_1x)^2}, \frac{x^2}{(1-u_1x)^3}, \ldots, \frac{x^{m-1}}{(1-u_1x)^m}, \ldots\right]^T. \tag{12}$$

Thus,

$$F_m(x) = [s_{m,1}, s_{m,2}, \ldots, s_{m,m}, 0, 0, \ldots]$$

$$\left[\frac{1}{(1-u_1x)}, \frac{x}{(1-u_1x)^2}, \frac{x^2}{(1-u_1x)^3}, \ldots, \frac{x^{m-1}}{(1-u_1x)^m}, \ldots\right]^T. \tag{13}$$

We summarize this discussion in the following theorem.

Theorem 2: The generating function of the m^{th} row of the convolution matrix C of two sequences $\{v_n\}$ and $\{u_n\}$ is

$$F_m(x) = \sum_{k=1}^{m} \frac{s_{m,k}\, x^{k-1}}{(1-u_1x)^k}, \tag{14}$$

where $s_{m,k}$ is the element in the m^{th} row and the k^{th} column of S and S is the convolution matrix of the sequences $\{v_n\}$ and $\{0, u_2, u_3, u_4, \ldots\}$.

Combining equations (8) and (14) we obtain the generating function of the m^{th} row of the convolution matrix C of the sequence $\{u_n\}$

$$F_m(x) = \sum_{k=1}^{m} \left\{ \sum_{\substack{l_2+l_3+\cdots+l_m=k-1\\ l_i\geq 0;\ i=2,3,\ldots,m;\\ 1l_2+2l_3+\cdots+(m-1)l_m=m-1}} \frac{(k-1)!}{l_2!l_3!\ldots l_n!} u_2^{l_2}u_3^{l_3}\ldots u_m^{l_m}, \right.$$

$$\left. + \sum_{\substack{l_2+l_3+\cdots+l_m=k\\ l_i\geq 0;\ i=2,3,\ldots,m;\\ 1l_2+2l_3+\cdots+(m-1)l_m=m-1}} \frac{k!}{l_2!l_3!\ldots l_m!} u_2^{l_2}u_3^{l_3}\ldots u_m^{l_m} \right\} \frac{x^{k-1}}{(1-u_1x)^k}. \tag{15}$$

For example, using the formula in equation (15) we obtain the generating function of the n^{th} row of matrix M in equation (1)

$$F_n(x) = \frac{\sum_{k=1}^{\lceil n/2\rceil+1} \binom{n}{2k-1} x^{k-1}}{(1-x)^n}. \tag{16}$$

The reader may wish to fill in the details of the computation.

5. CONCLUSION

The Generalized Convolution Decomposition Theorem allows calculation of row generating functions of convolution matrices. Our matrix decomposition approach differs from the approach in [2]. Indeed we have found that our results yield much insight into the structure of generating functions and can be extended to find the generating functions of a rather broad category of matrices. We hope that the matrix decomposition approach developed here may shed some light on derivations of row or column generating functions of arithmetic-progression matrices [3] and the recursion relation matrices studied by Ollerton and Shannon [4].

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