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A generating function for Laguerre–Sobolev orthogonal polynomials

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Abstract

Let $\{S_n\}$ denote the sequence of polynomials orthogonal with respect to the Sobolev inner product

$$(f,g)_S = \int_0^{+\infty} f(x)g(x)x^\alpha e^{-x}\,dx + \lambda \int_0^{+\infty} f'(x)g'(x)x^\alpha e^{-x}\,dx,$$

where $\alpha > -1$, $\lambda > 0$ and the leading coefficient of the S_n is equal to the leading coefficient of the Laguerre polynomial $L_n^{(\alpha)}$. In this work, a generating function for the Sobolev–Laguerre polynomials is obtained.

Keywords: Laguerre polynomials; Sobolev orthogonal polynomials; Generating functions

1. Introduction

Consider the Sobolev inner product

$$(f,g)_S = \int_0^{+\infty} f(x)g(x)x^\alpha e^{-x}\,dx + \lambda \int_0^{+\infty} f'(x)g'(x)x^\alpha e^{-x}\,dx, \tag{1}$$

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with $\alpha > -1$ and $\lambda > 0$. Let $\{S_n\}$ denote the sequence of polynomials orthogonal with respect to (1), normalized by the condition that S_n and the Laguerre polynomial $L_n^{(\alpha)}$ have the same leading coefficient $(n = 0, 1, 2, \ldots)$.

The special case $\alpha = 0$ has already been studied by Brenner [1]. In [11], Schäfke and Wolf introduced *einfache verallgemeinerte klassische Orthogonalpolynome* and the above defined sequence $\{S_n\}$ is a special case of them. The inner product (1) can also be studied as a special case of inner products defined by a *coherent pair of measures* as introduced by Iserles et al. [4]. For a survey of possible applications and results on Sobolev orthogonal polynomials, see [5,9].

The most complete treatment of the sequence $\{S_n\}$ orthogonal with respect to (1) appears in a paper of Marcellán et al. [7]. The paper gives among others several algebraic and differential relations with $\{L_n^{(\alpha)}\}$, a four-term recurrence relation, a Rodrigues-type formula and some properties concerning the zeros. An asymptotic result for $S_n(x)$ with $x \in \mathscr{C} \setminus [0, +\infty)$ and $n \to \infty$, has been obtained by Marcellán et al. [6] in a recent paper.

Finally, we remark that asymptotic results for polynomials orthogonal with respect to a Sobolev inner product defined by a coherent pair of measures has been derived by Martínez-Finkelshtein et al. [8] in the Jacobi case and by Meijer et al. [10] in the Laguerre case.

The aim of the present paper is to derive a generating function for the Laguerre–Sobolev polynomials. Our result is a generalization of the generating function for the Laguerre polynomials $L_n^{(\alpha)}$

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\omega^n = (1-\omega)^{-\alpha-1} \exp\left(-\frac{x\omega}{1-\omega}\right) \tag{2}$$

(see Szegő [12, p. 101, (5.1.9)]). The particular case $\alpha = 0$ has been studied by Wimp and Kiesel [13] with a different technique.

Section 2, gives the basic relations on Laguerre–Sobolev polynomials. In particular, it is shown that a generating function for the Laguerre–Sobolev polynomials can be found from a generating function for the classical Laguerre polynomials (Lemma 2.5). As a consequence, we refind the result of Wimp and Kiesel (Theorem 2.1). In Section 3, a generating function for Laguerre–Sobolev polynomials if $\alpha \neq 0$ is derived. The main result is stated in Theorem 3.1. Finally, in Section 4, some generalizations are discussed.

2. Laguerre–Sobolev orthogonal polynomials

Let $\{S_n\}$ denote the sequence of polynomials orthogonal with respect to the Sobolev inner product

$$(f, g)_S = \int_0^{+\infty} f(x)g(x)x^{\alpha}e^{-x}\,dx + \lambda \int_0^{+\infty} f'(x)g'(x)x^{\alpha}e^{-x}\,dx, \tag{3}$$

with $\alpha > -1$ and $\lambda > 0$. The S_n are normalized by the condition that the leading coefficient of S_n equals the leading coefficient of $L_n^{(\alpha)}$.

