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BILATERAL GENERATING FUNCTION FOR A FUNCTION DEFINED BY GENERALIZED RODRIGUE'S FORMULA

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Earlier the authors (Patil and Thakare 1975*a*) have introduced the generalized function $P_n^\alpha(x;\ r,\ s,\ p,\ k,\ \lambda)$ defined by

$P_n^\alpha(x;r,s,p,k,\lambda) - x^\alpha \cdot e^{px^r} \cdot \theta^n(x^{\alpha+sn}e^{-px^r})$.

where $\theta = \lambda x^k + x^{k+1} \cdot D$.

In the present paper we obtain a bilateral generating function for the same which reduces as a special case a large variety of bilinear and bilateral generating function for the classical orthogonal polynomials.

1. INTRODUCTION

In a recent paper we (Patil and Thakare 1975*a*) have introduced the generalized function $P_n^\alpha(x;r,s,p,k,\lambda)$ defined by

$$P_n^\alpha(x;\ r,s,p,k,\lambda) = x^\alpha \cdot e^{px^r} \cdot \theta^n(x^{\alpha+sn} \cdot e^{-px^r}), \qquad \ldots(1\cdot 1)$$

where

$$\theta = \lambda x^k + x^{k+1} \cdot D, \qquad D = \frac{d}{dx}. \qquad \ldots(1\cdot 2)$$

The generalized function (1·1) provides as a special case, almost all the classical and non-classical polynomials. Some of which we give below:

(A) $$G_n^\alpha(x;\ r,\ p,k) = \frac{1}{n!}P_n^{\alpha+kn}(x;\ r,\ -k,p,k,0),$$

where $G_n^\alpha(x;\ r,p,k)$ are the polynomials of Srivastava and Singhal (1971).

(B) $$F_n^\alpha(x;\ r,s,p) = P_n^\alpha(x;\ r,s,p,-1,0),$$

where $F_n^\alpha(x;\ r,s,p)$ are the polynomials defined by Chatterjea (1966).

(C) $$F_n^{(r,m)}(x;\ a,k,p) = P_n^\alpha(x;\ r,s,p,k-1,\ 0),$$

where $F_n^{(r,m)}(x;\ a,k,p)$ are the polynomials of Shrivastava (1972).

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Similarly the generalized Truesdell polynomials, Generalized Laguerre polynomials, Hermite polynomials, Bessel polynomials and Humbert polynomials etc., are also special cases of our function defined by (1·1) with (1·2) holding.

In the previous papers we have obtained the operational formulas, generating functions and explicit form for the function defined by (1.1) (see Patil and Thakare 1975*b*, 1976). The present note is concerned only with more general generating function and a bilateral generating function for the function (1·1).

2. Generating Function

From (1·1) we write,

$$\sum_{n=0}^{\infty} \frac{1}{m\,!\,n\,!}\, P_{m+n}^{\alpha-s\,(m+n)}(x;\ r, s, p, k, \lambda)\cdot t^n$$

$$=\sum_{n=0}^{\infty} \frac{x^{-\alpha+s\,(m+n)}}{m\,!\,n\,!}\cdot e^{px^r}\cdot \theta^{n+m}(x^\alpha\cdot e^{-px^r})\cdot t^n$$

$$=\frac{x^{-\alpha+sm}\cdot e^{px^r}}{m\,!}\sum_{n=0}^{\infty}\frac{x^{sn}\cdot t^n\cdot\theta^n}{n\,!}\,\theta^m(x^\alpha\cdot e^{-px^r}).$$

Again by using (1·1) and the property of our operator (1·2) (see Patil and Thakare 1975*a*).

$$e^{t\theta}\{x^\alpha\cdot f(x)\}=\frac{x^\alpha}{(1-tkx^k)^{(\alpha+\lambda)/k}}\cdot f\left\{\frac{x}{(1-tkx^k)^{1/k}}\right\}.$$

We get the required generating function:

$$\sum_{n=0}^{\infty} \frac{1}{m\,!\,n\,!}\, P_{m+n}^{\alpha-s\,(m+n)}(x;\ r, s, p, k, \lambda)\cdot t^n$$

$$=\frac{1}{m\,!}(1-tk\,x^{s+k})^{-(\alpha+\lambda-sm)/k}\cdot \exp.[px^r\{1-(1-tkx^{s+k})^{-r/k}\}]$$

$$\times P_m^{\alpha-sm}(x\,(1-tk\,x^{s+k})^{-1/k};\ r, s, p, k, \lambda) \qquad ...(2\cdot 1)$$

which can be put in the form:

$$\sum_{n=0}^{\infty} P_{m+n}^{\alpha-sn}(x;\ r, s, p, k, \lambda)\cdot\frac{t^n}{n\,!}$$

$$=(1-tkx^{s+k})^{-(\alpha+\lambda)k}\cdot \exp[px^r\{1-(1-tkx^{s+k})^{-r/k}\}]$$

$$\times P_m^{\alpha}(x\,(1-tkx^{s+k})^{-1/k},\ r, s, p, k, \lambda) \qquad ...(2\cdot 2)$$

where m is any integer $\geqslant 0$.

