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An Extension of Bilateral Generating Function of Certain Special Function-II

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Abstract

In this paper an extension of the bilateral generating function involving Jacobi polynomial derived by Chongdar [2] is well presented by group-theoretic method [6]. A nice application of our theorem is also pointed out.

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1. Introduction

While extending the general theorem on bilateral generating function for Jacobi polynomial we use the term "Quasi Bilinear Generating Function" [6] for Jacobi polynomial by means of the relation

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) P_m^{(n,\beta)}(u) w^n \tag{1}$$

and we prove the existence of a quasi bilinear generating function implies the existence of a more general generating function. In [2], A. K. Chongar proved the following theorem on bilateral generating function involving Jacobi polynomial as inroduced by G.K. Goyal [5].

Theorem 1 *If*

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) w^n \tag{2}$$

then

$$\sum_{n=0}^{\infty} \sigma_n(x,y) w^n = (1-w)^k (1-\omega)^{-(1+\alpha+k)} P\left(\frac{x-\omega}{1-\omega}, \frac{wy}{1-\omega}\right). \tag{3}$$

where

$$\sigma_n(x,y) = \sum_{p=0}^{n} a_p \frac{(p+1)_{n-p}}{(n-p)!} P_n^{(\alpha,k-n+p)}(x)\, y^p \qquad \textit{and} \qquad \omega = \frac{w}{2}(1+x)$$

Now we can state from the above discussion that the "Theorem-1" of [2] can be extended in the following way:

Theorem 2 *If there exists a generating relation of the form*

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) P_m^{(n,\beta)}(u) w^n \tag{4}$$

then

$$(1-wt)^{-(1+\beta+m)}(y+2wz)^{\beta}(y+2\omega z)^{-(1+\alpha+\beta)} y^{\alpha+1} G\left(\tfrac{xy+2\omega z}{y+2\omega z}, \frac{u+wt}{1-wt}, \frac{wztv}{(1-wt)(y+2\omega z)}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{p+q}}{p!q!}(n+1)_q (wzvt)^n (-2y^{-1}z)^q t^p (1+n+\beta+m)_p P_{n+q}^{(\alpha.\beta-q)}(x) P_m^{(n+p,\beta)}(u) \tag{5}$$

The importance of our theorem is that one can get a large number of bilinear generating relations from (5) by attributing different suitable values to a_n in (4).

Proof of the theorem

Let us consider now the following two linear partial differential operators [3, 4],

$$R_1 = (1-x^2)y^{-1}z\frac{\partial}{\partial x} - z(x-1)\frac{\partial}{\partial y} - (1+x)y^{-1}z^2\frac{\partial}{\partial z} - (1+\alpha)(1+x)y^{-1}z \tag{6}$$

and

$$R_2 = (1+u)t\frac{\partial}{\partial u} + t^2\frac{\partial}{\partial t} + (1+\beta+m)t \tag{7}$$

such that

$$R_1\left(P_n^{(\alpha,\beta)}(x)y^{\beta}z^n\right) = -2(n+1)P_{n+1}^{(\alpha,\beta-1)}(x)y^{\beta-1}z^{n+1} \tag{8}$$

and

$$R_2\left(P_n^{(n,\beta)}(u)t^n\right) = (1+n+\beta+m)P_m^{(n+1,\beta)}(u)t^{n+1} \tag{9}$$

and also

$$e^{wR_1}f(x,y,z) = \left(\frac{y}{y+2\omega z}\right)^{\alpha+1} f\left(\frac{xy+2\omega z}{y+2\omega z}, \frac{y(y+2wz)}{y+2\omega z}, \frac{yz}{y+2\omega z}\right) \tag{10}$$

and

$$e^{wR_2}f(u,t) = (1-wt)^{-(1+\beta+m)} f\left(\frac{u+wt}{1-wt}, \frac{t}{1-wt}\right). \tag{11}$$

Now we consider the following generating relation

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) P_m^{(n,\beta)}(u) w^n. \tag{12}$$