Observe that $S_0 = L_0^{(\alpha)}$ and $S_1 = L_1^{(\alpha)}$.

Several authors obtained the following result, see e.g. [7].

Lemma 2.1. *There exist positive constants a_n depending on α and λ, such that*

$$L_n^{(\alpha-1)}(x) = L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x) = S_n(x) - a_{n-1}S_{n-1}(x), \quad n \geqslant 1. \tag{4}$$

Marcellán et al. [7] found the following recurrence relation.

Lemma 2.2. *The sequence $\{a_n\}$ in* (4) *satisfies*

$$a_n = \frac{n+\alpha}{n(2+\lambda)+\alpha-na_{n-1}}, \quad n \geqslant 1 \tag{5}$$

with

$$a_0 = 1.$$

In order to derive a generating function for S_n we need more information on the sequence $\{a_n\}$.

Lemma 2.3. *The sequence $\{a_n\}$ is convergent, and*

$$a = \lim_{n\to\infty} a_n = \frac{\lambda+2-\sqrt{\lambda^2+4\lambda}}{2} < 1. \tag{6}$$

Proof. First, we observe that a simple induction argument applied on Lemma 2.2 gives $a_n \leqslant 1$ for all $n \geqslant 0$.

Suppose that $a = \lim_{n\to\infty} a_n$ exists, then (5) implies

$$a^2 - a(2+\lambda) + 1 = 0.$$

Since $a_n \leqslant 1$ for all $n \geqslant 0$, we have $a \leqslant 1$. Hence

$$a = \frac{\lambda+2-\sqrt{\lambda^2+4\lambda}}{2} < 1.$$

Now, we prove that $\{a_n\}$ is indeed convergent to a.

With (5) and $a(2+\lambda) = a^2+1$ we have

$$a_n - a = \frac{\alpha - \alpha a + na(a_{n-1}-a)}{n(2+\lambda)+\alpha-na_{n-1}}.$$

Then, using $a_{n-1} \leqslant 1$,

$$|a_n - a| \leqslant \frac{|\alpha-\alpha a|}{n(1+\lambda)+\alpha} + \frac{na|a_{n-1}-a|}{n(1+\lambda)+\alpha}.$$

Hence

$$\limsup |a_n - a| \leqslant \frac{a}{1+\lambda} \limsup |a_n - a|.$$

Since $\frac{a}{1+\lambda} < 1$, the lemma follows. $\square$

From the sequence $\{a_n\}$ we construct a sequence $\{q_n(\lambda)\}$ of polynomials in λ.

Lemma 2.4. *Define the sequence* $\{q_n(\lambda)\}$ *by*

$$q_0(\lambda) = 1, \quad q_{n+1}(\lambda) = \frac{q_n(\lambda)}{a_n}, \quad n \geqslant 0.$$

Then $q_n(\lambda)$ *is a polynomial in* λ, $\deg q_n = n-1$ *if* $n \geqslant 1$, *satisfying the three-term recurrence relation*

$$(n+\alpha)q_{n+1}(\lambda) = (n(\lambda+2)+\alpha)q_n(\lambda) - nq_{n-1}(\lambda), \quad n \geqslant 1 \tag{7}$$

with initial conditions $q_0(\lambda) = q_1(\lambda) = 1$.

Proof. The recurrence relation (7) is just relation (5) rewritten in terms of q_n. Since $a_0 = 1$, $q_1 = 1$ and then (7) implies that, for $n \geqslant 1$, q_n is a polynomial in λ of degree $n-1$. $\square$

The convergence of a series involving the Laguerre–Sobolev orthogonal polynomials can be reduced to the convergence of a series involving Laguerre polynomials.

Lemma 2.5. *For* $|\omega| < a < 1$ *we have*

$$\sum_{n=0}^{\infty} q_n(\lambda) S_n(x) \omega^n = \frac{1}{1-\omega} \sum_{n=0}^{\infty} q_n(\lambda) L_n^{(\alpha-1)}(x) \omega^n. \tag{8}$$

Proof. Since

$$\lim_{n\to\infty} \frac{q_{n+1}(\lambda)}{q_n(\lambda)} = \frac{1}{a}$$

and the series in (2) converges for $|\omega| < 1$, the series in the right-hand side of (8) is convergent for $|\omega| < a$.

