RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Tomo XLVIII (1999), pp. 591-596

GENERATING FUNCTIONS OF THE INCOMPLETE FIBONACCI AND LUCAS NUMBERS

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For the incomplete Fibonacci and incomplete Lucas numbers, which were introduced and studied recently by P. Filliponi [*Rend. Circ. Math. Palermo* (2) **45** (1996), 37-56], the authors derive two classes of generating functions in terms of the familiar Fibonacci and Lucas numbers, respectively.

1. Introduction and the Main Result.

The incomplete Fibonacci and incomplete Lucas numbers were introduced recently by Filipponi [1]. The incomplete Fibonacci numbers $F_n(k)$ are defined by

$$(1) \qquad F_n(k) = \sum_{j=0}^{k} \binom{n-1-j}{j} \left(n = 1, 2, 3, \ldots; 0 \leq k \leq \left[\frac{n-1}{2} \right] \right)$$

and the incomplete Lucas numbers $L_n(k)$ are defined by

$$(2) \qquad L_n(k) = \sum_{j=0}^{k} \frac{n}{n-j} \binom{n-j}{j} \left(n = 1, 2, 3, \ldots; 0 \leq k \leq \left[\frac{n}{2} \right] \right),$$

where $[s]$ denotes the greatest integer in s.

It is easy to see that

$$F_n\left(\left[\frac{n-1}{2}\right]\right)=F_n \quad \text{and} \quad L_n\left(\left[\frac{n}{2}\right]\right)=L_n,$$

where F_n and L_n are the nth Fibonacci and Lucas number, respectively. The purpose of this note is to derive the generating functions for these classes of numbers.

THEOREM Let k be a natural number. Then

$$R_k(t):=\sum_{j=0}^{\infty} F_k(j)t^j = t^{2k+1}\frac{(F_{2k}+F_{2k-1}t)(1-t)^{k+1}-t^2}{(1-t)^{k+1}(1-t-t^2)}, \tag{3}$$

and

$$S_k(t):\sum_{j=0}^{\infty} L_k(j)t^j = t^{2k}\frac{(L_{2k-1}+L_{2k-2}t)(1-t)^{k+1}-t^2(2-t)}{(1-t)^{k+1}(1-t-t^2)}. \tag{4}$$

In the trivial case when $k=0$, we have

$$F_0(j)=L_0(j)=1 \ (j\in\mathbb{N}_0:=\{0,1,2,\ldots\}),$$

which immediately yields

$$R_0(t)=S_0(t)=\frac{1}{1-t}.$$

2. Proof of the Theorem.

The proof of our Theorem is based upon the following

LEMMA *Let* $\{s_n\}_{n=0}^{\infty}$ *be a complex sequence satisfying the non-homogeneous second-order recurrence relation:*

$$s_n = as_{n-1}+bs_{n-2}+r_n, \ (n\in\mathbb{N}\backslash\{1\};\mathbb{N}:=\mathbb{N}_0\backslash\{0\}), \tag{5}$$

where $a, b \in \mathbb{C}$ and $r_n : \mathbb{N} \to \mathbb{C}$ is a given sequence. Then the generating function $U(t)$ of s_n is

$$U(t) = \frac{G(t) + s_0 - r_0 + (s_1 - s_0 a - r_1)t}{1 - at - bt^2} \tag{6}$$

where $G(t)$ denotes the generating function of r_n.

Proof. Indeed, by using a fairly standard technique, we obtain

$$U(t) = s_0 + s_1 t + s_2 t^2 + \cdots + s_n t^n + \cdots, \tag{7}$$

$$atU(t) = s_0 at + s_1 at^2 + \cdots + as_{n-1} t^n + \cdots, \tag{8}$$

$$bt^2 U(t) = s_0 bt^2 + \cdots + bs_{n-2} t^n + \cdots, \tag{9}$$

and

$$G(t) = r_0 + r_1 t + r_2 t^2 + \cdots + r_n t^n + \cdots. \tag{10}$$

Therefore

$$U(t)(1 - at - bt^2) - G(t) = s_0 - r_0 + (s_1 - s_0 a - r_1)t, \tag{11}$$

which completes the proof of the Lemma.

