MATHEMATICS Proceedings A 83 (2), June 13, 1980

Some bilateral generating functions for a certain class of special functions. I*

by H. M. Srivastava

Department of Mathematics, University of Victoria
Victoria, British Columbia, Canada V8W 2Y2

Communicated by Prof. C. J. Bouwkamp at the meeting of June 16, 1979

SUMMARY

The object of this paper is to present a systematic introduction to and several interesting applications of a general method of obtaining bilinear, bilateral or mixed multilateral generating functions for a fairly wide variety of special functions in one, two and more variables. The main results, contained in Theorems 2 and 3 below, are shown to apply not only to the Bessel polynomials, the classical orthogonal polynomials including, for example, Hermite, Jacobi (and, of course, Gegenbauer, Legendre, and Tchebycheff), and Laguerre polynomials, and to their various generalizations studied in recent years, but indeed also to such other special functions as the Bessel functions, a class of generalized hypergeometric functions, the Lauricella polynomials in several variables, and the familiar Lagrange polynomials which arise in certain problems in statistics. It is also indicated how these general results are related to a number of known results scattered in the literature.

1. INTRODUCTION

In her 1971 monograph [13] McBride discussed a number of useful methods of obtaining generating functions. Every special function, which she considered as the coefficient set in a bilinear (or bilateral) generating

* This work was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant A-7353.

For a preliminary report on this paper see *Notices Amer. Math. Soc.* 26 (1979), pp. A213–A214, Abstract # 79T–B48.

relationship, belongs to a class of functions $\{S_n(x)|n=0, 1, 2, \ldots\}$ generated by [17, p. 755, Eq. (1)]

$$(1.1) \qquad \sum_{n=0}^{\infty} A_{m,n}\, S_{m+n}(x)t^n = f(x, t)\{g(x, t)\}^{-m}\, S_m\left(h(x, t)\right),$$

where m is a non-negative integer, the coefficients $A_{m,n}$ are arbitrary constants, real or complex, and f, g, h are suitable functions of x and t. Indeed, the sequence $\{S_n(x)\}$, generated by (1.1), can be specialized to obtain a fairly wide variety of other known special functions (and polynomials) which were not considered by McBride [13]. With this objective in view, Singhal and Srivastava [17] presented a general class of bilateral generating functions for the $S_n(x)$ defined by (1.1) and showed how their results would easily apply to derive a large variety of bilateral generating functions for the Bessel, Jacobi, Hermite, Laguerre and ultraspherical polynomials, as well as for their numerous generalizations considered in the literature.

Since the publication of the aforementioned paper by Singhal and Srivastava [17], several independent attempts have been made in the literature towards generalizations of the Singhal-Srivastava theorem [17, p. 755] in a number of seemingly different directions. Chatterjea ([3] and [4]) extends the definition (1.1) to include cases when m is an arbitrary complex number and also when the sequence generated depends upon two complex variables. Notice, however, that if in (1.1) we replace the sequence $\{S_n(x)\}$ by another sequence $\{S^*_{\mu+n}(x)\}$, where μ is an arbitrary complex number, and set

$$(1.2) \qquad A_{m,n} = A^*_{\mu+m,n}, \quad f(x, t) = f^*(x, t)\{g(x, t)\}^{-\mu},$$

then (1.1) assumes the form:

$$(1.3) \qquad \sum_{n=0}^{\infty} A^*_{\mu+m,n}\, S^*_{\mu+m+n}(x)t^n = f^*(x, t)\{g(x, t)\}^{-\mu-m}\, S^*_{\mu+m}\left(h(x, t)\right),$$

or equivalently,

$$(1.4) \qquad \sum_{n=0}^{\infty} A^*_{\mu,n}\, S^*_{\mu+n}(x)t^n = f^*(x, t)\{g(x, t)\}^{-\mu}\, S^*_{\mu}\left(h(x, t)\right),$$

which is a generating relationship of the type considered by Chatterjea ([3] and [4]). Thus, Chatterjea's extensions ([3, p. 117, Proposition I] and [4, p. 2, Theorem II]) are derivable by merely applying† the Singhal-Srivastava theorem [17, p. 755] to the sequence $\{S^*_{\mu+n}(x)\}_{n=0}^{\infty}$, where μ is an arbitrary complex number. {As a matter of fact, Proposition I in Chatterjea's paper [3] is essentially the same as the main result (Theorem II) in his subsequent paper [4].}

---

† See also the review of S. K. Chatterjea's paper [3] in *Zentral. Math.* 348 (1977), pp. 160–161, # 33012 (by J. P. Singhal).

