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Certain Bilateral Generating Relations for a Class of Generalized Hypergeometric Functions of Two Variables

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Abstract In [10] we defined and studied a class of generalized hypergeometric functions $B_n^{(\alpha,\beta)}(x,y,w)$. In this paper an attempt has been made to obtain some bilateral generating relations with $B_n^{(\alpha,\beta)}(x,y,w)$. Each result is followed by its applications to the classical orthogonal polynomials.

Keywords Bilateral Generating Relations, Generalized Hypergeometric Functions, Classical Orthogonal Polynomials

1. Introduction

In the previous paper [10], we introduced a class of generalized hypergeometric functions $B_n^{(\alpha,\beta)}(x,y,w)$ defined as follows:

$$B_n^{(\alpha,\beta)}(x,y,w)=\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(n!)^2}\times$$

$$\sum_{r=0}^{n}\frac{(-1)^r y^{[rw]}}{r!\Gamma(n+\alpha-r+1)\Gamma(\beta+r+1)}J_{n-r}^{\alpha}(x,w) \tag{1.1}$$

where $J_n^{\alpha}(x,w)$ is modified Jacobi polynomial (see Parihar and Patel [6] and also see Lahiri and Satyanarayana [3]-[5]). We also derived the following relation

$$B_n^{(\alpha,\beta)}(x,y,w)=\frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2}\times,$$

$$F_{-:1;1}^{1:1;1}\left[\begin{array}{l}-n:-\frac{y}{w};\frac{x}{w}; \\ --:1+\beta;1+\alpha;\end{array} -w,w\right] \tag{1.2}$$

Taking the limit $w\to 0$ in (1.1), we obtain

$$\lim_{w\to 0}B_n^{(\alpha,\beta)}(x,y,w)=L_n^{(\alpha,\beta)}(x,y), \tag{1.3}$$

where $L_n^{(\alpha,\beta)}(x,y)$ is Laguerre polynomial of two variables [7].

In Satyanarayana [9] (also see [5, p.326(1.8)] defined generalized hypergeometric functions $I_{n;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w)$ and also proved that [9, p.65(3.3.3)]

$$\sum_{n=0}^{\infty}\binom{n+r}{r}\frac{(\rho+r)_n}{(1+\alpha+r)_n}I_{n+r;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w)t^n$$

$$=(1-t)^{-\rho-r}\binom{\alpha+r}{r}$$

$$F^{(3)}\left[\begin{array}{l}(a_p)::\frac{x}{w}-\mu+1;--;--; \\ (b_q)::1+\alpha;--;--;\end{array}\right.$$

$$\left.\begin{array}{l}\rho+r;-r;\frac{-x}{w}+\lambda \\ --;--;--;\end{array} -\frac{wt}{1-t},w,w\right] \tag{1.4}$$

where $F^{(3)}$ is generalized hypergeometric functions of three variables (see Srivastava and Karlsson [11]).
In particular for $\lambda=0$ and $\mu=1$, we have

$$I_{n;0;(a_p)}^{\alpha;1;(a_p)}(x,w)=(1-w)^{\frac{x}{w}}J_n^{\alpha}(x,w), \tag{1.5}$$

where $J_n^{\alpha}(x,w)$ is modified Jacobi polynomial (see Parihar and Patel [6]) and

$$\lim_{w\to 0}I_{n;\lambda;(a_p)}^{\alpha;\mu;(a_p)}(x,w)=e^{-x}L_n^{\alpha}(x), \tag{1.6}$$

where $L_n^{\alpha}(x)$ is Laguerre polynomial [8].
The following definitions and results given by Rainville [8 ,

p.302] Gottlieb polynomial

$$\phi_n(x;\lambda) = e^{-n\lambda}\, {}_2F_1(-n,-x;1;1-e^{\lambda}), \tag{1.7}$$

