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SOME GENERATING FUNCTIONS OF LAGUERRE POLYNOMIALS

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ABSTRACT. In this note a class of interesting generating relation, which is stated in the form of theorem, involving Laguerre polynomials is derived. Some applications of the theorem are also given here.

KEY WORDS AND PHRASES. Laguerre polynomials, generating functions.
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1. INDRODUCTION.

The Laguerre polynomials $L_n^{(\alpha)}(x)$ are defined by,

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} \; {}_1F_1(-n;\, 1+\alpha;\, x) \tag{1.1}$$

where n is a non-negative integer.

From [1] we have

$$\sum_{n=0}^{\infty} \binom{n+m}{n} L_{n+m}^{(\alpha)}(x)\, w^n = (1-w)^{-1-\alpha-m} \exp\left(\frac{-xw}{1-w}\right) L_m^{(\alpha)}\left(\frac{x}{1-w}\right), \tag{1.2}$$

Observing the existence of the above generating relation (1.2) the present author is interested to investigate the existence of more general generating relation by the group-theoretic method. In fact, the following theorem is obtained as the main result of our investigation.

THEOREM 1. If there exists a generating relation of the form

$$G(x,w) = \sum_{n=0}^{\infty} a_n\, w^n\, L_{n+m}^{(\alpha)}(x) \tag{1.3}$$

then

$$(1-w)^{-1-\alpha-m} \exp\left(\frac{-wx}{1-w}\right) G\left(\frac{x}{1-w}, \frac{wz}{1-w}\right) = \sum_{n=0}^{\infty} w^n\, f_n(z)\, L_{n+m}^{(\alpha)}(x) \tag{1.4}$$

where

$$f_n(z) = \sum_{k=0}^{n} \binom{n+m}{k+m} a_k\, z^k .$$

The importance of the above theorem lies in the fact that one can get a good number of generating relations from (1.4) by attributing different suitable values to a_n in the relation (1.3).

2. DERIVATION OF THE THEOREM.

THEOREM 1. Using the differential recurrence relation [2]

$$x \frac{d}{dx} \left(L_{n+m}^{(\alpha)}(x)\right) = (n+m+1)\, L_{n+m+1}^{(\alpha)}(x) - (n+m+\alpha+1-x)\, L_{n+m}^{(\alpha)}(x) \,. \tag{2.1}$$

We find the following partial differential operator,

$$\mathbb{R} = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + (-x+m+1)y$$

such that

$$\mathbb{R}\left(y^{\alpha+n} L_{n+m}^{(\alpha)}(x)\right) = (n+m+1)\, y^{\alpha+n+1} L_{n+m+1}^{(\alpha)}(x) \,. \tag{2.2}$$

The extended form of the group generated by $\mathbb{R}$ is given by,

$$e^{w\mathbb{R}} f(x,y) = (1-wy)^{-m-1} \exp\left(\frac{-wxy}{1-wy}\right) f\left(\frac{x}{1-wy}, \frac{y}{1-wy}\right) .$$

Let us consider the generating relation of the form:

$$G(x,w) = \sum_{n=o}^{\infty} a_n L_{n+m}^{(\alpha)}(x)\, w^n \,. \tag{2.3}$$

Replacing w by wyz and then multiplying both sides by y^{α}, we get

$$\begin{aligned} y^{\alpha}\, G(x,wyz) &= \sum_{n=o}^{\infty} a_n (wyz)^n\, y^{\alpha} L_{n+m}^{(\alpha)}(x) \\ &= \sum_{n=o}^{\infty} a_n (wz)^n\, y^{\alpha+n} L_{n+m}^{(\alpha)}(x) \,. \end{aligned}$$

Operating both sides of the above expression by (exp $w\mathbb{R}$), we get

$$(\exp w\mathbb{R})\left(y^{\alpha} G(x,wyz)\right) = (\exp w\mathbb{R})\left(\sum_{n=o}^{\infty} a_n (wz)^n y^{\alpha+n} L_{n+m}^{(\alpha)}(x)\right) \tag{2.4}$$