Now, Eq. (4) gives

$$q_n(\lambda) L_n^{(\alpha-1)}(x) = q_n(\lambda) S_n(x) - q_{n-1}(\lambda) S_{n-1}(x) \tag{9}$$

and therefore

$$q_n(\lambda) S_n(x) = \sum_{i=0}^{n} q_i(\lambda) L_i^{(\alpha-1)}(x).$$

In this way, we can write

$$q_n(\lambda)S_n(x)\omega^n = \sum_{i=0}^{n} (q_i(\lambda)L_i^{(\alpha-1)}(x)\omega^i)\omega^{n-i}$$

and thus, the series

$$\sum_{n=0}^{\infty} q_n(\lambda)S_n(x)\omega^n$$

converges, because it is the Cauchy product of the two convergent series

$$\sum_{n=0}^{\infty} \omega^n = \frac{1}{1-\omega}$$

and

$$\sum_{n=0}^{\infty} q_n(\lambda)L_n^{(\alpha-1)}(x)\omega^n.$$

Moreover, we conclude

$$\sum_{n=0}^{\infty} q_n(\lambda)S_n(x)\omega^n = \frac{1}{1-\omega}\sum_{n=0}^{\infty} q_n(\lambda)L_n^{(\alpha-1)}(x)\omega^n. \qquad \square$$

We have now to distinguish $\alpha = 0$ and $\alpha \neq 0$. The generating function if $\alpha \neq 0$ will be derived in the next section. The generating function if $\alpha = 0$ is stated in the following theorem; it is the result given by Wimp and Kiesel [13] using the expression of the Laguerre–Sobolev polynomials in terms of determinants.

Theorem 2.1. *Let $\alpha = 0$. Let $\{S_n\}$ denote the sequence of polynomials orthogonal with respect to the Sobolev inner product* (3) *with $\alpha = 0$ and normalized by the condition that the leading coefficient of S_n equals the leading coefficient of $L_n^{(0)}$. Let the sequence of polynomials $\{q_n(\lambda)\}$ be defined by the recurrence relation*

$$q_{n+1}(\lambda) = (\lambda+2)q_n(\lambda) - q_{n-1}(\lambda) \tag{10}$$

with the initial conditions $q_0(\lambda) = q_1(\lambda) = 1$.

Then, for $|\omega| < a < 1$,

$$\sum_{n=0}^{\infty} q_n(\lambda)S_n(x)\omega^n = \frac{1}{(1-\omega)(1+a)} \times \left[\exp\left(-\frac{x\omega a}{1-\omega a}\right) + a\exp\left(-\frac{x\omega/a}{1-\omega/a}\right)\right], \tag{11}$$

where

$$a = \frac{\lambda+2-\sqrt{\lambda^2+4\lambda}}{2}.$$

Proof. If $\alpha = 0$ the three-term recurrence relation (7) reduces to (10). Thus, we can give an explicit representation of $q_n(\lambda)$, in fact, we have

$$q_n(\lambda) = \frac{1}{1+a}(a^n + aa^{-n}).$$

Then the theorem follows from Lemma 2.5 and (2). □

3. Generating function if $\alpha \neq 0$

In this section, always $\alpha > -1$, $\alpha \neq 0$. We will derive a generating function for the polynomials $\{S_n\}$ starting from relation (8). It is possible to give an explicit representation for the polynomials $q_n(\lambda)$. However, we need a generating function for the $q_n(\lambda)$ rather than the $q_n(\lambda)$ itself.