Generalized Sylvester polynomial

$$f_n(x;a) = \frac{(ax)^n}{n!}\, {}_2F_0(-n,x;-;-\frac{1}{ax}) \tag{1.8}$$

Agarwal and Manocha [1, p.1372 (2.2)(5.5)]

$$\sum_{n=0}^{\infty}\binom{n+k}{k}\phi_{n+k}(x;\lambda)t^n = (1-t)^{x-k} \times (1-te^{-\lambda})^{-x-1}\, \phi_k\left(x;\log_e\left(\frac{e^{\lambda}-t}{1-t}\right)\right) \tag{1.9}$$

and

$$\sum_{n=0}^{\infty}\binom{n+k}{k} f_{n+k}(x;a)t^n = (1-t)^{-x-k} \times e^{axt} f_k(x;a(1-t)) \tag{1.10}$$

2. Main Results

Bilateral generating relations

We have derived the following bilateral generating relations for the class of generalized hypergeometric functions $B_n^{(\alpha,\beta)}(x,y,v)$:

$$\sum_{n=0}^{\infty}\frac{(\rho)_n(n!)^2}{\left[(1+\alpha)_n\right]^2(1+\beta)_n}\times B_n^{(\alpha,\beta)}(x,y,v)\, I_{n;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w)t^n$$

$$= (1-t)^{-\rho} F_{q+2:0;0;0;1;1}^{p+2:0;0;1;2;1}\left[\begin{matrix}[(a_p):1,1,1,0,1], [\rho:1,1,0,1,1],[\frac{x}{w}-\mu+1:1,1,0,0,1]: \\ [(b_q):1,1,1,0,1], [1+\alpha:1,1,0,0,1],[\rho:0,1,0,1,1,]:\end{matrix}\right.$$

$$\left.\begin{matrix}-;-;-\frac{x}{w}+\lambda;\rho,-\frac{y}{v};\frac{x}{v}; \\ -;-;-;1+\beta;1+\alpha;\end{matrix}\ \frac{-wt}{1-t},-w,w,\frac{vt}{1-t},wv\right] \tag{2.1}$$

Where F is a generalized Lauricella hypergeometric function of 5 variables.

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(1+\alpha)_n(1+\beta)_n}B_n^{(\alpha,\beta)}(x,y,v)\times \phi_n(x;\lambda)t^n$$

$$=(1-t)^x(1-te^{-\lambda})^{-x-1}\times F_{2:0;0;1}^{3:0;0;1}\left[\begin{matrix}[1:1,1,1],[-\frac{y}{v}:1,1,0],[-x:1,0,1]: & -;-;x/v; \\ [1+\beta:1,1,1],\ [1:1,0,1]\ : & -;-;1+\alpha;\end{matrix}\ t_1,t_2,t_3\right] \tag{2.2}$$

Where F is a generalized Lauricella hypergeometric function of 3 variables.

$$t_1 = \frac{v(e^{\lambda}-1)t}{(e^{\lambda}-t)(1-t)},\ t_2 = \frac{vt}{e^{\lambda}-t}$$

$$t_3 = \frac{v^2\left(e^{\lambda}-1\right)t}{\left(e^{\lambda}-1\right)(1-t)}.$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(1+\alpha)_n(1+\beta)_n}B_n^{(\alpha,\beta)}(x,y,v)\times f_n(x;a)t^n = (1-t)^{-x}e^{axt}F_{1:0;0;1}^{2:0;0;1}\left[\begin{matrix}[\frac{-y}{v}:1,1,1], [x:1,0,1]:-;-;\frac{x}{v}; \\ [1+\beta:1,1,1], - \quad :-;-;1+\alpha;\end{matrix}\ \left(\frac{vt}{1-t}\right)^n, vtax, \left(\frac{-v^2t}{1-t}\right)\right] \tag{2.3}$$