The left member of (2.4) becomes

$$(1-wy)^{-1-m} \exp\left(\frac{-wxy}{1-wy}\right) \left(\frac{y}{1-wy}\right)^{\alpha} G\left(\frac{x}{1-wy}, \frac{wyz}{1-wy}\right) . \tag{2.5}$$

The right member of (2.4) is equal to

$$\begin{aligned} &\sum_{n=o}^{\infty} \sum_{k=o}^{\infty} a_n (wz)^n \frac{w^k}{k!} \mathbb{R}^k \left(y^{n+\alpha} L_{n+m}^{(\alpha)}(x)\right) \\ &= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} a_n \frac{w^{n+k}}{k!} z^n (n+m+1)_k\, y^{n+\alpha+k} L_{n+m+k}^{(\alpha)}(x) \\ &= y^{\alpha} \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} a_n z^n (wy)^{n+k} \frac{(n+m+k)!}{k!(n+m)!} L_{n+m+k}^{(\alpha)}(x) \\ &= y^{\alpha} \sum_{n=o}^{\infty} \sum_{k=o}^{n} (wy)^n \left(a_{n-k}\, z^{n-k} \frac{(n+m)!}{k!\,(n-k+m)!}\right) L_{n+m}^{(\alpha)}(x) \\ &= y^{\alpha} \sum_{n=o}^{\infty} (wy)^n f_n(z)\, L_{n+m}^{(\alpha)}(x) \end{aligned} \tag{2.6}$$

where

$$f_n(z) = \sum_{k=o}^{n} \binom{n+m}{k+m} a_k\, z^k \,.$$

Equating (2.5) and (2.6) and then putting y=1, we get

$$\begin{aligned} (1-w)^{-1-\alpha-m} \exp\left(\frac{-wx}{1-w}\right) G\left(\frac{x}{1-w}, \frac{wz}{1-w}\right) \\ = \sum_{n=o}^{\infty} w^n f_n(z)\, L_{n+m}^{(\alpha)}(x) \end{aligned} \tag{2.7}$$

where,

$f_n(z) = \sum_{k=o}^{n} \binom{n+m}{k+m} a_k\, z^n$, this completes the proof of the theorem.

On the other hand, if we consider the continuous transformations group defined by the infinitesimal operator

$$\mathbb{R}_1 = e^t \left(\frac{\partial}{\partial t} - \frac{1}{2} x \frac{\partial}{\partial z} - x \frac{\partial}{\partial x}\right)$$

then the equations of finite transformations of the group are

$$x' = (\exp w\mathbb{R}_1)x,\ y' = (\exp w\mathbb{R}_1)y,\ z' = (\exp w\mathbb{R}_1)z \tag{2.8}$$

where w is the parameter of the group under consideration.

Also we know that

$$\begin{aligned}(\exp w\mathbb{R}_1)\, f(x,y,z) &= f((\exp w\mathbb{R}_1)x,\ (\exp w\mathbb{R}_1)y,\ (\exp w\mathbb{R}_1)z) \\ &= f\,(x',\ y',\ z')\end{aligned} \tag{2.9}$$

From [3] we see that the effect of the operator $(\exp w\mathbb{R}_1)$ on the variables are as follows:

$$\begin{aligned} x' &= x/\ (1-we^t) \\ y' &= t - \log\,(1-we^t) \\ z' &= z - x\, we^t/\ 2(1-we^t). \end{aligned} \tag{2.10}$$

and

$$\mathbb{R}_1\, F_{n+m}\,(x,t,z) = (n+m+1)\, F_{n+m+1}\,(x,t,z)\ , \tag{2.11}$$

where

$$F_{n+m}\,(x,t,z) = \exp\left[(n+m)t + \frac{\alpha+1}{2}\, t + z - \frac{x}{2}\right] x^{(\alpha+1)/2}\, L_{n+m}^{(\alpha)}\,(x).$$