Some non-trivial generalizations of the results of Singhal and Srivastava [17] (to hold for sequences of functions of one and more variables) have appeared in the works of Srivastava and Lavoie [18], Carlitz and Srivastava [2], and Panda [14]. We recall here one of the main results, due to Srivastava and Lavoie [18], in the following form:

THEOREM 1 (Srivastava and Lavoie [18, p. 319, Eq. (107)]). *Corresponding to the functions* $S_\mu^*(x)$, *generated by* (1.4), *let*

$$F_{q,\nu}[x, t] = \sum_{n=0}^{\infty} c_{\nu,n}\, S_{\nu+qn}^*(x) t^n, \quad c_{\nu,n} \neq 0, \tag{1.5}$$

where q *is a positive integer, and* ν *is an arbitrary complex number.*
Then

$$\begin{cases} \sum_{n=0}^{\infty} S_{\nu+n}^*(x) T_{n,\nu}^q(y) t^n = f^*(x, t)\{g(x, t)\}^{-\nu} \\ \qquad\qquad \cdot F_{q,\nu}[h(x, t), y\{t/g(x, t)\}^q], \end{cases} \tag{1.6}$$

where $T_{n,\nu}^q(y)$ *is a polynomial of degree* $[n/q]$ *in* y *defined by*

$$T_{n,\nu}^q(y) = \sum_{k=0}^{[n/q]} A_{\nu+qk,n-qk}^*\, c_{\nu,k}\, y^k. \tag{1.7}$$

Theorem 1 and its generalization to sequences of functions of r variables, given subsequently by Panda [14], can be specialized to yield all of the results of Chatterjea ([3] and [4]). Proposition I in Chatterjea's paper [3, p. 117], which indeed is the same as the main result in his latter paper [4, p. 2, Theorem II], follows from Theorem 1 in its special case when $\nu = m$ and $q = 1$, while his other result [3, p. 118, Proposition II] corresponds to the special case of Panda's theorem [14, p. 28, Eq. (5)] when $\nu = m$, $q = 1$ and $r = 2$. {See also the Srivastava-Lavoie theorem [18, p. 316, Theorem 2] which, when $r = 2$ and $q = 1$, can be applied to derive Chatterjea's result just cited [3, p. 118, Proposition II] in a manner analogous to the aforementioned derivation of Chatterjea's results [3, p. 117, Proposition I] and [4, p. 2, Theorem II] from the Singhal-Srivastava theorem [17, p. 755].} It may be of interest to observe in passing that Deductions 2 and 3 in Chatterjea's paper [3, p. 126], involving the familiar (two-variable) Lagrange polynomials defined by [6, p. 267, Eq. (1)]

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x, y) t^n = (1 - xt)^{-\alpha} (1 - yt)^{-\beta}, \tag{1.8}$$

are contained in a single bilateral generating function (due to Srivastava and Lavoie [18, p. 318, Eq. (100)]) with $q = 1$.

An interesting generalization of the bilateral generating function (1.6) is given by

THEOREM 2. *For the functions* $S_\mu^*(x)$ *defined by* (1.4), *let*

$$(1.9)\quad \begin{cases} F_{q,\nu}^{p,\mu}[x; y_1, \ldots, y_s; t] \\ \qquad = \sum_{n=0}^{\infty} c_n^{\mu,\nu} S_{\nu+qn}^*(x) \Omega_{\mu+pn}(y_1, \ldots, y_s) t^n, \quad c_n^{\mu,\nu} \neq 0, \end{cases}$$

where (and in what follows) μ *and* ν *are arbitrary complex numbers,* p *and* q *are positive integers, and* $\Omega_\mu(y_1, \ldots, y_s)$ *is a non-vanishing function of* s *variables* $y_1, \ldots, y_s$, $s \geqq 1$.

Then

$$(1.10)\quad \begin{cases} \sum_{n=0}^{\infty} S_{\nu+n}^*(x)\, W_{n,q,\nu}^{p,\mu}(y_1, \ldots, y_s; z) t^n = f^*(x, t)\{g(x, t)\}^{-\nu} \\ \qquad \cdot F_{q,\nu}^{p,\mu}[h(x, t); y_1, \ldots, y_s; z\{t/g(x, t)\}^q], \end{cases}$$

where $W_{n,q,\nu}^{p,\mu}(y_1, \ldots, y_s; z)$ *is a polynomial of degree* $[n/q]$ *in* z *(with coefficients dependent on* $y_1, \ldots, y_s$*) defined by*

$$(1.11)\quad W_{n,q,\nu}^{p,\mu}(y_1, \ldots, y_s; z) = \sum_{k=0}^{[n/q]} A_{\nu+qk,n-qk}^*\, c_k^{\mu,\nu}\, \Omega_{\mu+pk}(y_1, \ldots, y_s) z^k$$

in terms of the $A_{\mu,n}^*$ *occurring in* (1.4).