In the sequel, k is a fixed positive integer. It is known (see [1]) that

$$F_n(k) = 0 \text{ if } 0 \le n < 2k+1, F_{2k+1}(k) = F_{2k}, \text{ and } F_{2k+2}(k) = F_{2k+1},$$

and that

$$F_n(k) = F_{n-1}(k) + F_{n-2}(k) - \binom{n-3-k}{n-3-2k} \text{ if } n \ge 2k+3. \tag{12}$$

Set

$$s_0 = F_{2k+1}(k),\ s_1 = F_{2k+2}(k), \text{ and } s_n = F_{n+2k+1}(k)\ (n \in \mathbb{N}\backslash\{1\}).$$

Also let

$$r_0 = r_1 = 0 \text{ and } r_n = \binom{n-2+k}{n-2}.$$

The generating function of the sequence r_n is $t^2/(1-t)^{k+1}$ (cf. [2, p. 349, Problem 216] or [3, p. 355, Equation 7.1(5)]). Thus the generating function $U_k(t)$ of the sequence $\{s_n\}_{n=0}^{\infty}$ satisfies the equation:

$$U_k(t)(1-t-t^2)+\frac{t^2}{(1-t)^{k+1}}=F_{2k}+(F_{2k+1}-F_{2k})t. \tag{13}$$

Finally, the generating function $R_k(t)$ of $\{F(k)\}_{n=0}^{\infty}$ is $t^{2k+1}U_k(t)$.

For the following facts we also refer to [1]:

$$L_n(k)=0 \quad \text{if} \quad n<2k, \quad L_{2k}(k)=L_{2k-1}, \quad \text{and} \quad L_{2k+1}(k)=L_{2k},$$

and (in general)

$$\begin{aligned} L_n(k) &= L_{n-1}(k)+L_{n-2}(k) \\ &\quad -\frac{n-2}{n-2-k}\binom{n-2-k}{n-2-2k} \quad (n \geq 2k+2). \end{aligned} \tag{14}$$

Put

$$s_0=L_{2k}(k), \; s_1=L_{2k+1}(k), \quad \text{and} \quad s_n=L_{2k+n}(k) \; (n \in \mathbb{N}\setminus\{1\}).$$

Also set

$$r_0=r_1=0 \quad \text{and} \quad r_n=\frac{n-2+2k}{n-2+k}\binom{n-2+k}{n-2} \; (n \in \mathbb{N}\setminus\{1\}). \tag{15}$$

Then the generating function of r_n is (see [2, p. 348, Problem 212] or [3, p. 355, Equation 7.1 (9)])

$$\frac{t^2(2-t)}{(1-t)^{k+1}}, \tag{16}$$

and the generating function $U_k(t)$ of the sequence $\{s_n\}_{n=0}^{\infty}$ satisfies the equation:

$$\begin{aligned} U_k(t)(1-t-t^2)+\frac{t^2(2-t)}{(1-t)^{k+1}} &= L_{2k-1}+(L_{2k}-L_{2k-1})t \\ &= L_{2k-1}+L_{2k-2}t. \end{aligned} \tag{17}$$

Remark. Since (see [1, p. 47, Equation (3.8)]

$$L_n(k) = F_{n-1}(k-1) + F_{n+1}(k) \quad \left(0 \le k \le \left[\frac{n}{2}\right]\right), \tag{18}$$

we can give an alternative proof for the generating function (4) of the incomplete Lucas numbers. From the equation (18) above, we have

$$\sum_{n=0}^{\infty} L_n(k)t^n = \sum_{n=0}^{\infty} F_{n-1}(k-1)t^n + \sum_{n=0}^{\infty} F_{n+1}(k)t^n \tag{19}$$

and

$$S_k(t) = tR_{k-1}(t) + \frac{1}{t}R_k(t). \tag{20}$$

Applying the well-known identity:

$$F_{n-2} + F_n = L_{n-1} \; (n \in \mathbb{N}\backslash\{1\}) \tag{21}$$

our alternative proof of (4) is complete again.

Acknowledgments.

The present investigation was carried out during the first-named author's visit to the University of Victoria in the second half of the academic year 1996-1997. This work was supported, in part, by Eötvös Fellowship of Hungary (Grants 16975 and 19479 from HNFSR) and, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353.

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Pervenuto il 15 maggio 1998.

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