Lemma 3.1. *Let $\alpha > -1$, $\alpha \neq 0$ and let the polynomials $q_n(\lambda)$ be defined by the recurrence relation* (7) *with initial conditions $q_0 = q_1 = 1$. Put*

$$F(\omega) = \sum_{n=0}^{\infty} q_n(\lambda)\Gamma(n+\alpha)\frac{\omega^n}{n!}, \tag{12}$$

with $|\omega| < a < 1$. Then

$$F(\omega) = \Gamma(\alpha)(1-a\omega)^{-\beta}\left(1-\frac{\omega}{a}\right)^{-\gamma}, \tag{13}$$

where

$$\beta = \frac{\alpha}{1+a}, \quad \gamma = \frac{\alpha}{1+1/a} \tag{14}$$

and

$$a = \frac{\lambda + 2 - \sqrt{\lambda^2 + 4\lambda}}{2}.$$

Proof. Observe that the ratio test shows that the series in the right-hand side of (12) is convergent if $|\omega| < a < 1$.

To simplify write

$$h_n(\lambda) = \frac{q_n(\lambda)\Gamma(n+\alpha)}{n!}, \quad n \geq 0,$$

then

$$F(\omega) = \sum_{n=0}^{\infty} h_n(\lambda)\omega^n.$$

From the three-term recurrence relation (7) for the polynomials $q_n(\lambda)$ we obtain the recurrence relation for $h_n(\lambda)$

$$(n+1)h_{n+1}(\lambda) = \{n(\lambda+2)+\alpha\}h_n(\lambda) - (n+\alpha-1)h_{n-1}(\lambda), \quad n \geq 1, \tag{15}$$

with $h_0(\lambda) = \Gamma(\alpha)$, $h_1(\lambda) = \Gamma(\alpha+1)$.

Multiply (15) with ω^n and sum over $n = 1, 2, \ldots$ then

$$F'(\omega) - h_1(\lambda) = (\lambda + 2)\omega F'(\omega) + \alpha(F(\omega) - h_0(\lambda)) - \omega^2 F'(\omega) - \alpha\omega F(\omega).$$

Hence

$$F'(\omega)\{1 - (\lambda + 2)\omega + \omega^2\} = \alpha F(\omega)(1 - \omega).$$

Observe

$$\lambda + 2 = a + \frac{1}{a},$$

then

$$F'(\omega)(\omega - a)\left(\omega - \frac{1}{a}\right) = \alpha F(\omega)(1 - \omega)$$

and

$$\frac{F'(\omega)}{F(\omega)} = -\frac{\gamma}{(\omega - a)} - \frac{\beta}{(\omega - 1/a)},$$

where

$$\beta = \frac{\alpha}{1 + a}, \quad \gamma = \frac{\alpha}{1 + 1/a}$$

and the lemma follows from $F(0) = h_0(\lambda) = \Gamma(\alpha)$. □

Remark 3.1. Relation (15) is the recurrence relation for the Pollaczek polynomials with suitable choice of the parameters. In fact,

$$h_n = \Gamma(\alpha) P_n^{\alpha/2}\left(\frac{\lambda + 2}{2}; -\frac{\alpha}{2}, \frac{\alpha}{2}\right)$$

and Lemma 3.1 can be derived from the generating function of the Pollaczek polynomials, see [2, p. 184].

Lemma 3.2. *Let* $\alpha > -1$, $\alpha \neq 0$ *and* $|\omega| < a < 1$. *Then*

$$\sum_{n=0}^{\infty} q_n(\lambda) L_n^{(\alpha-1)}(x)\omega^n = \Gamma(\alpha)(1 - a\omega)^{-\beta}\left(1 - \frac{\omega}{a}\right)^{-\gamma} \sum_{l=0}^{\infty} \binom{-\beta}{l} \left(\frac{x\omega a}{1 - \omega a}\right)^l \times \sum_{m=0}^{\infty} \binom{-\gamma}{m} \left(\frac{x\omega/a}{1 - \omega/a}\right)^m \frac{1}{\Gamma(\alpha + l + m)},$$

where β *and* γ *are defined by* (14).