REMARK 1. For $s = 1$, the bilateral generating function (1.10) can at once be rewritten in the form:

$$(1.12)\quad \begin{cases} \sum_{n=0}^{\infty} S_{\nu+n}^*(x)\, Y_{n,q,\nu}^{p,\mu}(y, z) t^n = f^*(x, t)\{g(x, t)\}^{-\nu} \\ \qquad \cdot G_{q,\nu}^{p,\mu}[h(x, t), y, z\{t/g(x, t)\}^q], \end{cases}$$

where

$$(1.13)\quad Y_{n,q,\nu}^{p,\mu}(y, z) = \sum_{k=0}^{[n/q]} A_{\nu+qk,n-qk}^*\, c_k^{\mu,\nu}\, \Xi_{\mu+pk}(y) z^k,$$

$$(1.14)\quad G_{q,\nu}^{p,\mu}[x, y, t] = \sum_{n=0}^{\infty} c_n^{\mu,\nu} S_{\nu+qn}^*(x)\, \Xi_{\mu+pn}(y) t^n, \quad c_n^{\mu,\nu} \neq 0,$$

$\Xi_\mu(y) \neq 0$ is an arbitrary function of y, p and q are positive integers, and μ and ν are arbitrary complex numbers.

REMARK 2. Yet another interesting special form of the bilateral generating function (1.10) would occur when the multivariable function

$$\Omega_\mu(y_1, \ldots, y_s), \quad s > 1$$

can be expressed as a suitable product of several simpler functions. We omit the details involved in deriving such multilateral generating relations for the functions $S_\nu^*(x)$ defined by (1.4).

2. PROOF OF THEOREM 2

If we substitute for the polynomials $W_{n,q,\nu}^{p,\mu}(y_1, \ldots, y_s; z)$ from (1.11) into the left-hand side of (1.10), we shall get

$$\begin{aligned}
\Lambda &\equiv \sum_{n=0}^{\infty} S_{\nu+n}^{*}(x)\, W_{n,q,\nu}^{p,\mu}(y_1, \ldots, y_s; z)t^n \\
&= \sum_{n=0}^{\infty} S_{\nu+n}^{*}(x)t^n \sum_{k=0}^{[n/q]} A_{\nu+qk,n-qk}^{*}\, c_k^{\mu,\nu}\, \Omega_{\mu+pk}(y_1, \ldots, y_s)z^k \\
&= \sum_{k=0}^{\infty} c_k^{\mu,\nu}\, \Omega_{\mu+pk}(y_1, \ldots, y_s)(zt^q)^k \sum_{n=0}^{\infty} A_{\nu+qk,n}^{*}\, S_{\nu+qk+n}^{*}(x)t^n,
\end{aligned}$$

by interchanging the order of the double summation involved.

The inner series can be summed by appealing to the generating relation (1.4), with μ replaced by $\nu+qk$, and we thus obtain

$$\begin{aligned}
\Lambda = f^{*}(x, t)\{g(x, t)\}^{-\nu} \sum_{k=0}^{\infty} c_k^{\mu,\nu}\, S_{\nu+qk}^{*}\Big(h(x, t)\Big) \\
\cdot\, \Omega_{\mu+pk}(y_1, \ldots, y_s)[z\{t/g(x, t)\}^q]^k.
\end{aligned}$$

Now we interpret this last infinite series by means of the definition (1.9), and the second member of the bilateral generating relation (1.10) follows at once.

This evidently completes the proof of Theorem 2 under the hypothesis that the double series involved in the first two steps of our derivation are absolutely convergent. Thus, in general, Theorem 2 holds true for such values of the various parameters and variables involved for which each member of Equation (1.10) has a meaning.

3. FURTHER GENERALIZATIONS OF THEOREM 2

Consider a set of functions $\Delta_\mu(x_1, \ldots, x_r)$ of r variables $x_1, \ldots, x_r$, and of order μ, generated by [14, p. 28, Eq. (3)]

$$\text{(3.1)}\quad \left\{\begin{aligned}
&\sum_{n=0}^{\infty} \gamma_{\mu,n}\, \Delta_{\mu+n}(x_1, \ldots, x_r)t^n = \theta(x_1, \ldots, x_r; t)\{\phi(x_1, \ldots, x_r; t)\}^{-\mu} \\
&\qquad \cdot\, \Delta_\mu\Big(\psi_1(x_1, \ldots, x_r; t), \ldots, \psi_r(x_1, \ldots, x_r; t)\Big),
\end{aligned}\right.$$

where μ is an arbitrary complex number, the $\gamma_{\mu,n}$, $n \geqq 0$, are suitable constants, and θ, ϕ, $\psi_1, \ldots, \psi_r$ are suitable functions of $x_1, \ldots, x_r$ and t.