Particular Cases

On specialization of parameters we obtain the generating functions. Here we record only three cases:

(I) Putting $m = 0$ in (2·2) we get the generating function obtained by Patil and Thakare (1976*b*),

$$\sum_{n=0}^{\infty} P_n^{\alpha-sn}(x;\ r, s, p, k\ \lambda) \cdot \frac{t^n}{n\,!}$$
$$= (1 - tkx^{s+k})^{-(\alpha+\lambda)/k} \cdot \exp\left[px^r \left\{1 - (1 - tkx^{s+k})^{-r/k}\right\}\right].$$

(II) Put $s = -k$, $\lambda = 0$ and by (A) we get the generating function obtained by Srivastava and Singhal (1971),

$$\sum_{n=0}^{\infty} \binom{m+n}{n} \cdot G^{\alpha}_{m+n}(x;\ r, p, k)\, t^n$$
$$= (1 - tk)^{-(\alpha+km)/k} \cdot \exp\left[px^r \left\{1 - (1 - tk)^{-r/k}\right\}\right].$$
$$\times\ G^{\alpha}_m\left(x(1 - tk)^{-1/k};\ r, p, k\right).$$

(III) Put $\lambda = s = 0$ and $k \to 0$ and using the result

$$\lim_{b\to\infty}\left(1 - \frac{t}{b}\right)^b = e^{-t},$$

we get generating function obtained by Singh (1967),

$$\sum_{n=0}^{\infty} \frac{1}{n\,!} T^{\alpha}_{m+n}(x; r, p) \cdot t^n = \exp\left[\alpha t + px^r(1 - e^{rt})\right] \cdot T^{\alpha}_m(xe^t;\ r, p).$$

3. Bilinear Generating Function for $\phi_n(y)$

Let $\phi_n(y)$ be a polynomial of degree n in y given by

$$\phi_n(y) = \frac{1}{n\,!} \sum_{j=0}^{n} \binom{n}{j} \cdot \mu_j y^j, \qquad \ldots(3\cdot 1)$$

where $\mu_j \neq 0$ are arbitrary constants.

Consider,

$$\sum_{n=0}^{\infty} P_n^{\alpha-sn}(x;\ r, s, p, k, \lambda) \cdot \phi_n(y) \cdot t^n$$
$$= \sum_{n=0}^{\infty} P_n^{\alpha-sn}(x;\ r, s, p, k, \lambda) \cdot \frac{1}{n\,!} \sum_{j=0}^{n} \binom{n}{j} \cdot \mu_j y^j\, t^n$$

$$= \sum_{j=0}^{\infty} \mu_j (yt)^j \sum_{n=0}^{\infty} \binom{n+j}{j} \cdot \frac{1}{(n+j)!} \cdot P_{n+j}^{\alpha-s(n+j)} (x;\ r, s, p, k, \lambda)\, t^n,$$

$$= \sum_{j=0}^{\infty} \mu_j (yt)^j \frac{1}{j!} (1 - tk\, x^{s+k})^{-(\alpha+\lambda-sj)/k}\,.$$

$$\times \exp [px^r \{ 1 - (1 - tkx^{s+k})^{-r/k}\}].$$

$$\times P_j^{\alpha-sj} (x (1 - tk\, x^{s+k})^{-1/k}\ ;\ \ r, s, p, k, \lambda).$$

Hence we write

$$\sum_{n=0}^{\infty} P_n^{\alpha-sn} (x;\ \ r, s, p, k, \lambda) \cdot \phi_n (y) \cdot t^n$$

$$= (1 - tkx^{s+k})^{-(\alpha+\lambda)/k} \cdot \exp [p\, x^r \{1 - (1 - tkx^{s+k})^{-r/k}\}].$$

$$\times F (x (1 - tk\, x^{s+k})^{-1/k},\ \ yt (1 - tkx^{s+k})^{s/k}), \qquad \ldots(3.2)$$

where

$$F(x, t) = \sum_{n=0}^{\infty} \frac{1}{n!} \mu_n P_n^{\alpha-sn} (x;\ \ r, s, p, k, \lambda)\, t^n.$$

Particular Cases

(I) If we put $s = -k$, $\lambda = 0$ in (3.2) and because of (A) we get bilinear generating function obtained by Srivastava and Singhal (1971),

$$\sum_{n=0}^{\infty} n\,!\, G_n^{\alpha} (x;\ \ r, p, k) \cdot \phi_n (y) \cdot t^n$$

$$= (1 - tk)^{-\alpha/k} \cdot \exp [px^r \{1 - (1 - tk)^{-r/k}\}].$$

$$\times\ F [x (1 - tk)^{-1/k},\ \ yt (1 - tk)^{-1}],$$

where

$$F(x, t) = \sum_{n=0}^{\infty} \mu_n G_n^{\alpha}(x;\ \ r, p, k) \cdot t^n$$

(II) Put $\lambda = 0$ and $k = -1$ (3.2) and by (B) we write the bilinear generating function for the polynomials defind by Chatterjea (1966) as :

$$\sum_{n=0}^{\infty} F_n^{\alpha-sn} (x;\ \ r, s, p) \cdot \phi_n (y) \cdot t^n$$

$$= (1 + tx^{s-1})^{\alpha} \cdot \exp \cdot [px^r \{1 - (1 + tx^{s-1})^r\}]$$

$$\times F [x (1 + tx^{s-1}),\ \ yt (1 + tx^{s-1})^{-s}]. \qquad \ldots(3.3)$$

where

$$F(x, t) = \sum_{n=0}^{\infty} \mu_n F_n^{\alpha-sn}(x; \; r, s, p) \cdot t^n .$$

The result (3·3) seems to be new.

Similarly, as an application of (3.2) by assigning suitable values to the arbitrary coefficients μ_n, we can obtain a large variety of bilateral and also bilinear generating functions for the classical orthogonal polynomials.

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