Proof. Using the explicit representation of the Laguerre polynomials (see [12, p. 101, (5.1.6)]) we get

$$\begin{aligned} K &= \sum_{n=0}^{\infty} q_n(\lambda) L_n^{(\alpha-1)}(x)\omega^n \\ &= \sum_{n=0}^{\infty} q_n(\lambda)\omega^n \sum_{k=0}^{n} \frac{\Gamma(n+\alpha)}{(n-k)!\Gamma(\alpha+k)} \frac{(-x)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(-x)^k \omega^k}{k!\Gamma(\alpha+k)} \sum_{n=k}^{\infty} \frac{q_n(\lambda)\Gamma(n+\alpha)}{(n-k)!}\omega^{n-k}. \end{aligned}$$

We now apply Lemma 3.1

$$\begin{aligned} K &= \sum_{k=0}^{\infty} \frac{(-x)^k \omega^k}{k!\Gamma(\alpha+k)} F^{(k)}(\omega) \\ &= \Gamma(\alpha) \sum_{k=0}^{\infty} \frac{(-x)^k \omega^k}{k!\Gamma(\alpha+k)} \sum_{l=0}^{k} \binom{k}{l} D^l (1-a\omega)^{-\beta} D^{k-l}\Big(1-\frac{\omega}{a}\Big)^{-\gamma} \\ &= \Gamma(\alpha)(1-a\omega)^{-\beta}\Big(1-\frac{\omega}{a}\Big)^{-\gamma} \\ &\quad \times \sum_{k=0}^{\infty} \frac{x^k\omega^k}{\Gamma(\alpha+k)} \sum_{l=0}^{k} \binom{-\beta}{l}\Big(\frac{a}{1-a\omega}\Big)^l \binom{-\gamma}{k-l}\Big(\frac{1/a}{1-\omega/a}\Big)^{k-l} \\ &= \Gamma(\alpha)(1-a\omega)^{-\beta}\Big(1-\frac{\omega}{a}\Big)^{-\gamma} \\ &\quad \times \sum_{l=0}^{\infty} \binom{-\beta}{l}\Big(\frac{x\omega a}{1-\omega a}\Big)^l \sum_{k=l}^{\infty} \binom{-\gamma}{k-l}\Big(\frac{x\omega/a}{1-\omega/a}\Big)^{k-l} \frac{1}{\Gamma(\alpha+k)}. \end{aligned}$$

Substituting $k = l + m$ in the last series, we arrive at the lemma. □

The following lemma enables us to give the sum of the double series in Lemma 3.2.

Lemma 3.3. *Suppose* $\beta+\gamma \notin \{0,-1,-2,\dots\}$, *then*

$$\begin{aligned} &\sum_{l=0}^{\infty} \binom{-\beta}{l} u^l \sum_{m=0}^{\infty} \binom{-\gamma}{m} \frac{v^m}{\Gamma(\beta+\gamma+l+m)} \\ &\quad = \frac{e^{-v}}{\Gamma(\beta+\gamma)}\, {}_1F_1(\beta;\beta+\gamma;v-u) = \frac{e^{-u}}{\Gamma(\beta+\gamma)}\, {}_1F_1(\gamma;\beta+\gamma;u-v). \end{aligned}$$

Proof.

$$\sum_{l=0}^{\infty}\binom{-\beta}{l}u^l\sum_{m=0}^{\infty}\binom{-\gamma}{m}\frac{v^m}{\Gamma(\beta+\gamma+l+m)}$$
$$=\sum_{l=0}^{\infty}\binom{-\beta}{l}\frac{u^l}{\Gamma(\beta+\gamma+l)}\sum_{m=0}^{\infty}\frac{(\gamma)_m(-v)^m}{m!(\beta+\gamma+l)_m}$$
$$=\sum_{l=0}^{\infty}\binom{-\beta}{l}\frac{u^l}{\Gamma(\beta+\gamma+l)}\,{}_1F_1(\gamma;\beta+\gamma+l;-v).$$

Using Kummer's first relation

$$ {}_1F_1(a;c;z)=e^z{}_1F_1(c-a;c;-z), \tag{16}$$

we obtain

$$e^{-v}\sum_{l=0}^{\infty}\binom{-\beta}{l}\frac{u^l}{\Gamma(\beta+\gamma+l)}\,{}_1F_1(\beta+l;\beta+\gamma+l;v)$$
$$=\frac{e^{-v}}{\Gamma(\beta+\gamma)}\sum_{l=0}^{\infty}\frac{(\beta)_l(-u)^l}{l!(\beta+\gamma)_l}\,{}_1F_1(\beta+l;\beta+\gamma+l;v)$$
$$=\frac{e^{-v}}{\Gamma(\beta+\gamma)}\sum_{l=0}^{\infty}\frac{(-u)^l}{l!}\left(\frac{d}{dv}\right)^l{}_1F_1(\beta;\beta+\gamma;v).$$