The method of proof of Theorem 2 can be applied *mutatis mutandis* to derive the following generalization which would evidently yield bilateral (and, of course, multilateral) generating relations for a fairly wide variety of sequences of functions of several variables. {See Remark 2 above.}

THEOREM 3. *For the functions* $\Delta_\mu(x_1, \ldots, x_r)$ *defined by* (3.1), *let*

$$(3.2)\quad \begin{cases} \Phi^{p,\mu}_{q,\nu}[x_1, \ldots, x_r; y_1, \ldots, y_s; t] \\ = \sum_{n=0}^{\infty} \delta_n^{\mu,\nu} \Delta_{\nu+qn}(x_1, \ldots, x_r)\Omega_{\mu+pn}(y_1, \ldots, y_s)t^n, \quad \delta_n^{\mu,\nu} \neq 0, \end{cases}$$

where μ *and* ν *are arbitrary complex numbers,* p *and* q *are positive integers, and* $\Omega_\mu(y_1, \ldots, y_s)$ *is a non-vanishing function of* s *variables* $y_1, \ldots, y_s$, $s \geq 1$. *Then*

$$(3.3)\quad \begin{cases} \sum_{n=0}^{\infty} \Delta_{\nu+n}(x_1, \ldots, x_r) R^{p,\mu}_{n,q,\nu}(y_1, \ldots, y_s; z)t^n \\ = \theta(x_1, \ldots, x_r; t)\{\phi(x_1, \ldots, x_r; t)\}^{-\nu} \\ \cdot \Phi^{p,\mu}_{q,\nu}[\psi_1(x_1, \ldots, x_r; t), \ldots, \psi_r(x_1, \ldots, x_r; t); y_1, \ldots, y_s; \\ \qquad z\{t/\phi(x_1, \ldots, x_r; t)\}^q], \end{cases}$$

where $R^{p,\mu}_{n,q,\nu}(y_1, \ldots, y_s; z)$ *is a polynomial of degree* $[n/q]$ *in* z (*with coefficients involving* $y_1, \ldots, y_s$) *defined by*

$$(3.4)\quad R^{p,\mu}_{n,q,\nu}(y_1, \ldots, y_s; z) = \sum_{k=0}^{[n/q]} \gamma_{\nu+qk,n-qk}\, \delta_k^{\mu,\nu}\, \Omega_{\mu+pk}(y_1, \ldots, y_s)z^k,$$

in terms of the constants $\gamma_{\mu,n}$ *occurring in* (3.1).

REMARK 3. Obviously, Theorem 3 would reduce to Theorem 2 in its special case $r=1$, while the (Srivastava-Lavoie) bilateral generating function (1.6), contained in Theorem 1, follows from Theorem 2 and its consequence (1.12) when we set

$$(3.5)\quad \Omega_\mu(y_1, \ldots, y_s) \equiv 1 \text{ and } \Xi_\mu(y) \equiv 1,$$

respectively. Other known special cases of our theorems include the main results of the recent papers by Chatterjea [5, p. 325, Eq. (1.7)] and Panda [14, p. 28, Eq. (5)]; the former would follow from the bilateral generating function (1.12) when $p=q=1$, $\mu=0$ and $\nu=m$, where m is a positive integer, and the latter is readily derivable from Equation (3.3) above on setting $p=1$ and specializing $\Omega_\mu(y_1, \ldots, y_s)$ by means of (3.5).

4. APPLICATIONS TO FAMILIAR POLYNOMIALS

We begin by recalling some familiar instances of generating-function relationships of the type (1.1) or (1.4) with $\mu=m$. Many of these results are fairly well known; they may be found, for example, in the 1971 monograph by McBride [13]. {Others are readily derivable from known results. See also the recent work [8].}

Bessel polynomials:

$$(4.1)\quad \begin{cases} \sum_{n=0}^{\infty} y_{m+n}(x, \alpha-m-n, \beta)\frac{t^n}{n!} \\ \qquad = (1-xt/\beta)^{1-\alpha}\exp(t)\; y_m\Big(x(1-xt/\beta)^{-1}, \alpha-m, \beta\Big), \end{cases}$$

where $y_n(x, \alpha, \beta)$ denotes the (Krall-Frink) Bessel polynomials defined by [11, p. 108, Eq. (34)]

$$(4.2)\quad y_n(x, \alpha, \beta) = \sum_{k=0}^{n} \binom{n}{k}\binom{\alpha+n+k-2}{k} k! \left(\frac{x}{\beta}\right)^k .$$

For the *simple* Bessel polynomials $y_n(x)$ defined by

$$(4.3)\quad y_n(x) = y_n(x, 2, 2),$$

we have the relatively more familiar result [13, p. 50, Eq. (12)]

$$(4.4)\quad \begin{cases} \sum_{n=0}^{\infty} y_{m+n}(x)\frac{t^n}{n!} = (1-2xt)^{-(m+1)/2}\exp\Big(x^{-1}\{1-\sqrt{(1-2xt)}\}\Big) \\ \qquad\qquad \cdot\, y_m\Big(x/\sqrt{(1-2xt)}\Big). \end{cases}$$