Proof of (2.1). From (2.1), we have

$$\sum_{n=0}^{\infty}\frac{(\rho)_n(n!)^2}{\left[(1+\alpha)_n\right]^2(1+\beta)_n}\times B_n^{(\alpha,\beta)}(x,y,v)\, I_n^{\alpha}(x,w)t^n$$

$$=\sum_{n=0}^{\infty}\sum_{s=0}^{n-r}\sum_{r=0}^{n}\frac{(\rho)_n}{(1+\alpha)_n}\frac{(-n)_{r+s}\left(-\frac{y}{v}\right)_r}{(1+\alpha)_s(1+\beta)_r}\times$$

$$\frac{\left(\frac{x}{v}\right)_s}{r!}\frac{(-v)^r v^s}{s!}I_n^{\alpha}(x,w)t^n$$

$$=\sum_{s=0}^{n}\sum_{r=0}^{\infty}\frac{(-n)_s(\rho)_r\left(-\frac{y}{v}\right)_r\left(\frac{x}{v}\right)_s(vt)^r v^s}{(1+\alpha)_r(1+\alpha)_s(1+\beta)_r s!}\times$$

$$\sum_{n=0}^{\infty}\binom{n+r}{r}\frac{(\rho+r)_n}{(1+\alpha+r)_n}I_{n+r}^{\alpha}(x,w)t^n$$

By using (1.4), we get

$$=\sum_{s=0}^{n}\frac{(-n)_s\left(\frac{x}{v}\right)_s v^s}{(1+\alpha)_s s!}\sum_{r=0}^{\infty}\frac{\left(-\frac{y}{v}\right)_r(vt)^r(\rho)_r}{(1+\alpha)_r(1+\beta)_r}$$

$$(1-t)^{-\rho-r}\binom{\alpha+r}{r}F^{(3)}\left[\begin{matrix}(a_p)::\\(b_q)::\end{matrix}\right.$$

$$\left.\begin{matrix}\frac{x}{w}-\mu+1;-;-;\rho+r;-r;-\frac{x}{w}+\lambda;\frac{-wt}{1-t},w,w\\1+\alpha\ ;-;-;\ -\ ;-;\ \ -\ ;\end{matrix}\right]$$

$$=(1-t)^{-\rho}\sum_{m,k,n,r,s=0}^{\infty}\frac{(a_p)_{m+n+k+s}}{(b_q)_{m+n+k+s}}\times$$

$$\frac{(\rho)_{m+n+r+s}\left(\frac{x}{w}-\mu+1\right)_{m+n+s}}{(1+\alpha)_{m+n+s}(\rho)_{n+r+s}}\times$$

$$\frac{\left(-\frac{x}{w}+\lambda\right)_k\left(-\frac{y}{v}\right)_r(\rho)_r\left(\frac{x}{v}\right)_s}{(1+\alpha)_s(1+\beta)_r}\times$$

$$\left(\frac{-wt}{1-t}\right)^m\frac{(-w)^n w^k}{m!n!k!}\frac{\left(\frac{vt}{1-t}\right)^r}{r!}\frac{(wv)^s}{s!}$$

$$=(1-t)^{-\rho}F_{q+2:0;0;0;1;1}^{p+2:0;0;1;2;1}\left[\begin{matrix}[(a_p):1,1,1,0,1],\\ [(b_q):1,1,1,0,1],\end{matrix}\right.$$

$$[\rho:1,1,0,1,1],[\frac{x}{w}-\mu+1:1,1,0,0,1]:$$
$$[1+\alpha:1,1,0,0,1],[\rho:0,1,0,1,1,]:$$
$$-;-;-\frac{x}{w}+\lambda;\rho,-\frac{y}{v};\frac{x}{v};$$
$$-;-;-;1+\beta;1+\alpha;$$
$$\left.\frac{-wt}{1-t},-w,w,\frac{vt}{1-t},wv\right]$$

Hence complete the proof of (2.1).

Applications.