The last series is the Taylor expansion of ${}_1F_1(\beta;\beta+\gamma;v-u)$, which proves the first assertion of the lemma. The second equality follows with (16). □

Remark 3.2. Lemma 3.3 can also be derived from [3, Section 5.10 (1) and Section 5.7.1 (6)]. By the first relation

$$F_1\left(a,\beta,\gamma;\beta+\gamma;-\frac{u}{a},-\frac{v}{a}\right)=\left(1+\frac{v}{a}\right)^{-a}F\left(a,\beta;\beta+\gamma;\frac{-u/a+v/a}{1+v/a}\right)$$

and taking limit as $a\to\infty$ the first equality of Lemma 3.3 follows.

From Lemmas 2.5, 3.2 and 3.3 we obtain our main result. Observe that β and γ in (14) satisfy $\beta+\gamma=\alpha$.

Theorem 3.1. *Let* $\alpha>-1$, $\alpha\neq 0$. *Let* $\{S_n\}$ *denote the sequence of polynomials orthogonal with respect to the Sobolev inner product* (3), *normalized by the condition that the leading coefficient of* S_n *equals the leading coefficient of* $L_n^{(\alpha)}$. *Let the sequence of polynomials* $\{q_n(\lambda)\}$ *be defined by the recurrence relation* (7) *with* $q_0(\lambda)=q_1(\lambda)=1$.

Then, for $|\omega| < a < 1$,

$$(1-\omega)\sum_{n=0}^{\infty} q_n(\lambda)S_n(x)\omega^n = (1-a\omega)^{-\beta}\left(1-\frac{\omega}{a}\right)^{-\gamma} e^{-v}\ {}_1F_1(\beta;\alpha;v-u)$$
$$= (1-a\omega)^{-\beta}\left(1-\frac{\omega}{a}\right)^{-\gamma} e^{-u}\ {}_1F_1(\gamma;\alpha;u-v), \qquad (17)$$

where

$$\beta = \frac{\alpha}{1+a}, \quad \gamma = \frac{\alpha}{1+1/a}$$

and

$$u = \frac{x\omega a}{1-\omega a}, \quad v = \frac{x\omega/a}{1-\omega/a}, \quad a = \frac{\lambda+2-\sqrt{\lambda^2+4\lambda}}{2}. \qquad (18)$$

Remark 3.3. If $k \geqslant 1$, then substitution of $\alpha = 0$ in $\frac{(\beta)_k}{(\alpha)_k}$ reduces it to $\frac{1}{1+a}$. Hence, substitution of $\alpha = 0$ in ${}_1F_1(\beta;\alpha;v-u)$ gives

$$1 + \frac{1}{1+a}\sum_{k=1}^{\infty}\frac{1}{k!}(v-u)^k = \frac{1}{1+a}e^{v-u} + \frac{a}{1+a}$$

and we arrive at (11), the result of Wimp and Kiesel.

Remark 3.4. For $\lambda = 0$, we have $q_n = 1$ for all $n \geqslant 0$, $S_n = L_n^{(\alpha)}$, $a = 1$, so the confluent hypergeometric function reduces to ${}_1F_1(\frac{\alpha}{2};\alpha;0) = 1$ and the theorem reduces to (2), the generating function of the classical Laguerre polynomials.

4. Generalizations

The results of the preceding sections can be generalized to Sobolev inner products of the form

$$(f,g)_S = \int_0^{+\infty} f(x)g(x)\,d\psi_0(x) + \lambda\int_0^{+\infty} f'(x)g'(x)x^{\alpha}e^{-x}\,dx, \qquad (19)$$

with $\lambda > 0$, $\alpha \geqslant 0$ and

(a) if $\alpha = 0$, then $d\psi_0(x) = e^{-x}\,dx + M\delta(0)$, with $M \geqslant 0$;
(b) if $\alpha \neq 0$, then $d\psi_0(x) = (x-\xi)x^{\alpha-1}e^{-x}\,dx$, with $\xi \leqslant 0$.