Replacing w by $wztv$ and then multiplying both sides by y^{β} from we get

$$y^{\beta}(G(x,u,wztv)) = \sum_{n=0}^{\infty} a_n \left(P_n^{(\alpha,\beta)}(x)y^{\beta}z^n\right)\left(P_m^{(n,\beta)}(u)t^n\right)(wv)^n. \tag{13}$$

Operating e^{wR_1},e^{wR_2} on both sides of (13), we have

$$e^{wR_1}e^{wR_2}\left[y^{\beta}(G(x,u,wztv)\right] = e^{wR_1}e^{wR_2}\left[\sum_{n=0}^{\infty} a_n(P_n^{(\alpha,\beta)}(x)y^{\beta}z^n)(P_m^{(n,\beta)}(u)t^n)(wv)^n\right]. \tag{14}$$

Now the left member of (14) becomes

$$\begin{aligned}
&e^{wR_1}e^{wR_2}\left[y^{\beta}(G(x,u,wztv)\right]\\
&= e^{wR_1}\left[(-wt)^{-(1+\beta+m)}y^{\beta}G\left(x,\frac{u+wt}{1-wt},\frac{wztv}{1-wt}\right)\right]\\
&y^{\alpha+\beta+1}(1-wt)^{-(1+\beta+m)}(y+2wt)^{\beta}(y+2\omega z)^{-(\alpha+\beta+1)}\\
&G\left(\frac{xy+2\omega z}{y+2\omega z},\frac{u+wt}{1-wt},\frac{wyztv}{(1-wt)(y+2\omega z)}\right).
\end{aligned} \tag{15}$$

On the other hand the right member of (14) becomes

$$\begin{aligned}
&e^{wR_1}e^{wR_2}\left[\sum_{n=0}^{\infty} a_n(P_n^{(\alpha,\beta)}(x)y^{\beta}z^n)(P_m^{(n,\beta)}(u)t^n)(wv)^n\right]\\
&= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{(p+q+n)}}{p!q!}v^n(-2)^q(n+1)_q P_{n+q}^{(\alpha,\beta,-q)}(x)y^{\beta-q}z^{n+q}\\
&(1+n+\beta+m)_p P_m^{(n+p+\beta)}(u)t^{n+p}.
\end{aligned} \tag{16}$$

Equating (15) and (16) we get the following

$$\begin{aligned}
&(1-wt)^{-(1+\beta+m)}(y+2wz)^{\beta}(y+2\omega z)^{-(1+\alpha+\beta)}y^{\alpha+1}G\left(\frac{xy+2\omega z}{y+2\omega z},\frac{u+wt}{1-wt},\frac{wztv}{(1-wt)(y+2\omega z)}\right)\\
&= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{p+q}}{p!q!}(n+1)_q(wzvt)^n(-2y^{-1}z)^q t^p(1+n+\beta+m)_p P_{n+q}^{(\alpha.\beta-q)}(x)P_m^{(n+p,\beta)}(u)
\end{aligned} \tag{17}$$

which is our desired result.

2. Application

Putting $m=0$, $y=z=t=1$ in the above stated result (17), we obtain

$$\begin{aligned}
&(1-w)^{-(1+\beta)}(1+2w)^{\beta}(1+2\omega)^{-(1+\alpha+\beta)}G\left(\frac{x+2\omega}{1+2\omega},\frac{wv}{(1-w)(1+2\omega)}\right)\\
&= \sum_{n=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^q}{q!}(n+1)_q(wv)^n(-2)^q P_{n+q}^{(\alpha,\beta-q)}(x)\sum_{n=0}^{\infty}\frac{w^p}{p!}(1+n+\beta)_p.
\end{aligned} \tag{18}$$

Now, replacing $2w$ by $(-r)$, s by $-v/(2+r)$ and then simplifying we get

$$(1-r)^{\beta}[1-r(1+x)/2]^{-(1+\alpha+\beta)}G\left(\frac{x-r(1+x)/2}{1-r(1+x)/2}, \frac{sr}{1-r(1+x)/2}\right) = \sum_{n=0}^{\infty} r^n \sigma_n(x,s)$$

where

$$\sigma_n(x,s) = \sum_{q=0}^{n} \binom{n}{q} P_n^{(\alpha,\beta-n+q)}(x)\, s^q$$

which is found derived in [2].

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