Gegenbauer (or ultraspherical) polynomials:

$$(4.5)\quad \sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{\alpha}(x)t^n = \varrho^{-m-2\alpha}\, P_m^{\alpha}\left(\frac{x-t}{\varrho}\right),$$

where

$$\varrho = \sqrt{(1-2xt+t^2)}.$$

Hermite polynomials:

$$(4.6)\quad \sum_{n=0}^{\infty} H_{m+n}(x)\frac{t^n}{n!} = \exp(2xt-t^2)H_m(x-t).$$

Jacobi polynomials:

$$(4.7)\quad \begin{cases} \sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{(\alpha-m-n,\, \beta-m-n)}(x)t^n \\ = \Big\{1+\tfrac{1}{2}(x+1)t\Big\}^{\alpha-m}\Big\{1+\tfrac{1}{2}(x-1)t\Big\}^{\beta-m} P_m^{(\alpha-m,\, \beta-m)}\Big(\xi(x,t)\Big), \end{cases}$$

where, for convenience,

$$(4.8)\quad \xi(x, t) = x + \tfrac{1}{2}(x^2-1)t;$$

$$(4.9)\quad \begin{cases} \sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{(\alpha-m-n,\, \beta)}(x)t^n \\ = (1+t)^{\alpha-m}\Big\{1-\tfrac{1}{2}(x-1)t\Big\}^{-\alpha-\beta-1} P_m^{(\alpha-m,\, \beta)}\Big(\eta(x,t)\Big), \end{cases}$$

where

(4.10) $\eta(x, t) = \left\{ x + \tfrac{1}{2}(x-1)t \right\} \left\{ 1 - \tfrac{1}{2}(x-1)t \right\}^{-1};$

(4.11) $$\begin{cases} \sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{(\alpha, \beta-m-n)}(x) t^n \\ = (1-t)^{\beta-m} \left\{ 1 - \tfrac{1}{2}(x+1)t \right\}^{-\alpha-\beta-1} P_m^{(\alpha, \beta-m)} \Big(\zeta(x, t) \Big), \end{cases}$$

where

(4.12) $\zeta(x, t) = \left\{ x - \tfrac{1}{2}(x+1)t \right\} \left\{ 1 - \tfrac{1}{2}(x+1)t \right\}^{-1}.$

Laguerre polynomials:

(4.13) $$\sum_{n=0}^{\infty} \binom{m+n}{n} L_{m+n}^{(\alpha)}(x) t^n = (1-t)^{-1-\alpha-m} \exp\left(-\frac{xt}{1-t} \right) L_m^{(\alpha)}\left(\frac{x}{1-t} \right);$$

(4.14) $$\sum_{n=0}^{\infty} \binom{m+n}{n} L_{m+n}^{(\alpha-m-n)}(x) t^n = (1+t)^{\alpha-m} \exp(-xt) L_m^{(\alpha-m)}\Big(x(1+t) \Big).$$

Now we compare each of the above generating-function relationships with (1.1), that is, (1.4) with $\mu = m$, and we are led at once to the following interesting applications of Theorem 2:

COROLLARY 1. *If*

(4.15) $$\begin{cases} \Lambda_{m,q}^{(1)}[x; y_1, \ldots, y_s; t] \\ = \sum_{n=0}^{\infty} \frac{a_n}{(n)!} y_{m+qn}(x, \alpha - qn, \beta) \Omega_{\mu+pn}(y_1, \ldots, y_s) t^n, \quad a_n \neq 0, \end{cases}$$

then, for every non-negative integer m,

(4.16) $$\begin{cases} \sum_{n=0}^{\infty} y_{m+n}(x, \alpha - n, \beta) M_{n,q}^{p,\mu}(y_1, \ldots, y_s; z) \frac{t^n}{n!} \\ = (1 - xt/\beta)^{1-\alpha-m} \exp(t) \Lambda_{m,q}^{(1)}[x(1-xt/\beta)^{-1}; y_1, \ldots, y_s; zt^q], \\ \qquad \forall p, q, s \in \{1, 2, 3, \ldots\}, \end{cases}$$

where, and throughout this paper,

(4.17) $$M_{n,q}^{p,\mu}(y_1, \ldots, y_s; z) = \sum_{k=0}^{[n/q]} \binom{n}{qk} a_k \, \Omega_{\mu+pk}(y_1, \ldots, y_s) z^k,$$

$\Omega_{\mu+pk}(y_1, \ldots, y_s)$ *being, as before, a non-vanishing function of s variables* $y_1, \ldots, y_s$, $s \geqq 1$.