The pair $\{d\psi_0(x), x^{\alpha}e^{-x}\,dx\}$ is a coherent pair of Laguerre type I studied in [10].

Let $\{S_n\}$ denote the sequence of polynomials orthogonal with respect to (19) with leading coefficient of S_n equal to the leading coefficient of $L_n^{(\alpha)}$. Relations (4) and (6) are still satisfied; for a proof we refer to [10].

Lemma 4.1. *There exist positive constants a_n depending on α, λ and M or ξ, such that*

$$L_n^{(\alpha-1)}(x) = S_n(x) - a_{n-1}S_{n-1}(x), \quad n \geqslant 1; \tag{20}$$

the sequence $\{a_n\}$ is convergent and

$$a = \lim_{n\to\infty} a_n = \frac{\lambda + 2 - \sqrt{\lambda^2 + 4\lambda}}{2} < 1.$$

Define as in Section 2 the $\{q_n(\lambda)\}$ by

$$q_0(\lambda) = 1, \quad q_{n+1}(\lambda) = \frac{q_n(\lambda)}{a_n}, \quad n \geqslant 0.$$

Then (20) implies

$$q_n(\lambda)L_n^{(\alpha-1)}(x) = q_n(\lambda)S_n(x) - q_{n-1}(\lambda)S_{n-1}(x), \quad n \geqslant 1.$$

This is the starting formula (9) in the proof of Lemma 2.5. Hence, Lemma 2.5 is still satisfied.

The recurrence relation for the a_n in (20), however, is somewhat different from the recurrence relation in Lemma 2.2. We distinguish $\alpha = 0$ and $\alpha > 0$.

Theorem 4.1. *Let $\alpha = 0$. Let $\{S_n\}$ denote the sequence of polynomials orthogonal with respect to* (19) *with $d\psi_0(x) = e^{-x}\,dx + M\delta(0)$ where $M \geqslant 0$ and the leading coefficient of S_n be equal to the leading coefficient of $L_n^{(0)}$. Let the sequence of polynomials $\{q_n(\lambda)\}$ be defined by*

$$q_{n+1}(\lambda) = (\lambda + 2)q_n(\lambda) - q_{n-1}(\lambda), \quad n \geqslant 1, \tag{21}$$

with the initial conditions $q_0(\lambda) = 1$, $q_1(\lambda) = M + 1$.

Then, for $|\omega| < a < 1$,

$$\sum_{n=0}^{\infty} q_n(\lambda)S_n(x)\omega^n = \frac{1}{1-\omega}\left[A\exp\left(-\frac{x\omega a}{1-\omega a}\right) + B\exp\left(-\frac{x\omega/a}{1-\omega/a}\right)\right],$$

where

$$A = \frac{1}{1+a} - M\frac{a}{1-a^2}, \quad B = \frac{a}{1+a} + M\frac{a}{1-a^2}. \tag{22}$$

Proof. The recurrence relation for the a_n in (20) reads (see [10])

$$a_n = \frac{1}{2 + \lambda - a_{n-1}}, \quad n \geqslant 1, \quad a_0 = \frac{1}{M+1},$$

then the recurrence relation for the q_n becomes (21) with $q_0(\lambda) = 1$, $q_1(\lambda) = M + 1$. The recurrence relation (21) can be solved explicitly and

$$q_n(\lambda) = Aa^n + Ba^{-n}$$

with A and B given by (22). Then (2) and Lemma 2.5 give the desired result. □

Remark 4.1. For $M = 0$ Theorem 4.1 reduces to Theorem 2.1, the result of Wimp and Kiesel [13].

We now turn to the case $\alpha > 0$, $d\psi_0(x) = (x - \xi)x^{\alpha-1}e^{-x}\,dx$, with $\xi \leqslant 0$. The recurrence relation for the a_n in (20) reads (see [10])

$$a_n = \frac{n + \alpha}{n(2 + \lambda) + \alpha - \xi - na_{n-1}}, \quad n \geqslant 1$$

and $a_0 = \frac{\alpha}{\alpha - \xi}$.