(i) By setting p = q, $a_j = b_j$, j = 1, 2… p, $\mu = 1$ and $\lambda = 0$ in (2.1), we get

$$\sum_{n=0}^{\infty}\frac{(\rho)_n(n!)^2}{\left[(1+\alpha)_n\right]^2(1+\beta)_n}\times$$
$$B_n^{(\alpha,\beta)}{}_{(x,y,v)}J_n^{\alpha}(x,w)t^n=(1-t)^{-\rho}\times$$
$$F_{2:0;0;1;1}^{2:0;0;2;1}\left[\begin{matrix}[\rho:1,1,1,1],[\frac{x}{w}:1,1,0,1]:\\ [1+\alpha:1,1,0,1],[\rho:0,1,1,1,]:\end{matrix}\right.$$
$$\left.\begin{matrix}-;-;\rho,-\frac{y}{v};\frac{x}{v};\\-;-;1+\beta;1+\alpha;\end{matrix}\ \frac{-wt}{1-t},-w,\frac{vt}{1-t},wv\right] \tag{2.4}$$

(ii) Applying p = q, $a_j = b_j$, j = 1, 2, …, p and writting $w \to 0$ in (2.1), we get

$$\sum_{n=0}^{\infty}\frac{(\rho)_n(n!)^2}{\left[(1+\alpha)_n\right]^2(1+\beta)_n}B_n^{(\alpha,\beta)}{}_{(x,y,v)}L_n^{\alpha}(x)t^n$$
$$=(1-t)^{-\rho}F_{2:0;0;1;1}^{1:0;0;2;1}\left[\begin{matrix}[\rho:1,1,1,1]\qquad :\\ [1+\alpha:1,1,0,1],[\rho:0,1,1,1,]:\end{matrix}\right.$$
$$\left.\begin{matrix}-;-;\rho,-\frac{y}{v};\frac{x}{v};\\-;-;1+\beta;1+\alpha;\end{matrix}\ \frac{-xt}{1-t},-x,\frac{vt}{1-t},xv\right] \tag{2.5}$$

(iii) On taking $v \to 0$ in (2.1), we get

$$\sum_{n=0}^{\infty}\frac{(\rho)_n(n!)^2}{\left[(1+\alpha)_n\right]^2(1+\beta)_n}\times$$
$$L_n^{(\alpha,\beta)}{}_{(x,y)}I_{n;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w)t^n$$

$$= (1-t)^{-\rho} F^{p+2:0;0;1;1;0}_{q+2:0;0;0;1;1}\left[\begin{array}{l}[(a_p):1,1,1,0,1],\\ [(b_q):1,1,1,0,1],\end{array}\right.$$

$$[\rho:1,1,0,1,1],[\frac{x}{w}-\mu+1:1,1,0,0,1]:$$

$$[1+\alpha:1,1,0,0,1],[\rho:0,1,0,1,1,]:$$

$$\left.\begin{array}{l}-;-;-\dfrac{x}{w}+\lambda;\rho;-;\\ -;-;-;1+\beta;1+\alpha;\end{array}\frac{-wt}{1-t},-w,w,\frac{-yt}{1-t},wx\right] \quad (2.6)$$

(iv) By writing $\mu = 1$, $\lambda = 0$, p = q and $a_j = b_j$, j = 1, 2, …, p and letting $v \to 0$, in (2.1), we have (2.6)

$$\sum_{n=0}^{\infty}\frac{(\rho)_n (n!)^2}{\left[(1+\alpha)_n\right]^2 (1+\beta)_n}\times$$

$$L_n^{(\alpha,\beta)}(x,y)\, J_n^{\alpha}(x,w)t^n$$

$$= (1-t)^{-\rho} F^{2:0;0;1;0}_{2:0;0;1;1}\left[\begin{array}{l}[\rho:1,1,1,1],\\ [1+\alpha:1,1,0,1],\end{array}\right.$$

$$[\frac{x}{w}:1,1,0,1]:-;-;\rho;-;$$

$$[\rho:0,1,1,1]:-;-;1+\beta;1+\alpha;$$

$$\left.\frac{-wt}{1-t},-w,\frac{yt}{1-t},wx\right]$$

(v) Taking $v \to 0$, $w \to 0$, p = q and $a_j = b_j$, j = 1, 2, …, p in (2.1), we get