COROLLARY 2. *If*

$$(4.18)\quad \Lambda^{(2)}_{m,q}[x; y_1, \ldots, y_s; t] = \sum_{n=0}^{\infty} \frac{a_n}{(qn)!} y_{m+qn}(x)\Omega_{\mu+pn}(y_1, \ldots, y_s)t^n, \quad a_n \neq 0,$$

then, for every non-negative integer m,

$$(4.19)\quad \begin{cases} \sum_{n=0}^{\infty} y_{m+n}(x) M^{p,\mu}_{n,q}(y_1, \ldots, y_s; z)\frac{t^n}{n!} = (1-2xt)^{-(m+1)/2} \\ \qquad \cdot \exp\left(x^{-1}\{1-\sqrt{(1-2xt)}\}\right) \\ \qquad \cdot \Lambda^{(2)}_{m,q}[x/\sqrt{(1-2xt)}; y_1, \ldots, y_s; zt^q(1-2xt)^{-q/2}]. \end{cases}$$

COROLLARY 3. *If*

$$(4.20)\quad \Lambda^{(3)}_{m,q}[x; y_1, \ldots, y_s; t] = \sum_{n=0}^{\infty} \frac{a_n}{(qn)!} H_{m+qn}(x)\Omega_{\mu+pn}(y_1, \ldots, y_s)t^n, \quad a_n \neq 0,$$

then, for every non-negative integer m,

$$(4.21)\quad \begin{cases} \sum_{n=0}^{\infty} H_{m+n}(x) M^{p,\mu}_{n,q}(y_1, \ldots, y_s; z)\frac{t^n}{n!} \\ = \exp(2xt-t^2)\, \Lambda^{(3)}_{m,q}[x-t; y_1, \ldots, y_s; zt^q]. \end{cases}$$

COROLLARY 4. *If*

$$(4.22)\quad \Lambda^{(4)}_{m,q}[x; y_1, \ldots, y_s; t] = \sum_{n=0}^{\infty} a_n P^{\alpha}_{m+qn}(x)\Omega_{\mu+pn}(y_1, \ldots, y_s)t^n, \quad a_n \neq 0,$$

then, for every non-negative integer m,

$$(4.23)\quad \begin{cases} \sum_{n=0}^{\infty} P^{\alpha}_{m+n}(x) N^{p,\mu}_{n,m,q}(y_1, \ldots, y_s; z)t^n \\ = \varrho^{-m-2\alpha}\, \Lambda^{(4)}_{m,q}[(x-t)/\varrho; y_1, \ldots, y_s; z(t/\varrho)^q], \end{cases}$$

where, and in what follows,

$$(4.24)\quad N^{p,\mu}_{n,m,q}(y_1, \ldots, y_s; z) = \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} a_k\, \Omega_{\mu+pk}(y_1, \ldots, y_s)z^k,$$

and, as before, $\varrho = \sqrt{(1-2xt+t^2)}$.

REMARK 4. For $m=0$, this last definition (4.24) would obviously reduce to that in (4.17), and we have the equivalence

$$(4.25)\quad M^{p,\mu}_{n,q}(y_1, \ldots, y_s; z) \equiv N^{p,\mu}_{n,0,q}(y_1, \ldots, y_s; z).$$

REMARK 5. Corollary 4 can readily be specialized to yield the corresponding bilateral generating functions for the well-known Legendre polynomials

$$P_n(x) = P_n^{\frac{1}{2}}(x), \tag{4.26}$$

and for the Tchebycheff polynomials of the first and second kinds:

$$T_n(x) = \tfrac{1}{2}n \lim_{|\alpha|\to 0} \{\alpha^{-1}P_n^{\alpha}(x)\}, \quad U_n(x) = P_n^1(x). \tag{4.27}$$

We omit the details involved.

COROLLARY 5. *If*

$$\begin{cases} \Lambda_{m,q}^{(5)}[x; y_1, \ldots, y_s; t] \\ = \sum_{n=0}^{\infty} a_n P_{m+qn}^{(\alpha-qn, \beta-qn)}(x)\Omega_{\mu+pn}(y_1, \ldots, y_s)t^n, \quad a_n \neq 0, \end{cases} \tag{4.28}$$

then, for every non-negative integer m,

$$\begin{cases} \sum_{n=0}^{\infty} P_{m+n}^{(\alpha-n, \beta-n)}(x)\, N_{n,m,q}^{p,\mu}(y_1, \ldots, y_s; z)t^n \\ = \left\{1 + \tfrac{1}{2}(x+1)t\right\}^{\alpha} \left\{1 + \tfrac{1}{2}(x-1)t\right\}^{\beta} \Lambda_{m,q}^{(5)}\Big[\xi(x, t); \\ \qquad y_1, \ldots, y_s; zt^q \Big/ \left\{1 + \tfrac{1}{2}(x+1)t\right\}^q \left\{1 + \tfrac{1}{2}(x-1)t\right\}^q\Big], \end{cases} \tag{4.29}$$

where $\xi(x, t)$ *is defined, as before, by* (4.8).