This implies the recurrence relation for the $q_n(\lambda)$:

$$(n + \alpha)q_{n+1}(\lambda) = \{n(\lambda + 2) + \alpha - \xi\}q_n(\lambda) - nq_{n-1}(\lambda), \quad n \geqslant 1, \tag{23}$$

with initial conditions $q_0(\lambda) = 1,\ q_1(\lambda) = 1 - \frac{\xi}{\alpha}$. Lemma 3.1 on the generating function of the $\{q_n(\lambda)\}$ has to be modified.

Lemma 4.2. *Let $\alpha > 0$ and let the polynomials $q_n(\lambda)$ be defined by the recurrence relation* (23) *with initial conditions $q_0(\lambda) = 1,\ q_1(\lambda) = 1 - \frac{\xi}{\alpha}$. Put*

$$F(\omega) = \sum_{n=0}^{\infty} q_n(\lambda)\Gamma(n + \alpha)\frac{\omega^n}{n!}$$

with $|\omega| < a < 1$. Then

$$F(\omega) = \Gamma(\alpha)(1 - a\omega)^{-\beta}\left(1 - \frac{\omega}{a}\right)^{-\gamma}, \tag{24}$$

where

$$\beta = \frac{\alpha}{1 + a} + \frac{\xi a}{1 - a^2}, \quad \gamma = \frac{\alpha}{1 + 1/a} - \frac{\xi a}{1 - a^2} \tag{25}$$

and

$$a = \frac{\lambda + 2 - \sqrt{\lambda^2 + 4\lambda}}{2}.$$

Proof. With

$$h_n(\lambda) = \frac{q_n(\lambda)\Gamma(n + \alpha)}{n!}, \quad n \geqslant 0,$$

relation (23) is transformed in

$$(n + 1)h_{n+1}(\lambda) = \{n(\lambda + 2) + \alpha - \xi\}h_n(\lambda) - (n + \alpha - 1)h_{n-1}(\lambda), \quad n \geqslant 1,$$

with $h_0(\lambda) = \Gamma(\alpha),\ h_1(\lambda) = \Gamma(\alpha)(\alpha - \xi)$.

This implies

$$F'(\omega) - h_1(\lambda) = (\lambda + 2)\omega F'(\omega) + (\alpha - \xi)(F(\omega) - h_0(\lambda)) - \omega^2 F'(\omega) - \alpha\omega F(\omega).$$

Hence

$$F'(\omega)\{1 - (\lambda + 2)\omega + \omega^2\} = F(\omega)(\alpha - \xi - \alpha\omega).$$

Then

$$\frac{F'(\omega)}{F(\omega)} = -\frac{\gamma}{(\omega - a)} - \frac{\beta}{(\omega - 1/a)},$$

where β and γ are defined in (25). With $F(0) = \Gamma(\alpha)$ we arrive at (24). □

Relation (24) equals (13) with the β and γ in (14) replaced by their values in (25). Observe that they still satisfy $\beta + \gamma = \alpha$. The calculations in the proof of Lemma 3.2 do not depend on the special values β and γ. So Lemma 3.2 is still satisfied with the values of β and γ given in (25). Finally, we arrive at the generating function for S_n stated in the following theorem.

Theorem 4.2. *Let $\alpha > 0$. Let $\{S_n\}$ denote the sequence of polynomials orthogonal with respect to* (19) *with $d\psi_0(x) = (x - \xi)x^{\alpha-1}e^{-x}\,dx$, where $\xi \leqslant 0$, and the leading coefficient of S_n be equal to the leading coefficient of $L_n^{(\alpha)}$. Let the sequence of polynomials $\{q_n(\lambda)\}$ be defined by* (23) *with $q_0(\lambda) = 1$, $q_1(\lambda) = 1 - \frac{\xi}{\alpha}$. Then, for $|\omega| < a < 1$, the generating function relation* (17) *is satisfied with β and γ given by* (25) *and u, v and a given by* (18).

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