$$\sum_{n=0}^{\infty}\frac{(\rho)_n (n!)^2}{\left[(1+\alpha)_n\right]^2 (1+\beta)_n} L_n^{(\alpha,\beta)}(x,y)\, L_n^{\alpha}(x)t^n$$

$$= (1-t)^{-\rho} F^{1:0;0;1;0}_{2:0;0;1;1}\left[\begin{array}{l}[\rho:1,1,1,1] \quad :\\ [1+\alpha:1,1,0,1],[\rho:0,1,1,1]:\end{array}\right.$$

$$\left.\begin{array}{l}-;-;\rho;-;\\ -;-;1+\beta;1+\alpha;\end{array}\frac{-xt}{1-t},-x,\frac{yt}{1-t},x^2\right] \quad (2.7)$$

(vi) Taking $\beta = 0$, y = 0 in (2.7), we get

$$\sum_{n=0}^{\infty}\frac{(\rho)_n (n!)}{\left[(1+\alpha)_n\right]^2} L_{n(x)}^{\alpha}\, L_n^{\alpha}(x)t^n$$

$$= (1-t)^{-\rho} F^{1:0;0;0}_{2:0;0;1}\left[\begin{array}{l}[\rho:1,1,1] \quad :\\ [1+\alpha:1,1,1],[\rho:0,1,1]:\end{array}\right.$$

$$\left.\begin{array}{l}-;-;-;\\ -;-;1+\alpha;\end{array}\frac{-xt}{1-t},-x,x^2\right] \quad (2.8)$$

These are all the bilateral (bilinear) generating relations for the class of generalized hypergeo-metric functions (1.1), whereas the results for the modified Jacobi polynomial, Laguerre poly-nomial of two variables and Laguerre poly-nomial are believed to be new.

Proof of (2.2). The result (2.2) can also be deduced by using (1.9) and the same techniques as followed in the previous result.

(i). By letting limit $v \to 0$ in (2.2), we obtain

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(1+\alpha)_n (1+\beta)_n} L_n^{(\alpha,\beta)}(x,y)\, \phi_n(x;\lambda)t^n$$

$$= (1-t)^{x}(1-te^{-\lambda})^{-x-1} F^{2:0;0;0}_{2:0;0;1}\left[\begin{array}{l}[1:1,1,1],\\ [1+\beta:1,1,1],\end{array}\right.$$

$$\left.\begin{array}{l}[-x:1,0,1]:-;-;-;\\ [1:1,0,1] \ :-;-;1+\alpha;\end{array}t_1,t_2,t_3\right] \quad (2.9)$$

where

$$t_1 = \frac{y(e^{\lambda}-1)t}{(e^{\lambda}-t)(1-t)},\ t_2 = \frac{-yt}{e^{\lambda}-t},$$

$$t_3 = \frac{yx\left(e^{\lambda}-1\right)t}{\left(e^{\lambda}-1\right)(1-t)}.$$

is the bilateral generating relation for the Laguerre polynomial of two variables, which is believed to be new.

Proof of (2.3). The result (2.3) can also be deduced by using (1.10) and the same techniques as followed in the result (2.1).

Application. Applying limit $v \to 0$ on (2.3), we obtain the result

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(1+\alpha)_n (1+\beta)_n} L_n^{(\alpha,\beta)}(x,y) f_n(x;a)t^n$$

$$= (1-t)^{-x} e^{axt} F^{1:0;0;0}_{1:0;0;1}\left[\begin{array}{l}[x:1,0,1]:\\ [1+\beta:1,1,1],- \ :\end{array}\right.$$

$$\left.\begin{array}{l}-;-;\dfrac{x}{v} \quad ;\\ -;-;1+\alpha;\end{array}\left(\frac{-yt}{1-t}\right)^n, ytax, \left(\frac{-xyt}{1-t}\right)\right]$$

is the bilateral generating relation for the Laguerre polynomial of two variables, which is believed to be new.

3. Conclusion

By employing the technique used in the proof of (2.1) and adjusting the parameters one can easily get the bilateral generating relations.

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