COROLLARY 6. *If*

$$\Lambda_{m,q}^{(6)}[x; y_1, \ldots, y_s; t] = \sum_{n=0}^{\infty} a_n P_{m+qn}^{(\alpha-qn, \beta)}(x)\Omega_{\mu+pn}(y_1, \ldots, y_s)t^n, \ a_n \neq 0, \tag{4.30}$$

then, for every non-negative integer m,

$$\begin{cases} \sum_{n=0}^{\infty} P_{m+n}^{(\alpha-n, \beta)}(x)\, N_{n,m,q}^{p,\mu}(y_1, \ldots, y_s; z)t^n \\ = (1+t)^{\alpha} \left\{1 - \tfrac{1}{2}(x-1)t\right\}^{-\alpha-\beta-1} \Lambda_{m,q}^{(6)}[\eta(x, t); y_1, \ldots, y_s; zt^q/(1+t)^q], \end{cases} \tag{4.31}$$

where $\eta(x, t)$ *is given by* (4.10).

COROLLARY 7. *If*

$$\begin{cases} \Lambda_{m,q}^{(7)}[x; y_1, \ldots, y_s; t] \\ = \sum_{n=0}^{\infty} a_n P_{m+qn}^{(\alpha, \beta-qn)}(x)\Omega_{\mu+pn}(y_1, \ldots, y_s)t^n, \quad a_n \neq 0, \end{cases} \tag{4.32}$$

then, for every non-negative integer m,

$$(4.33)\quad \begin{cases} \sum_{n=0}^{\infty} P_{m+n}^{(\alpha,\beta-n)}(x)\, N_{n,m,q}^{p,\mu}(y_1, \ldots, y_s; z)t^n \\ = (1-t)^{\beta}\left\{1-\tfrac{1}{2}(x+1)t\right\}^{-\alpha-\beta-1} \Lambda_{m,q}^{(7)}[\zeta(x,t); y_1, \ldots, y_s; zt^q/(1-t)^q], \end{cases}$$

where $\zeta(x, t)$ is defined by (4.12).

REMARK 6. Corollaries 5, 6 and 7 are essentially equivalent, since we have the familiar relationships [22, p. 59, Eq. (4.1.3)]

$$(4.34)\quad P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$$

and [*op. cit.*, p. 64, Eq. (4.22.1)]

$$(4.35)\quad P_n^{(\alpha,\beta-n)}(x) = \left(\frac{1-x}{2}\right)^n P_n^{(-\alpha-\beta-1-n,\beta-n)}\left(\frac{x+3}{x-1}\right).$$

Indeed, the equivalence of Corollaries 6 and 7 follows immediately from (4.34), Corollaries 5 and 7 are equivalent in view of (4.35), while Corollaries 5 and 6 can be shown to be equivalent by appealing to both (4.34) and (4.35).

COROLLARY 8. *If*

$$(4.36)\quad \begin{cases} \Lambda_{m,q}^{(8)}[x; y_1, \ldots, y_s; t] \\ = \sum_{n=0}^{\infty} a_n L_{m+qn}^{(\alpha)}(x)\, \Omega_{\mu+pn}(y_1, \ldots, y_s)t^n, \quad a_n \neq 0, \end{cases}$$

then, for every non-negative integer m,

$$(4.37)\quad \begin{cases} \sum_{n=0}^{\infty} L_{m+n}^{(\alpha)}(x)\, N_{n,m,q}^{p,\mu}(y_1, \ldots, y_s; z)t^n \\ = (1-t)^{-1-\alpha-m} \exp\left(-\frac{xt}{1-t}\right) \Lambda_{m,q}^{(8)}[x/(1-t); y_1, \ldots, y_s; zt^q/(1-t)^q], \end{cases}$$

where, as before, $N_{n,m,q}^{p,\mu}(y_1, \ldots, y_s; z)$ is defined by (4.24).

COROLLARY 9. *If*

$$(4.38)\quad \begin{cases} \Lambda_{m,q}^{(9)}[x; y_1, \ldots, y_s; t] \\ = \sum_{n=0}^{\infty} a_n L_{m+qn}^{(\alpha-qn)}(x)\, \Omega_{\mu+pn}(y_1, \ldots, y_s)t^n, \quad a_n \neq 0, \end{cases}$$

then, for every non-negative integer m,

$$(4.39)\quad \begin{cases} \sum_{n=0}^{\infty} L_{m+n}^{(\alpha-n)}(x)\, N_{n,m,q}^{p,\mu}(y_1, \ldots, y_s; z)t^n \\ = (1+t)^{\alpha} \exp(-xt)\, \Lambda_{m,q}^{(9)}[x(1+t); y_1, \ldots, y_s; zt^q/(1+t)^q]. \end{cases}$$

REMARK 7. Since [22, p. 103, Eq. (5.3.4)]

$$(4.40)\quad L_n^{(\alpha)}(x) = \lim_{|\beta|\to\infty} P_n^{(\alpha,\beta)}(1-2x/\beta), \quad \forall n \in \{0, 1, 2, \ldots\},$$

this last Corollary 9 can be derived as the confluent case of Corollary 6 (or Corollary 5) when $|\beta| \to \infty$, while Corollary 8 is similarly derivable from Corollary 7.