Now replacing w by wye^t in (2.3) and then multiplying both members by

$$\exp\left[\, mt + \frac{\alpha+1}{2}\, t + z - \frac{x}{2}\,\right] x^{(\alpha+1)/2}$$

we get

$$\begin{aligned} G(x,\ wye^t)\ \exp\{mt + \frac{\alpha+1}{2}\, t + z - \frac{x}{2}\}\ x^{(\alpha+1)/2} \\ = \sum_{n=o}^{\infty} a_n\ (wy)^n\ F_{n+m}(x,t,z)\ . \end{aligned} \tag{2.12}$$

Operating both members of the above expression by $(\exp w\,\mathbb{R}_1)$ and using (2.8), (2.9) and (2.11), we get

$$\begin{aligned} G(x',\ wye^{t'})\ \exp\{mt' + \frac{\alpha+1}{2}\, t' + z' - \frac{x'}{2}\}\ (x')^{(\alpha+1)/2} \\ = \sum_{n=o}^{\infty} w^n\ f_n(y)\ F_{n+m}\,(x,t,z)\ , \end{aligned} \tag{2.13}$$

where

$$f_n(y) = \sum_{k=o}^{n} \binom{n+m}{k+m} a_k\ y^k\ .$$

Putting the values of x', y', z' from (2.10) and then substituting $t = z = o$ we finally obtain

if

$$G(x,w) = \sum_{n=o}^{\infty} a_n\ w^n\ L_{n+m}^{(\alpha)}\,(x)$$

then

$$\begin{aligned} (1-w)^{-1-\alpha-m}\ \exp\left(\frac{-xw}{1-w}\right) G\left(\frac{x}{1-w}\ ,\ \frac{wy}{1-w}\right) \\ = \sum_{n=o}^{\infty} w^n\ f_n(y)\, L_{n+m}^{(\alpha)}\,(x) \end{aligned} \tag{2.14}$$

where

$$f_n(y) = \sum_{k=o}^{n} \binom{n+m}{k+m} a_k\, y^k\ ,$$

which is same as (2.7).

From above we see that if $\mathbb{R}_1$ be used the calculation becomes much harder than when $\mathbb{R}$ is used.

COROLLARY 1. If we put m=o in the above theorem, we get the following well-known theorem derived by W.A. Al-Salam [4], and the second author [5].

"If there exists a generating relation of the form

$$G(x,w) = \sum_{n=o}^{\infty} a_n w^n L_n^{(\alpha)}(x)$$

then

$$(1-w)^{-\alpha-1} \exp\left(\frac{-wx}{1-w}\right) G\left(\frac{x}{1-w}, \frac{wz}{1-w}\right) = \sum_{n=o}^{\infty} w^n f_n(z) L_n^{(\alpha)}(x)$$

where

$$f_n(z) = \sum_{k=o}^{n} \binom{n}{k} a_k z^k .$$

APPLICATION. As a nice application of our theorem, we consider the generating relation given in (1.2), i.e.,

$$\sum_{n=o}^{\infty} \binom{n+m}{n} L_{n+m}^{(\alpha)}(x) w^n = (1-w)^{-1-\alpha-m} \exp\left(\frac{-xw}{1-w}\right) L_m^{(\alpha)}\left(\frac{x}{1-w}\right) .$$

If we put $a_n = \binom{m+n}{n}$ in our theorem, we get

$$(1-w-wz)^{-1-\alpha-m} \exp\left(\frac{-wx(1+z)}{1-w-wz}\right) L_m^{(\alpha)}\left(\frac{x}{1-w-wz}\right) = \sum_{n=o}^{\infty} w^n f_n(z) L_{n+m}^{(\alpha)}(x)$$

where

$$f_n(z) = \sum_{k=o}^{m} \binom{n+m}{k+m} \binom{m+n}{n} z^k .$$

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