REMARK 8. In view of the known relationship $\Big($*cf.*, *e.g.*, [18], p. 311, Eq. (53)$\Big)$

$$(4.41)\quad y_n(x, \alpha-n, \beta) = n!(-x/\beta)^n L_n^{(1-\alpha-n)}(\beta/x),$$

Corollary 9 can easily be shown to be *equivalent* to Corollary 1 involving Bessel polynomials.

REMARK 9. Since [13, pp. 68–69]

$$(4.42)\quad f_n^{-\alpha}(x) = (-1)^n L_n^{(\alpha-n)}(x) = \frac{x^n}{n!} c_n(\alpha; x),$$

Corollary 9 can alternatively be stated as follows in terms of the *modified* Laguerre polynomials $f_n^{\alpha}(x)$ defined by [*op. cit.*, p. 4, Eq. (9)]

$$(4.43)\quad (1-t)^{-\alpha} \exp(xt) = \sum_{n=0}^{\infty} f_n^{\alpha}(x)t^n,$$

or the Poisson-Charlier polynomials $c_n(x; \alpha)$ defined by $\Big($[*op. cit.*, p. 68]; see also [22], p. 35, Eq. (2.81.2)$\Big)$

$$(4.44)\quad c_n(x; \alpha) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{x}{k} k!\, \alpha^{-k}, \quad \alpha > 0,\ x = 0, 1, 2, \ldots.$$

COROLLARY 10. *If*

$$(4.45)\quad \begin{cases} \Lambda_{m,q}^{(10)}[x; y_1, \ldots, y_s; t] \\ = \sum_{n=0}^{\infty} a_n f_{m+qn}^{\alpha}(x) \Omega_{\mu+pn}(y_1, \ldots, y_s)t^n, \quad a_n \neq 0, \end{cases}$$

then, for every non-negative integer m,

$$(4.46)\quad \begin{cases} \sum_{n=0}^{\infty} f_{m+n}^{\alpha}(x)\, N_{n,m,q}^{p,\mu}(y_1, \ldots, y_s; z)t^n \\ = (1-t)^{-\alpha-m} \exp(xt) \Lambda_{m,q}^{(10)}[x(1-t); y_1, \ldots, y_s; zt^q/(1-t)^q], \end{cases}$$

where the functions $N_{n,m,q}^{p,\mu}(y_1, \ldots, y_s; z)$, $n \geq 0$, *are given by* (4.24).

COROLLARY 11. *Let*

$$(4.47)\quad \begin{cases} \Lambda_{m,q}^{(11)}[x; y_1, \ldots, y_s; t] \\ = \sum_{n=0}^{\infty} \frac{a_n}{(qn)!} c_{m+qn}(\alpha; x) \Omega_{\mu+pn}(y_1, \ldots, y_s) t^n, \quad a_n \neq 0, \end{cases}$$

where $x > 0$ *and* $\alpha = 0, 1, 2, \ldots$.

Then, for every non-negative integer m,

$$(4.48)\quad \begin{cases} \sum_{n=0}^{\infty} c_{m+n}(\alpha; x)\, M_{n,q}^{p,\mu}(y_1, \ldots, y_s; z) \frac{t^n}{n!} \\ = (1-t/x)^{\alpha} \exp(t)\, \Lambda_{m,q}^{(11)}[x-t; y_1, \ldots, y_s; zt^q], \end{cases}$$

where the functions $M_{n,q}^{p,\mu}(y_1, \ldots, y_s; z)$, $n \geq 0$, *are defined, as in Corollaries* 1, 2 *and* 3, *by* (4.17).

Incidentally, these last Corollaries 10 and 11 can be derived *directly* from Theorem 2 by appealing to the generating-function relationships

$$(4.49)\quad \sum_{n=0}^{\infty} \binom{m+n}{n} f_{m+n}^{\alpha}(x) t^n = (1-t)^{-\alpha-m} \exp(xt)\, f_m^{\alpha}\Big(x(1-t)\Big)$$

and

$$(4.50)\quad \sum_{n=0}^{\infty} c_{m+n}(\alpha; x) \frac{t^n}{n!} = (1-t/x)^{\alpha} \exp(t)\, c_m(\alpha; x-t),$$

respectively. Indeed, by virtue of (4.42), both (4.49) and (4.50) are rather straightforward consequences of the familiar result (4.13).

(To be continued)